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ON VERTEX STABILITY OF COMPLETE k-PARTITE GRAPHS

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Abstract. Let H be any graph. We say that graph G is H-stable if $G-u$ contains a subgraph isomorphic to H for an arbitrary chosen $u \in V(G)$. We characterize all H-stable graphs of minimal size where H is any complete k-partite graph. Thus, we generalize the results of Dudek and Żak regarding complete bipartite graphs.

Keywords: vertex stability, minimal stable graphs, complete k-partite graphs.

Mathematics Subject Classification: 05C35, 05C60.

1. INTRODUCTION

Consider a network of sensors (processors, transmitters, etc.). We require that a given configuration of connections between the sensors is assured even in the case of failure of one of them. We assume that the only substantial cost is related to the connections between the sensors. Obviously, we are interested in finding a network of minimum cost which is fault-tolerant with respect to the given configuration.

More formally, we consider only simple graphs without loops, multiple edges and isolated vertices. We are using the standard notation of graph theory [3] and some of the notation introduced in [4]. Let H be any graph with set of vertices $V(H)$ and set of edges $E(H)$. A graph G is said to be (H,k)- vertex stable if G contains a subgraph isomorphic to H after removing any k of its vertices. If $k=1$ we say shortly that G is H-stable. Moreover, $\operatorname{stab}(H)$ denotes the minimum of sizes of all H-stable graphs. The order and the size of H are denoted by n and m, respectively.

The exact values of $\operatorname{stab}(H)$ are known for some basic classes of graphs. In particular, it is known that $\operatorname{stab}(K_n) = \binom{n+1}{2}$ ([7]), and that $\operatorname{stab}(C_n) = n + 2\lceil\sqrt{n-1}\rceil$ for infinitely many n's ([2]). The known results ([4–6]) regarding bipartite complete graphs are presented in details in next section.

There are also more general result giving a lower bound of $\operatorname{stab}(H)$ for any H with given connectivity κ and minimal degree δ ([1]). Furthermore, for any even $\kappa = \delta \geq 2$

there are examples of graphs for which this lower bound is approximately equal to the upper one ([1]).

The problem was also considered in the more general case, i.e. for any $k \geq 1$. The (H, k)-stable graphs of minimum size were characterized for H being $C_3, C_4, K_4, K_{1,m}$ ([4]), K_5 ([7]), K_n for k large enough regarding fixed n ([9]) and K_n for n large enough regarding fixed k ([8]). Moreover, some estimations of the parameter $stab(H; k)$ were obtained for H being a cycle ([2]) or any graph of minimal degree δ and connectivity κ ([10]).

In this paper we study $\mathrm{stab}(H)$ for H being a k-partite complete graph. One of the substantial parts of the proof of the main result (Theorem 3.2) is to show that the minimum size of H-stable graphs is achieved (not always exclusively) by graphs of order $|H| + 1$. Notice that if G is a H-stable graph of minimum size with no isolated vertices then $\delta(G) \geq \delta(H)$. This is used to show some lower bounds of $\|G\|$ as a function of s, where $s := |G| - |H|$ (Lemmas 2.3–2.5). Finally, we apply those lemmas to show that for H being a k-partite complete graph considering $s = 1$ is enough to show the lower bounds and (in almost all cases) the uniqueness of the construction of H-stable graphs of minimum size.

2. GENERAL OBSERVATIONS

The general bounds of the value of $\mathrm{stab}(H)$ are the following.

Proposition 2.1 ([6]). *Let H be any graph with n vertices and m edges. Then*

$$m + \Delta(H) \leq \mathrm{stab}(H) \leq \min\{m + n, 2m\} \tag{2.1}$$

Remark 2.2. A star ($K_{m,1}$) is the only graph for which the general lower bound is equal to the upper one (2.1). Therefore, $\mathrm{stab}(K_{m,1}) = 2m$ ([4]).

Let G be a H-stable graph of minimum size. Now we show some lower bounds of $\|G\| = \mathrm{stab}(H)$, dependently on the order of G.

Observe at first that there exist a copy of H, say H', being a subgraph of G such that each of the vertices of $V(G) \setminus V(H')$ have degree at least $\delta(H)$ and moreover (at least) one of them has degree greater or equal to $\Delta(H)$. Therefore,

$$\sum_{v \in V(G) \setminus V(H')} \deg_G(v) \geq \Delta(H) + (s-1)\delta(H). \tag{2.2}$$

Lemma 2.3. *If G is an H-stable graph of minimum size such that $|G| = |H| + s$, where $s \geq \Delta(H) + 1$, then*

$$\mathrm{stab}(H) = \|G\| \geq \|H\| + \frac{1}{2}\left(\Delta(H) + (s-1)\delta(H)\right). \tag{2.3}$$

Moreover, equality in (2.3) *may hold only if $s = \Delta(H) + 1$.*

Proof. Since each edge incident to vertices of $V(G) \setminus V(H')$ can be counted twice in (2.2), then

$$\|G\| \geq \|H\| + \frac{1}{2}\left(\Delta(H) + (s-1)\delta(H)\right). \qquad \square$$

Lemma 2.4. *Let H be a graph such that $\Delta(H) > \delta(H) > 1$. If G is an H-stable graph of minimum size such that $|G| = n + s$, where $s \leq \Delta(H)$. Then*

$$\|G\| \geq \begin{cases} \|H\| + \Delta(H) & \text{if } s = 1, \\ \|H\| + \Delta(H) + (s-1)\delta(H) - \binom{s}{2} & \text{if } 2 \leq s \leq \delta(H), \\ \|H\| + \Delta(H) + \frac{1}{2}(s-1)(\delta(H) - 1) & \text{if } \delta(H) + 1 \leq s \leq \Delta(H), \end{cases} \tag{2.4}$$

where the eqaulity in (2.4) *is possible only for $s = 1, s = 2$, and $s = \delta(H) + 1$.*

Proof. First we prove the correctness of the inequality (2.4). The case $s = 1$ is a straightforward consequence of (2.1). Then we only have to show the cases with $s > 1$.

If $2 \leq s \leq \delta(H)+1$, then at most $\binom{s}{2}$ edges incident to the vertices of $V(G) \setminus V(H')$ are counted twice in (2.2).

If $\delta(H)+1 \leq s \leq \Delta(H)$, then there exists a copy of H, say H', being a subgraph of G such that there exists vertex $u \in V(G) \setminus V(H')$ of degree at least $\Delta(G)$. Notice that there are at least $\frac{1}{2}(s-1)(\delta(H)-1)$ edges incident with the vertices of $V(G-u) \setminus V(H')$. Summing up those two we get $\|G\| \geq \|H\| + \Delta(H) + \frac{1}{2}(s-1)(\delta(H)-1)$, as desired.

The fact that the expressions of the right-side part of (2.4) are increasing (with respect to the domain) functions of variable s completes the proof. $\square$

Lemma 2.5. *Let H be a d-regular graph such that $d > 1$. If G is a H-stable graph of minimum size such that $s := |G| - |H| \leq d$, then*

$$\|G\| = \operatorname{stab}(H) \geq \begin{cases} \|H\| + \frac{|H|+d+1}{2} & \text{if } s = 1, \\ \|H\| + sd - \binom{s}{2} + 1 & \text{if } 2 \leq s \leq d. \end{cases} \tag{2.5}$$

Proof. Consider the case $s = 1$. Suppose for the contrary that

$$\|G\| \leq \|H\| + \frac{|H| + d}{2} = \frac{(|H|+1)(d+1) - 1}{2}.$$

Therefore, there exists a vertex $y \in V(G)$ such that $\deg_G(y) \leq d$. Consider now $G - y'$, where y' a neighbour of y. Observe that the degree of y in $G - y'$ is smaller than d, hence y does not belong to any copy of H in $G - y'$. Since $|G - y'| = |H|$, it can be easily seen that $G - y'$ does not contain any subgraph of H.

Now consider the case $2 \leq s \leq d$. Suppose for the contrary that $\|G\| = \|H\| + sd - \binom{s}{2}$. Let H_0 be any copy of H being a subgraph of G. Then all vertices of $V(G) \setminus V(H_0)$ are adjacent to each other and their degree in G equals d. Moreover, each of s vertices of $V(G) \setminus V(H_0)$ has a neighbour in $V(H_0)$ (since $s \leq d$). Let $u \in V(H_0) \cap N(x)$, where x is some vertex of $V(G) \setminus V(H_0)$. We are going to show that at least s vertices of $V(G - u)$ are useless, i.e. cannot be contained in any copy of H, which ends the proof since $|G-u| = |H|+s-1$. To this aim observe that the vertex $x \in N(u) \setminus V(H_0)$ is not contained in any copy of H in $G-u$, because its degree in $G-u$ is smaller than d. Consequently, all the remaining vertices of $V(G) \setminus V(H_0)$ cannot be in any copy of H in $G-u$. Indeed, their degree ignoring (already useless) neighbour x is lower than d. Therefore, we show, as required, that $s = |G| - |H|$ vertices of $V(G-u)$ are useless which completes the proof. □

Remark 2.6. It can be shown that the inequalities (2.3), (2.4) and (2.5) are tight.

3. COMPLETE k-PARTITE GRAPHS

The $K_{n,n}$-stable and $K_{n,n+1}$-stable graphs of minimum size were characterized by Dudek and Zwonek ([5]). This results were generalized by Dudek and Żak ([6]) to the case of any complete bipartite graph.

Theorem 3.1 ([6]). *Let $p \geq q \geq 2$. Then for $H = K_{p,q}$*

$$\operatorname{stab}(H) = \begin{cases} pq+p & \text{for } p-q=1, \\ pq+p+q & \text{for } p-q \neq 1. \end{cases} \tag{3.1}$$

Moreover, all $K_{p,q}$-stable graphs of minimum size were characterized ([6]). Namely, if $p = q+1 > 2$, then $K_{p,p}$ is the only $K_{p,q}$-stable graph of minimum size. Otherwise, if $p \geq 4, q \geq 2$ and $p \geq q$, then the only $K_{p,q}$-stable graph of minimum size are $G_1 = K_{p,q} * K_1$ and $G_2 = K_{p+1,q+1} - e$, where e is any edge of $K_{p+1,q+1}$.

Keeping the assumption that $H = K_{p,q}$ with $p \geq q \geq 2$ we can formulate (3.1) in the following way:

$$\operatorname{stab}(H) = \begin{cases} \|H\| + \Delta(H) & \text{for } p-q=1, \\ \|H\| + |H| & \text{for } p-q \neq 1. \end{cases}$$

Observe that $\operatorname{stab}(K_{p,q})$ achieves exactly the lower or the upper bound of (2.1) and no in-between value is possible. This property holds also in the general case of any complete k-partite graphs as follows.

Theorem 3.2. *Let H be a complete k-partite graph $H = K_{n_1 n_2 \ldots n_k}$ with $k \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_k$ such that $H \neq K_{m,1}$. Then*

$$\operatorname{stab}(H) = \begin{cases} \|H\| + \Delta(H) & \text{for } n_1 = n_2 = \ldots = n_{k-1} = n_k + 1, \\ \|H\| + |H| & \text{otherwise}. \end{cases}$$

Proof. Since the theorem is proved for $k = 2$ ([6]), we focus on the case $k \geq 3$.

I. Let $n_1 = n_2 = \ldots = n_{k-1} = n_k + 1$. It can be easily verified that $G = K_{n_1 n_1 \ldots n_1}$ is H-stable and $\|G\| = \|H\| + \Delta(H)$. By (2.1) we know that there is no H-stable graph of smaller size which completes the proof of this case.

II. Now consider any k-partite complete graph H different from that defined in I. Let G be an H-stable graph. Let $n = |H|$ and $m = \|H\|$. Due to Lemmas 2.3-2.5 and the facts that $\delta(H) = n - n_1$ and $\Delta(H) = n - n_k$, we can observe that if $|G| \geq n+2$ then $\|G\| \geq \|H\| + n$. Therefore, we assume that $|G| = n+1$.

Now let us transform our problem to the equivalent one. Since we are assuming that $|G-x| = n$ then, in fact, $V(G-x) = V(H')$ for any $x \in V(G)$, where $H' \subset G-x$ is isomorphic to H. Therefore,

$$(G - x \supset H') \Leftrightarrow (\overline{H'} \supset \overline{G-x}).$$

Now we are interested in maximizing the size of $\overline{G}$ such that $\overline{G-x}$ is isomorphic with some subgraph of $\overline{H}$ for arbitrary chosen x.

$\overline{H}$ is a union of k cliques (of orders $n_1, n_2, \ldots, n_k$), hence for any $x \in V(G)$ graph $\overline{G-x}$ has at least k components of connectivity. Since each connected graph of order greater than one contains a vertex which can be removed without loosing connectivity, we conclude that graph $\overline{G}$ also consists of at least k components of connectivity (of orders, say, $r_1, \ldots, r_{k+t}$ such that $r_1 \geq \ldots \geq r_{k+t}$ with $t \geq 0$).

Of course

$$n + 1 = n_1 + \ldots + n_k + 1 = r_1 + \ldots r_{k+t}. \tag{3.2}$$

Consider multiset $R_j := \{r_1, \ldots, r_{j-1}, r_j - 1, r_{j+1}, \ldots, r_{k+t}\}$. For each $j \in \{1, \ldots, k+t\}$ there exists a partition of R_j into k multisubsets $R_j^1, \ldots, R_j^k$ such that the sum of elements of R_j^i is equal to n_i for all $i = 1, \ldots, k$.

First consider the case that $\overline{G}$ consists of exactly k components of connectivity. In that case the partition of R_j into k subsets is unique - each subset consists just of one element. Then the equality $r_1 = \ldots = r_k$ must be satisfied. Indeed, if $r_j \neq r_l$, then $R_j \neq R_l$ and at least one of R_j, R_l does not correspond to a given sequence of orders of components of $\overline{H}$. The equality of all r_i's implies that $n_1 = \ldots = n_{k-1} = n_k + 1$, but this is exactly the case already considered in I, which is excluded in II.

Let us move to the case $s > 0$. Obviously

$$\|\overline{G}\| \leq \binom{r_1}{2} + \ldots + \binom{r_{k+t}}{2}. \tag{3.3}$$

Let x belong to the $(k+t)$th component of $\overline{G}$. It is clear that for each $i = 1, \ldots, k$ no more than $\binom{n_i}{2}$ edges of $\overline{G-x}$ can be included in a component of $\overline{H}$ of order n_i.

Therefore,

$$\|\overline{G-x}\| \leq \binom{n_1}{2} + \ldots + \binom{n_k}{2}. \tag{3.4}$$

(i) If each component of $\overline{G}$ is a clique, $t = 1$, $r_{k+1} = 1$ and $r_i = n_i$ for $i = 1, \ldots, k$, we obtain

$$\|\overline{G}\| = \|\overline{G-x}\| = \binom{n_1}{2} + \ldots + \binom{n_k}{2},$$

and, consequently,

$$\|G\| = \binom{n+1}{2} - \|\overline{G}\| = n + \binom{n}{2} - \binom{n_1}{2} - \ldots - \binom{n_k}{2} = m + n.$$

The graph G constructed in that way is isomorphic with $K_1 * H$ which is H-stable (see [6]).

(ii) If (i) is not satisfied then some R^l_{k+t} consists of two (or more) elements. Consequently, in l-th component of $\overline{H}$ two (or more) disjoint components of $\overline{G-x}$ are included, leaving unused edges of $\overline{H}$ between them. The number of that unused edges is minimal if there are only two disjoint components being cliques. Assuming that the orders of that disjoint cliques are, say a and b, then ab unused edges are in $\overline{H}$. Since the two smallest cliques in $\overline{G-x}$ are of orders not less than 1 and r_{k+t}, we obtain that

$$\|\overline{G-x}\| \leq \binom{n_1}{2} + \ldots + \binom{n_k}{2} - r_{k+t},$$

and, consequently,

$$\|\overline{G}\| = \|\overline{G-x}\| + r_{k+t} - 1 \leq \binom{n_1}{2} + \ldots + \binom{n_k}{2} - 1$$

which is less than in case (i). This shows that in case II there is no H-stable graph G containing less than $m+n$ edges which ends the proof. □

Theorem 3.3. *Let H be a complete k-partite graph $H = K_{n_1 n_2 \ldots n_k}$ with $k \geq 3$ and $n_1 \geq n_2 \geq \ldots \geq n_k$ such that $H \neq K_{m,1}$ and $H \neq K_3$. Then the only H-stable graph of minimum size is $K_{n_1,\ldots,n_1}$ if $n_1 = n_2 = \ldots = n_{k-1} = n_k + 1$ and $H * K_1$ otherwise.*

Proof. Case I: $n_1 = n_2 = \ldots = n_{k-1} = n_k + 1$. Let G be an H-stable graph of minimum size, i.e. $\|G\| = m + \Delta(H)$. If $|G| > n+1$, then, as it was already showed in the proof of Theorem 3.2, $\|G\| \geq m + n > m + \Delta(H)$ – a contradiction. Therefore, we assume that $|G| = n+1$. It is easy to observe that $\delta(G) = \Delta(G) = \Delta(H)$. Indeed, if $\delta(G) \leq \Delta(H) - 1 = \delta(H)$, then removing some neighbour of a vertex of degree $\delta(G)$ we obtain a graph of minimum degree less than $\delta(H)$, which cannot contain H as a subgraph. On the other hand, if $\Delta(G) > \Delta(H)$, then

$$\|G\| > \frac{1}{2}(n+1)\Delta(H) = m + \Delta(H),$$

a contradiction. Therefore, $\|G - u\| = m$ and, in consequence, $G - u$ is isomorphic to H for arbitrary chosen vertex u. It is clear that the only graph satisfying this property is the complete k-partite graph with all components of partition of order n_1.

Case II: (Case I not satisfied). Let G be a H-stable graph of minimum size, i.e. $\|G\| = m + n$.

1. If $|G| = n+1$, then, accordingly to the proof of Theorem 3.2, $G = H * K_1$ is the only H-stable graph of minimum size.

2. Suppose for the contrary that $|G| = n + s$ with $s > 1$. Due to Lemmas 2.3–2.5 we can observe that there may exists a H-stable graph of size $m+n$ only if $k = 3$ and $n_2 = n_3 = 1$ and if one of the following cases is satisfied:

a) $s = 2$,
b) $s = 3$ with $\Delta(H) \geq 3$,
c) $s = 3$ with $\Delta(H) = 2$.

We show that, in fact, even in these cases there is no H-stable graph of size $m+n$ and order greater than $n+1$. First, observe that since $k=3$ and $n_2=n_3=1$ then $\delta(H)=2$ and $\Delta(H)=n-1$. If $\Delta(H)=\delta(H)=2$, then, in fact, $H=K_3$ which is excluded in the theorem's assumptions. (It is easy to observe that the only K_3-stable graph of minimum size are K_4 and $2K_3$.) Therefore, it is enough to focus only on the cases a) and b) assuming that $\Delta(H)\geq 3$.

Case a) If $\Delta(G)\geq\Delta(H)+1=n$, then $\|G-\bar{u}\|\leq(m+n)-n$, where $\deg_G(\bar{u})=\Delta(G)$. Since $G-\bar{u}$ contains no more than m edges incident with $n+1$ non-isolated vertices, it cannot contain H as a subgraph.

Consider now the case $\Delta(G)=\Delta(H)=n-1$. Observe that since $n_2=n_3=1$, then there exists vertices u and u' of degree $\Delta(H)$ in H. Let x be a vertex of degree at least $\Delta(H)$ in G. Then there is a copy of H such that $V(H)=V(G)\setminus\{x,y\}$, where y is some vertex of degree at least two in G. Notice, that $ux, u'x, uy, u'y\notin E(G)$ (otherwise $\Delta(G)>\Delta(H)$). Then x is connected with all vertices of except u and u', hence each vertex of G except y is of degree at least three. Consequently, $\delta_{G-u}\geq 2$. Finally,

$$\|G\|\geq\|H\|+\Delta(H)+2=m+n+1>m+n,$$

a contradiction.

Case b) There exists such a copy of H in G that $V(G)=V(H)\cup\{x,y,z\}$, where $\deg_G(x)\geq 2$, $\deg_G(y)\geq 2$ and $\deg_G(z)\geq\Delta(H)=n-1$. If $xy\notin E(G)$ then

$$\|G\|\geq m+2+2+n-1-2>m+n,$$

a contradiction.

Therefore, assume that $xy\in E(G)$. Since G is a H-stable graph of minimum size then xy is included in some copy of H, say H'', being a subgraph of G. In that case x and y are not in the same component of a partition of H'', hence at least one of the vertices x, y has degree at least $\Delta(H)\geq 3$. Therefore,

$$\|G\|\geq m+2(n-1)+2-\binom{3}{2}>m+n,$$

a contradiction. □

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