Feasible star – delta and delta – star transformations for reliability networks

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Summary: Consider the problem of transforming a star (delta) into an equivalent delta (star) in a reliability network with imperfect undirected edges and perfect vertices. It is believed that such transformations are not possible in general if the probabilities of the elements of the given star / delta are rational numbers. Contrary to this, it is shown here that star – delta and delta – star transformations are possible under certain conditions. Further the probability of success of an element of an equivalent star (delta) is shown to be equal to the probability of failure of the corresponding element of the given delta (star).

Key words:
star – delta transformation, delta – star transformation, series parallel reliability networks

I. INTRODUCTION

Consider the problem of evaluating the reliability of a network with respect to a pair of terminals, s (source/input) and t (load / output). All elements are bidirectional and imperfect and the probabilities of success/failure of elements are known. All nodes are perfect. We wish to calculate the probability of success of the output i.e., there is connectivity between input and output. This problem is important for most systems including many electronic, electrical and power networks [1], [2]. It is well known that the evaluation of reliability of such networks is a NP hard problem [14]. On the other hand evaluation of reliability of series parallel networks is polynomial [14]. Therefore series parallel techniques [1] – [14] are important for reliability analysis. Star (Delta) — Delta (Star) transformations help us to transform some networks into series parallel networks. Further they help us to reduce computation even when series parallel techniques do not work on the entire network. Thus star ↔ delta transformations are important. They attracted the attention of many researchers in the last few decades [2] – [11]. Wang and Sun [10] studied this problem extensively and gave conditions under which star — delta and delta — star transformations are possible. But they argued that these conditions are never satisfied in practice except in special cases in which there are flow restrictions as in Gupta and Sharma [7] and Prasad [11]. Thus at present it is widely believed that star — delta and delta — star transformations are not possible in real life networks. Contrary to this it is shown in this paper that these transformations are possible. Therefore Wang and Sun's conditions are useful in practice. They are not just theoretical. Their conditions are greatly simplified to achieve this. It is further shown that the probability of success of an element of an equivalent delta (star) is same as the probability of failure of the corresponding element of the given star (delta).

II. STAR — DELTA, DELTA — STAR TRANSFORMATIONS

The star and delta networks under consideration are shown in Figure 1. Throughout this paper we assume that no element of star or delta has a probability zero or one. There is no loss of generality in this assumption because transformation of a star or delta with zero or one probability for one or more elements is trivial [10].

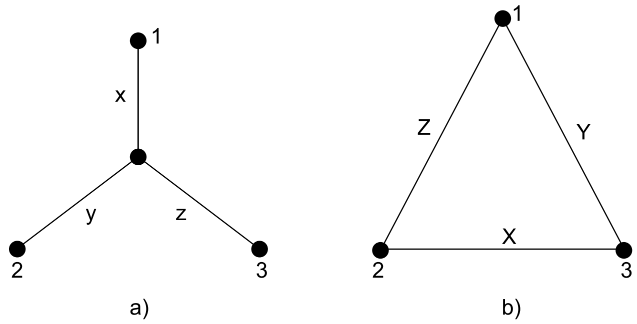

Fig. 1 (a) Star network, (b) Delta network.

2.1. Star — Delta transformation

Given the star of Figure 1 (a) with probabilities of success p_x, p_y and p_z, we want to determine p_X, p_Y and p_Z, the probabilities of success of the elements of the equivalent delta. Prasad [11] studied this problem when there are restrictions in flows. Wang and Sun [10] studied the general case. They partitioned the connection states into four classes for each node, derived equations, simplified them and got the following expressions.

$$p_X = \frac{S_3}{K + S_3} \tag{1}$$

$$p_Y = \frac{S_5}{K + S_5} \tag{2}$$

$$p_Z = \frac{S_6}{K + S_6} \quad (3)$$

where

$$K = S_0 + S_1 + S_2 + S_4 \quad (4)$$

S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7 are the eight possible states of star. S_0 is $q_x\, q_y\, q_z$. It corresponds to 000. S_1 is $q_x\, q_y\, p_z$. It corresponds to 001. Other states are also understood similarly. Wang and Sun [10] showed that equations (1), (2) and (3) can be used provided the probabilities p_x, p_y and p_z of the elements of star satisfy the equation

$$K^2 = (K + S_3)(K + S_5)(K + S_6) \quad (5)$$

i.e., star to delta transformation is possible if equation (5) holds.

We will first simplify this equation so that it is easy to use. It can be written as: $K^3 + K^2(S_3 + S_5 + S_6 - 1) + K(S_3S_5 + S_5S_6 + S_3S_6) + S_3S_5S_6 = 0$.

But $K = S_0 + S_1 + S_2 + S_4 = 1 - (S_3 + S_5 + S_6 + S_7)$ i.e., $S_3 + S_5 + S_6 - 1 = -(K + S_7)$. Substitute into the above equation and simplify. This gives:

$$S_7K^2 - BK - C = 0 \quad (6)$$

where $B = S_3S_5 + S_5S_6 + S_3S_6$ and $C = S_3S_5S_6$.

$B = (q_x\, p_y\, p_z)(p_x\, q_y\, p_z) + (p_x\, q_y\, p_z)(p_x\, p_y\, q_z) + (q_x\, p_y\, p_z)(p_x\, p_y\, q_z) = p_x\, p_y\, p_z(q_x\, q_y\, p_z + p_x\, q_y\, q_z + q_x\, p_y\, q_z) = = S_7(S_1 + S_4 + S_2)$.

Similarly $C = (q_x\, p_y\, p_z)(p_x\, q_y\, p_z)(p_x\, p_y\, q_z) = S_7^2 S_0$

Substitute into equation (6) and cancel S_7 throughout. This gives: $K^2 - K(S_1 + S_2 + S_4) - S_7S_0 = 0$
i.e., $K[K - (S_1 + S_2 + S_4)] - S_7S_0 = 0$. This implies that $KS_0 - S_7S_0 = 0$. Thus equivalent delta exists if the condition $K = S_7$ is satisfied.

But $K = (S_0 + S_1 + S_2 + S_3) - S_3 + S_4 = q_x - q_x\, p_y\, p_z + p_x\, q_y\, q_z$. $K = S_7$ means that $q_x + p_x\, q_y\, qz = p_y\, p_z$. This gives:

$$q_x + q_y + q_z = 1 + q_x\, q_y\, q_z \quad (7)$$

This equation is a simplified form of equation (5). $q_x = \frac{2}{7},\ q_y = \frac{1}{2},\ q_z = \frac{1}{4}$ satisfies this equation. Thus a star with these probabilities has an equivalent delta showing that Wang and Sun's condition for star delta transformation is useful in practice. Further Theorem 2 of Wang and Sun which says that star to delta transformation is not possible for rational probabilities of star is wrong. Substituting these probabilities into equations (1), (2) and (3), the probabilities of equivalent delta are $p_X = \frac{2}{7},\ p_Y = \frac{1}{2}$ and $p_Z = \frac{1}{4}$.

Some irrational numbers also satisfy equation (7).

$q_x = \frac{1}{\sqrt{2}},\ q_y = \frac{\sqrt{2}}{7},\ q_z = \frac{7\sqrt{2} - 9}{6\sqrt{2}}$ is an example.

Equations (1), (2) and (3) help us to calculate the probabilities of the elements of delta from those of star. But they can be simplified as follows.

<u>Theorem 1:</u> $p_X = q_x$ if equation (7) is satisfied.

Proof: We want to prove that $\frac{S_3}{K + S_3} = q_x$ i.e., $K + S_3 = p_y\, p_z$ as $S_3 = q_x\, p_y\, p_z$. But $K + S_3 = S_0 + S_1 + S_2 + S_3 + S_4 = q_x + p_x\, q_y\, q_z$ as $q_x = S_0 + S_1 + S_2 + S_3$. $p_y\, p_z = = (1 - q_y)(1 - q_z) = 1 - q_y - q_z + q_y\, q_z$. So we want $q_x + q_y + q_z + p_x\, q_y\, q_z = 1 + q_y\, q_z$. Using equation (7), L.H.S $= 1 + q_x\, q_y\, q_z + p_x\, q_y\, q_z$ = R.H.S. Hence the result.

<u>Theorem 2:</u> $p_Y = q_y$ if equation (7) is satisfied.

<u>Theorem 3:</u> $p_Z = q_z$ if equation (7) is satisfied.

The proofs of Theorems 2 and 3 are similar to Theorem 1.

2.2 Delta – Star Transformation:

Given the delta of Figure 1(b) with probabilities p_X, p_Y, p_Z, we want to determine the probabilities p_x, p_y, p_z of the equivalent star. Gupta and Sharma [7] studied this problem when there are flow restrictions. Wang and Sun [10] considered the general case and derived the following expressions.

$$p_x = \frac{N}{N + D_4} \quad (8)$$

$$p_y = \frac{N}{N + D_2} \quad (9)$$

$$p_z = \frac{N}{N + D_1} \quad (10)$$

where $N = D_3 + D_5 + D_6 + D_7$ (11)

and D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7 are states of delta. D_0 is $q_X\, q_Y\, q_Z$. It corresponds to 000. D_2 is $q_X\, p_Y\, q_Z$ corresponding to 010. Other states are also understood in a similar way. Wang and Sun [10] showed that equivalent star exists if

$$N^2 = (N + D_1)(N + D_2)(N + D_4) \quad (12)$$

Simplifying this we get

$$D_0N^2 - BN - C = 0 \quad (13)$$

where $B = D_1D_2 + D_2D_4 + D_1D_4$ and $C = D_1D_2D_4$. It can be shown that $B = D_0(D_3 + D_6 + D_5)$ and $C = D_0^2 D_7$. Substitute in equation (13) and simplify. This gives $N = D_0$. But $N = (D_4 + D_5 + D_6 + D_7) - D_4 + D_3 = p_X - p_X\, q_Y\, q_Z + q_X\, p_Y\, p_Z$. Substitute for N and D_0 and simplify this equation. We can then show that the required condition for the existence of an equivalent star is

$$p_X + p_Y + p_Z = 1 + p_X p_Y p_Z \qquad (14)$$

This equation is a simplified form of equation (12). $p_X = \frac{2}{7}$, $p_Y = \frac{1}{2}$ and $p_Z = \frac{1}{4}$ satisfy this equation. Thus a delta with these probabilities has an equivalent star showing that Wang and Sun's condition for delta star transformation is useful in practice. Further Theorem 3 of Wang and Sun which says that delta to star transformation is not possible for rational probabilities of delta is wrong. Substituting these probabilities into equations (8), (9) and (10), the probabilities of equivalent star are $p_x = \frac{5}{7}$, $p_y = \frac{1}{2}$, and $p_z = \frac{3}{4}$.

$p_X = \frac{1}{\sqrt{2}}$, $p_Y = \frac{\sqrt{2}}{7}$, $p_Z = \frac{7\sqrt{2}-9}{6\sqrt{2}}$ are an example of irrational numbers which satisfy equation (14).

Equations (8), (9) and (10) can be used to calculate the probabilities of the elements of star from those of delta. But these expressions can be simplified as shown below.

Theorem 4: $p_x = q_X$ if equation (14) is satisfied.
Theorem 5: $p_y = q_Y$ if equation (14) is satisfied.

Proof: $p_y = q_Y$ if $\frac{D_2}{N + D_2} = p_Y$.

So we want to prove that $N + D_2 = q_X q_Z$ as $D_2 = q_X p_Y q_Z$. $N + D_2 = D_3 + D_5 + D_6 + D_7 + D_2 = p_Y + p_X q_Y p_Z$ where $D_2 + D_3 + D_6 + D_7 = p_Y$. $q_X q_Z = (1 - p_X)(1 - p_Z) = 1 - p_X - p_Z + p_X p_Z$. Substituting we want to prove that $p_X + p_Y + p_Z + p_X q_Y p_Z = 1 + p_X p_Z$.

Using equation (14), L.H.S. $= 1 + p_X p_Y p_Z + p_X q_Y p_Z$ = R.H.S. Hence the result.

Theorem 6: $p_z = q_Z$ if equation (14) is satisfied.

Proofs of Theorems 4 and 6 are similar to Theorem 5. It is clear from Theorems (1) to (6) that equations (7) and (14) imply each other. Let S_D (S_Y) be the set of all deltas (stars) satisfying equations (14), (7). Let S'_Y (S'_D) be the set of all equivalent stars (deltas) obtained using the star – delta and delta – star transformations described above. Then $S_D = S'_D$ and $S_Y = S'_Y$ because equations (7) and (14) are equivalent.

The following example illustrates the use of star ↔ delta transformations.

Example: Consider the IEEE 5 Bus power network of Figure 2 (a). It has generators at buses I and II. Loads are present at buses II, III, IV, V. Seven transmission lines numbered 3 to 9 connect the various buses as shown in the figure. The generators and the transmission lines are not perfect i.e., they are subject to failure. We wish to know how these imperfections affect the availability of power at the loads. To do this we must first know the imperfections of these elements in terms of probability of success / failure. Billington and Allan [1] suggested

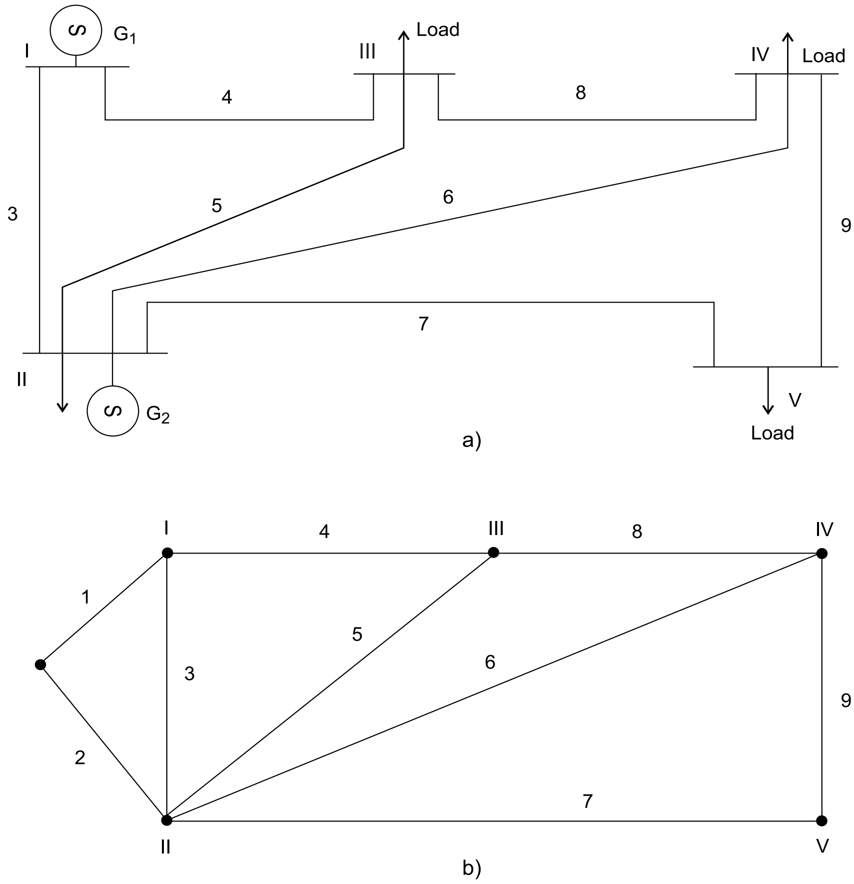

Fig. 2. (a) IEEE 5 Bus network, (b) its Graph.

concepts like failure rate, repair rate, repair time etc to calculate the probability of a component being in up (success) or down (failure) state. Using such ideas let the probabilities of the various elements of the 5 bus system be $p_1 = p_2 = 3/4$, $p_3 = 7/15$, $p_4 = 3/4$, $p_5 = p_6 = p_7 = p_8 = p_9 = 7/8$ where p_i (q_i) denotes probability of success (failure) of i^{th} element. The graph of the system is shown in Figure 2 (b).

Edges 3 to 9 correspond to the transmission lines. Edges 1 and 2 correspond to generators G_1 and G_2. The probabilities of these edges are the probabilities of the corresponding elements of the system. Node s is the source vertex which combines both generators into a single source. Therefore there will be power to a load point if there is at least one functional path from s to it. All vertices of this graph are perfect and all edges are imperfect. We wish to determine the probability of availability of power to load point 3. The graph is not series parallel. Therefore series parallel techniques cannot be applied. However it can be converted into a series parallel network by transforming the star (1, 3, 4) into a delta. Prasad's [11] star – delta transformation cannot be used because there are no restrictions in the direction of flows in the elements of star. q_1, q_2, q_3 satisfy equation (7). So this star can be transformed into a delta using the results of this paper. Let 1', 3', 4' be the corresponding edges of the delta. Edges 1', 3', 4' are between nodes (II, III), (III, s) and (II, s) respectively. Let edge 10 (11) denote the parallel combination of edges 2 and 4' (5 and 1'). The resulting graph is shown in Figure 3. Let edge 14 denote the result of the series parallel combinations of edges 7, 9, 6, 8 and 11. 10 and 14 are in series and the combination is in parallel with 3'. Combining them we get the probability of failure of load point 3. Calculations of these probabilities are shown below. From the theorems

$$p_{1'} = q_1 = \frac{1}{4},\ p_{3'} = q_3 = \frac{8}{15},\ p_{4'} = q_4 = \frac{1}{4},\ q_{10} = q_2\, q_{4'} = \frac{3}{16},$$

$$q_{11} = q_{1'}\, q_5 = \frac{3}{32},\ p_{14} = \frac{128021}{131072} \cong 0.977.$$

Failure probability of load point III = $q_{III} = q_{3'}(1 - p_{10}\, p_{14}) = 0.096$. Direct calculations also give the same result.

This clearly shows that a star can be transformed into a delta in real life networks. Therefore Wang and Sun's conditions are useful in practice. They are not just theoretical.

Consider the problem of determining the probability of supplying power to load point IV. Referring to Figure 3, this requires transforming the star (3', 8, 11) into delta. This star does not satisfy equation (7). So it cannot be transformed into a delta and the network cannot be reduced to a single element using series parallel techniques. But comparing Figure 2 (b) and Figure 3 we notice that Figure 3 has fewer paths and fewer elementary events to load point IV. Therefore calculation of probability for load point IV using conventional methods is more easily done with Figure 3 than Figure 2 (b). Thus star delta transformation is useful even when the network is not reducible to a single element.

Next consider the problem of transforming a star with

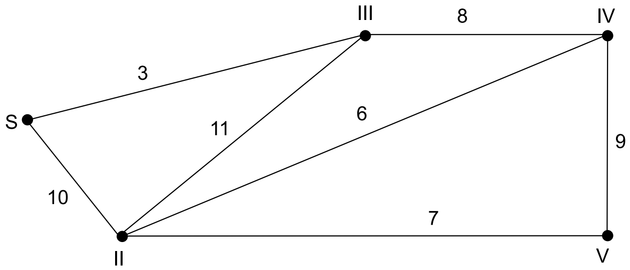

Fig. 3. Simplified graph of Figure 2 (b).

imperfect centre vertex 'd' into a delta. Wang and Sun showed that this is possible if

$$M^2 = (M + Y_3)(M + Y_5)(M + Y_6) \qquad (15)$$

where $Y_i = p_d S_i$, i = 0, 1, 2 ,…..,7, p_d is the probability of success of the centre vertex d and $M = (Y_0 + Y_1 + Y_2 + Y_4) + q_d$. Following the approach presented here, we can simplify equation (15) to

$$M(Y_0 + q_d) = Y_7 Y_0 \qquad (16)$$

When $p_d = 1$ this reduces to $K = S_7$. Equation (16) can be written as

$$S_0 p_d^2 (K - S_7) + S_0 p_d q_d + q_d^2 = 0$$

If $K = S_7$, $S_0 = -\, q_d / p_d$ which is not possible. This proves that a star with an imperfect centre vertex has no equivalent delta if equation (7) is satisfied.

III. CONCLUSIONS

Star delta and delta star transformations are shown to be feasible in reliability networks. Well known conditions of the literature are applicable. The probabilities of the edges of the equivalent star/delta are trivially calculated. It is in fact much simpler than calculating the equivalent resistances of the corresponding star delta problem of electrical circuits. Star ↔ delta transformations are useful whether series parallel techniques work on the entire network or not.

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