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Some Useful Results Related with Sampling Theorem and Reconstruction Formula

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***Abstract*—In this paper, we present some useful results related with the sampling theorem and the reconstruction formula. The first of them regards a relation existing between bandwidths of interpolating functions different from a perfect-reconstruction one and the bandwidth of the latter. Furthermore, we prove here that two non-identical interpolating functions can have the same bandwidths if and only if their (same) bandwidth is a multiple of the bandwidth of an original unsampled signal. The next result shows that sets of sampling points of two non-identical (but not necessarily interpolating) functions possessing different bandwidths are unique for all sampling periods smaller or equal to a given period (calculated in a theorem provided). These results are completed by the following one: in case of two different signals possessing the same bandwidth but different spectra shapes, their sets of sampling points must differ from each other.**

***Keywords*—sampling theorem, cardinal series, reconstruction formula**

I. Introduction

IT seems that everything has already been said about sampling of signals, sampling theorem, and reconstruction formula. Everything seems to have been fully explained in thousands of articles and textbooks published on the above subjects. However, that is not entirely true, as we will see in this paper. We will show here that there are still some properties of the sampling operation and reconstruction formula that we did not get to know yet.

The paper is organized as follows. Section II contains a short description of basics of the sampling theorem and of the reconstruction formula [1]-[6], which we will need in derivations of the next sections. In section III, we show that bandwidths of interpolating functions different from the perfect-reconstruction one are greater than the bandwidth of the latter. Three interesting observations are presented in section IV. First, we prove that two non-identical interpolating functions can have the same bandwidths if and only if their (same) bandwidth is a multiple of the bandwidth of an original unsampled signal. Moreover, we show that sets of sampling points of two non-identical (but not necessarily interpolating) functions possessing different bandwidths are unique for all sampling periods smaller or equal to a given one (calculated in a theorem provided). Finally, we complete the above two results with the following one: in case of two different signals possessing the same bandwidth but different spectra shapes, their sets of sampling points must differ from each other. The paper ends with section V that contains conclusions.

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II. Preliminary Material

Let a signal $x(t)$ of a continuous time variable $t \in R$, where R denotes the set of real numbers, be sampled uniformly with a sampling period *T*, what leads to receiving an infinite set of signal samples. We denote it here by $\{..., x(-T), x(0), x(T), ...\} = x\{kT\}, k = ..., -1, 0, 1, ...$.

Assume now that only the set of samples, as defined above, of a signal $x(t)$ and the sampling period *T* are available. And having this, we want to recover an unknown form of the signal mentioned. That is we want to deduce from this data a function of a continuous time variable t that would be an original signal $x(t)$.

Before proceeding further, let us however note that the operation of signal recovery from its samples, as stated above, can be viewed as an inverse operation with respect to the signal sampling. So, as such, it can be formulated as searching for an appropriate inverse operator. However, this operator can exist or not. Obviously, when it exists this leads to achieving a perfect signal reconstruction from its samples. However, in cases it does not exist, it will be always possible to find an imperfect version of the original signal. Then, in terms of operators' terminology, we will speak about searching for a good pseudo-inverse operator.

As we know, with regard to the problem stated above, there exists a highly celebrated sampling theorem, which uses the following formula:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\mathrm{sinc}(t/T - k) \,, \tag{1}$$

where the function $\mathrm{sinc}(t)$ is defined as

$$\mathrm{sinc}(t) = \sin(\pi t)/\pi t \quad \text{for } t \neq 0 \quad \text{and} \quad 1 \text{ for } t = 0 \tag{2}$$

and $\hat{x}(t)$ means a function being an approximation of $x(t)$ that exploits just the set $x\{kT\}$ defined above. Further, the sampling theorem is formulated as follows: Let $x(t)$ be such a real function, whose bandwidth is finite. That is it denotes a bandlimited signal having a Fourier transform $X(f)$ satisfying the following equation:

$$X(f) \equiv 0 \text{ for } |f| > f_m > 0, \tag{3}$$

where f means a continuous frequency variable.

By assuming (3), we say that the Fourier transform $X(f)$ of $x(t)$ is identically zero outside a closed frequency interval $< -f_m, f_m >$. So, the bandwidth of the signal $x(t)$ is equal to $B = f_m - 0 = f_m$, where f_m means a real number. And, assume additionally that $X(f)$ is a piecewise continuous function on the set $< -f_m, f_m >$.

Then, the function $x(t)$ can be exactly approximated (or, in other words, it can be perfectly reconstructed from its samples defined above) at every point $t \in R$ with the use of (1), when the following:

$$T \le 1/(2 f_m) \tag{4}$$

is fulfilled. As we know this constitutes the so-called sampling theorem [2]. Furthermore, as it is also well known, (1) is called a reconstruction formula or a cardinal series.

We do not often realize how powerful is the signal sampling theorem. Its powerfulness follows from the fact that it really expresses an equivalence between a series of discrete indexed values and a certain function of a continuous variable. However, we underscore that this is only true, when an additional condition (4) is fulfilled. So, we see that the knowledge regarding fulfillment or non-fulfillment of (4) is crucial.

III. Sampling of Different Signals Leading to Identical Sets of Samples and Repercussions of This

Assume that we sample different signals of a continuous time *t* and get exactly the same sets of samples $x\{kT\}$ in all these cases. This is illustrated in Fig. 1.

Note that it is easy to get such a situation as that sketched in Fig. 1. For instance, imagine you have a set of discrete values $\{.., x(-T), x(0), x(T),..\} = x\{kT\}$, $k = ..,-1,0,1,..$, coming from an unknown analog signal, Assume, however, that you know its bandwidth, which is equal to $B = f_m - 0 = 1/(2T)$, and you wish to interpolate this signal. As illustrated in Fig. 1, you can do this in many ways, also with the use of the formula (1). Note that in the case considered it allows you to perform a perfect reconstruction (because the condition (4) is then fulfilled).

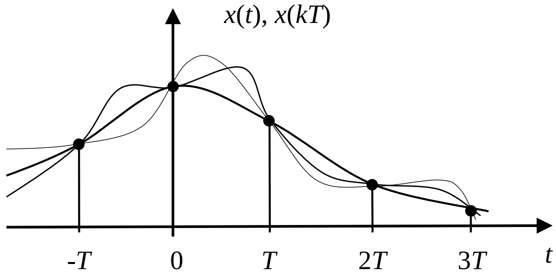

Fig. 1. Example showing three different signals of a continuous time variable *t* sampled in such a way that the sets of samples $x(kT)$, $k = ...,-1,0,1,...$, obtained are exactly the same in these cases.

An interesting question can arise regarding the above interpolating curves: Are the bandwidths of the interpolating functions different from that of the perfect-reconstruction one, or are lesser, equal to, or greater than the bandwidth of the latter? In what follows, we will answer this question.

Theorem 1. Bandwidths of interpolating functions different from the perfect-reconstruction one are greater than the bandwidth of the latter.

Proof: Obviously, the bandwidth of any of the interpolating functions differing from the perfect-reconstruction one cannot be equal to $1/(2T)$ because the latter is unique. So, their bandwidths must be lesser or greater than $1/(2T)$. And, let us start first with checking whether they can be lesser than $1/(2T)$. To this end, assume that there exists at least one such the function. Denote it as $x_1(t)$. Moreover, denote its bandwidth and maximal frequency as B_1 and f_{m1}, respectively. Further, let use T_1 for denoting the sampling period equal to $1/(2 f_{m1})$. Then, under the assumptions made above, we have

$$B_1 = f_{m1} = 1/(2T_1) < 1/(2T) = f_m = B \ . \tag{5}$$

This gives $T < T_1$. So, by virtue of the sampling theorem invoked in section II, the interpolating signal $x_1(t)$, when sampled at the rates $1/T_1 = 2 f_{m1}$ and $1/T = 2 f_m$, can be perfectly reconstructed using formula (1) in both these cases. Therefore, we can write

$$\begin{aligned} x_1(t) &= \sum_{k=-\infty}^{\infty} x_1(kT_1) \operatorname{sinc}(t/T_1 - k) = \\ &= \sum_{k=-\infty}^{\infty} x_1(kT) \operatorname{sinc}(t/T - k) \ . \end{aligned} \tag{6}$$

Now, on the other hand, note that we assumed that the samples of the interpolating signals $x(t)$ and $x_1(t)$ at the points kT, $k = ..., -1, 0, 1, ...$, are equal to each other. That is we have $x(kT) \equiv x_1(kT)$, $k = .., -1, 0, 1, ...$ Taking this into account in (6), we get

$$x_1(t) = \sum_{k=-\infty}^{\infty} x_1(kT_1)\ \mathrm{sinc}(t/T_1 - k) = \\ = \sum_{k=-\infty}^{\infty} x(kT)\ \mathrm{sinc}(t/T - k) = x(t) \quad . \tag{7}$$

However, this contradicts our assumption that $x_1(t)$ does not identically equal $x(t)$ (see also Fig. 1 for illustration). So, because of this fact, we must conclude that it is not possible to have an interpolating signal possessing the bandwidth smaller than $1/(2T)$. And this ends our proof. □

By the way, note that the problems of the kind mentioned above do not occur, when an interpolating signal, say $x_2(t)$, is assumed to have its bandwidth greater than $1/(2T)$. With an equivalent of (5) now in the form

$$B_2 = f_{m2} = 1/(2T_2) > 1/(2T) = f_m = B \ , \tag{8}$$

where B_2, f_{m2}, and T_2 have the same meaning as B_1, f_{m1}, and T_1 for the signal $x_1(t)$, we can write

$$\Delta x(t) = x_2(t) - x(t) = \\ = \sum_{k=-\infty}^{\infty} x_2(kT_2)\ \mathrm{sinc}(t/T_2 - k) - \\ - \sum_{k=-\infty}^{\infty} x(kT)\ \mathrm{sinc}(t/T - k) = \\ = \sum_{k=-\infty}^{\infty} x_2(kT_2)\ \mathrm{sinc}(t/T_2 - k) - \\ - \sum_{k=-\infty}^{\infty} x(kT_2)\ \mathrm{sinc}(t/T_2 - k) = \\ = \sum_{k=-\infty}^{\infty} \left[x_2(kT_2) - x(kT_2)\right] \mathrm{sinc}(t/T_2 - k) \ . \tag{9}$$

Note now that because the difference $x_2(kT_2) - x(kT_2)$ in (9) for $k = ..., -1, 0, 1, ...$ does not identically equal zero the most right-hand side sum in (9) is not identically equal to zero, too. That is $\Delta x(t)$ does not equal zero for all values of *t*, what is true (once again see Fig. 1 for illustration).

Finally in this section, we stress the importance - for the validity of considerations presented above - of the assumption that the signal $x(t)$ was sampled with such a sampling period *T* that fulfilled the condition (4). Note that this was crucial.

IV. Three Observations More

In this section, further three statements that regard the sampling theorem and the reconstruction formula are discussed. They are presented in form of short theorems, in a similar way as Theorem 1. Moreover, their proofs are also similar.

So, let us now begin with the first observation:

Theorem 2. Two non-identical interpolating functions that were defined in the previous section can have the same bandwidths if and only if their (same) bandwidth, let denote it by f_{me}, is a multiple of the bandwidth f_m of the signal $x(t)$.

Proof: We will prove this theorem by showing that assuming two non-identical interpolating functions possessing the same bandwidths leads to contradiction when f_{me} is not a multiple of f_m. To this end, assume that we found two different interpolating functions $x_1(t)$ and $x_2(t)$ having the same bandwidths $f_{m1} = f_{m2} = f_{me}$. Sampling both of them with the same sampling period, say T_e, that is given by $T_e = 1/(2 f_{me})$, and applying then the sampling theorem and the reconstruction formula (1) allows us to write

$$x_1(t) = \sum_{k=-\infty}^{\infty} x_1(kT_e)\ \mathrm{sinc}(t/T_e - k) \tag{10}$$

and

$$x_2(t) = \sum_{k=-\infty}^{\infty} x_2(kT_e)\ \mathrm{sinc}(t/T_e - k) \ . \tag{11}$$

In the next step, observe that because we assumed that $x_1(t)$ and $x_2(t)$ are interpolating functions of a function $x(t)$, the following equalities:

$$x_1(nT) = x_2(nT) = x(nT), \ ..., -1, 0, 1, ..., \tag{12}$$

must hold. So, applying (10) and (11) in (12) gives

$$\sum_{k=-\infty}^{\infty} \left[x_1(kT_e) - x_2(kT_e)\right] \mathrm{sinc}(nT/T_e - k) = 0 \ , \tag{13}$$

which must hold for all $n = ..., -1, 0, 1, ...$.

To proceed further, observe first that (13) holds for $n = 0$. That is because it follows from (13) that

$$\sum_{k=-\infty}^{\infty}\left[x_1(kT_e)-x_2(kT_e)\right]\operatorname{sinc}(-k)= \\ =..+0+0+\left[x_1(0)-x_2(0)\right]+0+0+...= \\ =x_1(0)-x_2(0)\ . \tag{14}$$

However, $x_1(0)=x_2(0)$ (meaning that all the interpolating functions have the same value at the point $t=0$; see also Fig. 1). Applying this in (14) leads to the conclusion that really (13) holds for $n=0$. Note also that the validity of (13) for $n=0$ does not depend upon values of the ratio T/T_e occurring in (13).

In what follows, we will check validity of (13) for all the other values of n. Now, however, we will need to distinguish between two cases: first one when the ratio T/T_e is not a natural number, and second when it is.

We begin with the first one. Observe that then the values of $\operatorname{sinc}(nT/T_e-k)$ do not identically equal zero for all the possible combinations of $n\in Z-\{0\}$, where Z denotes the set of integers, and $k\in Z$. So, only way to satisfy equations (13) is to require fulfillment of the following: $x_1(kT_e)=x_2(kT_e),\ k=...,-1,0,1,...$. Note however that this, in view of the reconstruction formula (1), is equivalent to saying that $x_1(t)=x_2(t)$. But, we assumed that the functions $x_1(t)$ and $x_2(t)$ are not identical. So, we arrived at a contradiction. That is the occurrence of this case is not possible.

Let us now consider the second case when the ratio T/T_e is a natural number, and denote it by c_e. (By the way, note that in view of theorem 1 of section III this natural number will be always greater than 1, ie. $c_e\in N-\{1\}$, where N denotes the set of natural numbers.) Further, see that we can express equivalently the above ratio in terms of the bandwidths as

$$\frac{T}{T_e}=\frac{1/(2f_m)}{1/(2f_{me})}=c_e \quad\Rightarrow\quad f_{me}=c_e f_m\ . \tag{15}$$

Substituting (15) into (13) gives

$$\sum_{k=-\infty}^{\infty}\left[x_1(kT_e)-x_2(kT_e)\right]\operatorname{sinc}(nc_e-k)= \\ =..+0+0+\left[x_1(nc_eT_e)-x_2(nc_eT_e)\right]+ \\ +0+0+...= \\ =x_1\left(n\frac{T}{T_e}T_e\right)-x_2\left(n\frac{T}{T_e}T_e\right) = \\ =x_1(nT)-x_2(nT) \tag{16}$$

for all $n\in Z-\{0\}$.

However, by virtue of that $x_1(t)$ and $x_2(t)$ are the interpolating functions assuming the same value at the points nT, $n\in Z$, $x_1(nT)=x_2(nT)$. Applying this in (16) leads to the conclusion that really (13) holds for all $n\in Z-\{0\}$, when (15) is satisfied. And this ends the proof. □

Our second observation is the following:

Theorem 3. Sets of sampling points of two non-identical (but not necessarily interpolating) functions possessing different bandwidths $B_1=f_{m1}$ and $B_2=f_{m2}$, respectively, are unique (in the sense that they are not identical) for all sampling periods $T_i\le\min\left(1/(2f_{m1}),1/(2f_{m2})\right)$.

Proof: To prove this theorem, consider two functions $x_1(t)$ and $x_2(t)$ having the bandwidths $B_1=f_{m1}$ and $B_2=f_{m2}$, respectively. Note that sampling them with the same sampling period T_i fulfilling the following inequality:

$$T_i\le\min\left(T_1=1/(2f_{m1}),T_2=1/(2f_{m2})\right), \tag{17}$$

and applying then the sampling theorem and the reconstruction formula (1) allows us to write

$$x_1(t)=\sum_{k=-\infty}^{\infty}x_1(kT_1)\operatorname{sinc}(t/T_1-k)= \\ =\sum_{k=-\infty}^{\infty}x_1(kT_i)\operatorname{sinc}(t/T_i-k) \tag{18}$$

and

$$x_2(t)=\sum_{k=-\infty}^{\infty}x_2(kT_2)\operatorname{sinc}(t/T_2-k)= \\ =\sum_{k=-\infty}^{\infty}x_2(kT_i)\operatorname{sinc}(t/T_i-k)\quad . \tag{19}$$

Consider now the most right-hand side expressions in (18) and (19). It follows from them, the reconstruction formula (1), and from the fact that the functions $x_1(t)$ and $x_2(t)$ were assumed to be not identical that the sequences $x_1\{kT_i\}$ and $x_2\{kT_i\}$, $k=...,-1,0,1,...$, are not identical, too. That is they are unique. And this ends the proof. □

We remark that theorem 3 holds also when two non-identical (but not necessarily interpolating) functions possess the same bandwidth, say f_{me}. Then, the condition (17) reduces simply to $T_i\le T_e=1/(2f_{me})$.

Let us now consider our third observation. We express it in a form of the following theorem:

Theorem 4. In case of two different signals possessing the same bandwidth but different spectra shapes, their sets of sampling points must differ from each other. That is they cannot be identical.

Proof: Consider two functions $x_1(t)$ and $x_2(t)$ that have the same bandwidths $f_{m1} = f_{m2} = f_{me}$ but different spectra shapes. Let us sample these signals with the same sampling period $T_1 = T_2 = T_e = 1/(2 f_{me})$. Then, see that because this period satisfies the condition (4) of the sampling theorem, we can express them using the reconstruction formula (1) as

$$x_1(t) = \sum_{k=-\infty}^{\infty} x_1(kT_e)\ \mathrm{sinc}(t/T_e - k) \tag{20}$$

and

$$x_2(t) = \sum_{k=-\infty}^{\infty} x_2(kT_e)\ \mathrm{sinc}(t/T_e - k)\quad . \tag{21}$$

Observe now that according to the sampling theorem the expressions on the right-hand sides of (20) and (21) are unique. So, because $x_1(t) \neq x_2(t)$ holds, it follows from the above that $x_1\{kT\} \neq x_2\{kT\}$ as well. And this ends the proof. □

The latter observation, maybe, may seem for many obvious. However, in our opinion, it is worthy to recall it also in this paper to complete the remaining ones discussed here.

V. CONCLUSIONS

Among the most fundamental tools of the digital signal processing are the sampling theorem and the reconstruction formula. Their history is long and dates, after [1], to 1841 and to 1897. In 1841, A. Cauchy recognized something what is called today a minimal sampling rate (rediscovered by H. Nyquist and named after him the Nyquist rate). Several years later, in 1897, another famous mathematician E. Borel recognized possibility of recovering a bandlimited signal from its samples. In the 20th century, E. T. Whittaker (1915), H. Nyquist (1928), V. A. Kotelnikov (1933), and C. Shannon (1948) published their works, in which they formulated the sampling theorem and the reconstruction formula (called also a cardinal series) in the form we know today. They introduced the aforementioned tools to the theory of signals and modern telecommunications. The literature on these topics is huge. Let us only mention, at the end of this paper, some of the most prominent publications in these areas, articles and books [1]-[26].

And finally, let us say the following: Nowadays, it seems that such topics like sampling of signals, sampling theorem, and reconstruction formula are fully developed, as mentioned above. This paper shows however that there are still new intriguing and useful results that can be obtained in this highly matured area.

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