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The generalized S- and σ-inverse – a comparative case study for right- and left-invertible plants

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Abstract. In this paper, an advanced study covering the comparison between two classes of generalized inverses is conducted. Two sets of instances, strictly derived from the recently introduced nonunique S- and σ-inverse, are analyzed, especially in terms of degrees of freedom-oriented interchangeable application in different engineering tasks. Henceforth, the respective collections of right and left inverses can be combined in order to achieve a complex tool for robustification of a plethora of real processes. The great potential of two S- and σ-inverse, in particular in robust control and signal recovery as well as complex optimal tasks, is confirmed in the manuscript and supported by the recently carried out research investigations.

Key words: nonunique right/left inverses, nonsquare polynomial matrices, inverse theory, robustification of processes, practical implementation.

1. Introduction

In the field of modern control systems covering the communication plants, the MIMO structures play a significant role, especially in terms of entire process robustification [1–5]. Through an application of a plethora techniques, strictly related to systems having different numbers of input and output variables, we can impact on the detrimental effects, often observed during various nominal operations [6]. In fact, for square plants with the same inputs and outputs, we can not effectively decrease the broadly understood parasitic outcomes, being in relation with, e.g., energy consumption or unsatisfactory data capacity of different industrial nets [6, 7]. Therefore, in order to fulfill all requirements guaranteeing the proper work of different systems, the special inverse model control-oriented calculus has recently been offered and efficiently applied to robust control plants [6] and telecommunications systems [7] as well as computer networks [8]. Thus, the newly proposed generalized inverses can be employed in various theoretical and practical branches, giving rise to meet the needed conditions specified by the engineering design [9–15]. Please note, that in case of nonsquare MIMO objects, the nonuniqueness property plays the important role, which usually effects the elimination of any drawbacks [7]. In the manuscript, the new parameter/polynomial generalized inverses are recalled and some connections between two crucial ones are indicated. The so-called nonunique S-inverse and σ-inverse are compared, in particular in terms of existing degrees of freedom [16]. A number of dependencies are formulated, what certainly proves the partial uselessness of typical Moore-Penrose inverse. Henceforth, the resulted unified approach can successfully predefine the robust properties of analyzed plant, this advantage is clearly demonstrated in the paper. Indeed, the application of the new combined method can offer the opportunities to improve the inverse model control-originated algorithm devoted to any physical nonsquare plants [17–24]. Replacing the Moore-Penrose inverse by the S and σ inverses will certainly impact on the performance indices, directly leading to design of efficient multivariable control schemes.

Therefore, the manuscript is organized in the following manner. In the Section 2, the nonunique right and left generalized S and σ inverses are observed. The inverse-oriented background involving the representative instances associated with the compared degrees of freedom is shown in the Section 3. Two relationship conjectures of the next section with the accompanying algorithms constitute the main accomplishment of the paper. The conclusions with appendices successfully end the innovative investigation proposed in the manuscript.

2. Generalized S and σ inverses of nonsquare polynomial matrices

Following the notions of the introduction section and, most importantly, in order to show the main achievement of the manuscript, the brief descriptions of two recently introduced nonunique inverses should immediately be discussed. We will start our investigation with the generalized Smith factorization-oriented S-inverse being ready to applied to any full normal rank matrix polynomial.

2.1. S-inverse. For a plant described by the polynomial matrix of full normal rank as follows

$$\underline{\mathbf{B}}(q^{-1}) = \underline{\mathbf{b}}_0 + \underline{\mathbf{b}}_1 q^{-1} + \ldots + \underline{\mathbf{b}}_k q^{-k}, \tag{1}$$

where $\underline{\mathbf{B}} \in \mathbb{R}^{m\times n}(q^{-1})$, $k = 0, 1, \ldots$ and q^{-1} stands for the one-step backward shift operator, the first step of finding the

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right/left S-inverse is to perform a matrix decomposition. Since the nonsquare matrix $\underline{\mathbf{B}}(q^{-1})$ covers the FIR-type polynomials of different degrees, the Smith factorization should be completed in the following manner

$$\underline{\mathbf{B}}(q^{-1}) = \underline{\mathbf{U}}(q^{-1})\underline{\Sigma}(q^{-1})\underline{\mathbf{V}}(q^{-1}), \tag{2}$$

under unimodular matrices $\underline{\mathbf{U}} \in \mathbb{R}^{m\times m}(q^{-1})$ and $\underline{\mathbf{V}} \in \mathbb{R}^{n\times n}(q^{-1})$, as well as, structure $\underline{\Sigma} \in \mathbb{R}^{m\times n}(q^{-1})$ involving the eigenvalue(s) of $\underline{\mathbf{B}} \in \mathbb{R}^{m\times n}(q^{-1})$.

Remark 1. In the case of $\underline{\mathbf{B}}(q^{-1})$ employing some IIR-type filters, we interchangeably use the Smith-McMillan formulation.

Finally, the respective right and left nonunique S-inverse can be presented as

$$\underline{\mathbf{B}}^{\mathrm{R}}_{S}(q^{-1}) = \underline{\mathbf{V}}^{-1}(q^{-1})\underline{\Sigma}^{\mathrm{R}}(q^{-1})\underline{\mathbf{U}}^{-1}(q^{-1}), \tag{3}$$

and

$$\underline{\mathbf{B}}^{\mathrm{L}}_{S}(q^{-1}) = \underline{\mathbf{V}}^{-1}(q^{-1})\underline{\Sigma}^{\mathrm{L}}(q^{-1})\underline{\mathbf{U}}^{-1}(q^{-1}), \tag{4}$$

where symbol $(.)^{\mathrm{R/L}}$ denotes any right/left inverse including the unique Moore-Penrose one. Naturally, the generalized inverses implement some so-called degrees of freedom, strictly derived from the $\underline{\mathbf{M}}(q^{-1})$ matrix polynomial being in relation with two corresponding forms (3) and (4) through the

$$\underline{\Sigma}^{\mathrm{R}}_{n\times m}(q^{-1}) = \begin{bmatrix} \underline{\widetilde{\mathbf{D}}}_{m\times m}(q^{-1}) \\ \underline{\mathbf{M}}_{(n-m)\times m}(q^{-1}) \end{bmatrix}, \tag{5}$$

and

$$\underline{\Sigma}^{\mathrm{L}}_{n\times m}(q^{-1}) = \begin{bmatrix} \underline{\widetilde{\mathbf{D}}}_{n\times n}(q^{-1}) & \underline{\mathbf{M}}_{n\times(m-n)}(q^{-1}) \end{bmatrix}, \tag{6}$$

having square diagonal matrix $\underline{\widetilde{\mathbf{D}}}(q^{-1})$ possibly arranges the inversions of the so-called transmission zeros of $\underline{\mathbf{B}}(q^{-1})$ [8].

Let us switch now to the generalized polynomial σ-inverse in the subsequent section.

2.2. σ-inverse. For a system defined in the FIR-related domain as in Formula (1), the nonunique right and left σ-inverses sound in the following way

$$\underline{\mathbf{B}}^{\mathrm{R}}_{\sigma}(q^{-1}) = \underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1})\left[\underline{\mathbf{B}}(q^{-1})\underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1})\right]^{-1}, \tag{7}$$

and

$$\underline{\mathbf{B}}^{\mathrm{L}}_{\sigma}(q^{-1}) = \left[\underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1})\underline{\mathbf{B}}(q^{-1})\right]^{-1}\underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1}), \tag{8}$$

respectively, considering that the full rank $\underline{\boldsymbol{\beta}}_{m\times n}(q^{-1})$ stands for the arbitrary (polynomial) degrees of freedom [6].

Remark 2. Note, that in the cases: $\underline{\mathbf{M}}(q^{-1}) = \mathbf{0}$ and $\underline{\boldsymbol{\beta}}(q^{-1}) = \underline{\mathbf{B}}(q^{-1})$, the right and left-oriented new methods reduce to the respective minimum-norm and least-squares instances of the classical Moore-Penrose inverse.

Remark 3. Observe, that two crucial generalized inverses have often a plethora of different behaviors, efficiently used in the various engineering tasks associated with the control and systems theory, as well [6, 7]. Therefore, in order to clarify these interesting properties, which can interchangeably be applied depending on the context, the unified framework, in particular in terms of employed degrees of freedom, is effectively postulated here. Although, the new a approach is shown for rather narrow set of multivariable plants, the conducted relationship can immediately be extended to a general solution covering all transfer-function-oriented nonsquare systems in the nearest future.

Hence, the next section shows the main goal of the paper resulted in the existing dependences of two inverses in sense of the degrees of freedom characteristics for arbitrary selected matrix polynomial $\underline{\mathbf{B}}(q^{-1})$.

3. Relationship between S- and σ-inverse: a motivation study

At the beginning, it should be indicated, that the preliminary investigation covering the broadly understood union of two compared S- and σ-related structures has been performed under Ref. [16]. Some similarities have been appointed there, which are effectively extended in the manuscript. Although, the right- and left-invertible scenarios seem to rather be identical, some peculiarities are risen, finally to create the useful compact tool for the scientific and engineering world societies.

In order to clarify the new issues that have never been seen before, two representative instances have separately been explored and explained in detail below.

3.1. Right-invertible scenario. Consider a nonsquare plant described by the matrix $\underline{\mathbf{B}}_{3\times 5}(q^{-1})$ in the form of

$$\underline{\mathbf{B}}(q^{-1}) = \begin{bmatrix} \mathrm{b}_{11} & \mathrm{b}_{12} & \mathrm{b}_{13} & \mathrm{b}_{14} & \mathrm{b}_{15} \\ \mathrm{b}_{21} & \mathrm{b}_{22} & \mathrm{b}_{23} & \mathrm{b}_{24} & \mathrm{b}_{25} \\ \mathrm{b}_{31} & \mathrm{b}_{32} & \mathrm{b}_{33} & \mathrm{b}_{34} & \mathrm{b}_{35} \end{bmatrix}, \tag{9}$$

employing the polynomial-related elements reduced to the parameter ones under some branch-oriented requirements.

Following the notion, after usage of the Smith decomposition, the generalized right S-inverse sounds as follows

$$\underline{\mathbf{B}}^{\mathrm{R}}_{S}(q^{-1}) = \underline{\mathbf{V}}^{-1}(q^{-1})\underline{\Sigma}^{\mathrm{R}}(q^{-1})\underline{\mathbf{U}}^{-1}(q^{-1}) =$$

$$= \begin{bmatrix} \mathrm{v}_{11} & \mathrm{v}_{12} & \mathrm{v}_{13} & \mathrm{v}_{14} & \mathrm{v}_{15} \\ \mathrm{v}_{21} & \mathrm{v}_{22} & \mathrm{v}_{23} & \mathrm{v}_{24} & \mathrm{v}_{25} \\ \mathrm{v}_{31} & \mathrm{v}_{32} & \mathrm{v}_{33} & \mathrm{v}_{34} & \mathrm{v}_{35} \\ \mathrm{v}_{41} & \mathrm{v}_{42} & \mathrm{v}_{43} & \mathrm{v}_{44} & \mathrm{v}_{45} \\ \mathrm{v}_{51} & \mathrm{v}_{52} & \mathrm{v}_{53} & \mathrm{v}_{54} & \mathrm{v}_{55} \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} 1/\mathrm{s}_{11} & 0 & 0 \\ 0 & 1/\mathrm{s}_{22} & 0 \\ ]0 & 0 & 1/\mathrm{s}_{33} \\ m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{bmatrix} \mathrm{u}_{11} & \mathrm{u}_{12} & \mathrm{u}_{13} \\ \mathrm{u}_{21} & \mathrm{u}_{22} & \mathrm{u}_{23} \\ \mathrm{u}_{31} & \mathrm{u}_{32} & \mathrm{u}_{33} \end{bmatrix}^{-1}, \tag{10}$$

where expressions: $s_{11},\ldots,s_{33}$ constitute the eigenvalues of $\underline{\mathbf{B}}(q^{-1})$, whereas: $m_{11},\ldots,m_{23}$ are associated with the inverse-derived degrees of freedom arranged by the pencil $\underline{\mathbf{M}}(q^{-1})$.

Remark 4. Let us remind, that due to the space limitation reason, the cells of matrix structures represent the polynomials of different degrees. Naturally, the studies can cover the case involving more general forms – the typical nature of the said degrees of freedom. In addition to the parameter and polynomial modes derived from the S and σ inverses, the degrees of freedom can be associated with other structures supported by e.g. the rational function expressions.

On the other hand, the consideration brought us to the specification of a σ-inverse-related example pursuing the matrix polynomial $\underline{\mathbf{B}}(q^{-1})$, in the following manner

$$\underline{\mathbf{B}}^{\mathrm{R}}_{\sigma}(q^{-1}) = \underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1})\left(\underline{\mathbf{B}}(q^{-1})\underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1})\right)^{-1} = \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \\ \beta_{13} & \beta_{23} & \beta_{33} \\ \beta_{14} & \beta_{24} & \beta_{34} \\ \beta_{15} & \beta_{52} & \beta_{35} \end{bmatrix} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}\right. \cdot \left.\begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \\ \beta_{13} & \beta_{23} & \beta_{33} \\ \beta_{14} & \beta_{24} & \beta_{34} \\ \beta_{15} & \beta_{52} & \beta_{35} \end{bmatrix}\right)^{-1}. \tag{11}$$

Thus, the fundamental merit of the paper should again be reappeared. We are trying now to give some formulation allowing to combine two aforementioned right-invertible approaches. Note, that since these inverses involve structurally different degrees of freedom, the operation seems to be rather complex, in general. Therefore, after solid mathematical way, employing the crucial expression

$$\underline{\mathbf{B}}^{\mathrm{R}}_{S}(q^{-1}) = \underline{\mathbf{B}}^{\mathrm{R}}_{\sigma}(q^{-1}), \tag{12}$$

we receive the essential results integrating two methods in context of arbitrary applied modes. The final observations are as follows

$$\underline{m}_{11}(q^{-1}) = \frac{C_{2,1}D_{1,1}+C_{2,2}D_{1,2}+C_{2,3}D_{1,3}+C_{2,4}D_{1,4}}{s_{11}(C_{1,1}D_{1,1}+C_{1,2}D_{1,2}+C_{1,3}D_{1,3}+C_{1,4}D_{1,4}} \frac{+C_{2,5}D_{1,5}+C_{2,6}D_{1,6}+C_{2,7}D_{1,7}+\cdots+C_{2,10}D_{1,10}}{+C_{1,5}D_{1,5}+C_{1,6}D_{1,6}+C_{1,7}D_{1,7}+\cdots+C_{1,10}D_{1,10})}, \tag{13}$$

$$\underline{m}_{12}(q^{-1}) = \frac{C_{3,1}D_{1,1}+C_{3,2}D_{1,2}+C_{3,3}D_{1,3}+C_{3,4}D_{1,4}}{s_{22}(C_{1,1}D_{1,1}+C_{1,2}D_{1,2}+C_{1,3}D_{1,3}+C_{1,4}D_{1,4}} \frac{+C_{3,5}D_{1,5}+C_{3,6}D_{1,6}+C_{3,7}D_{1,7}+\cdots+C_{3,10}D_{1,10}}{+C_{1,5}D_{1,5}+C_{1,6}D_{1,6}+C_{1,7}D_{1,7}+\cdots+C_{1,10}D_{1,10})}, \tag{14}$$

$$\vdots$$

$$\underline{m}_{23}(q^{-1}) = \frac{C_{7,1}D_{1,1}+C_{7,2}D_{1,2}+C_{7,3}D_{1,3}+C_{7,4}D_{1,4}}{s_{33}(C_{1,1}D_{1,1}+C_{1,2}D_{1,2}+C_{1,3}D_{1,3}+C_{1,4}D_{1,4}} \frac{+C_{7,5}D_{1,5}+C_{7,6}D_{1,6}+C_{7,7}D_{1,7}+\cdots+C_{7,10}D_{1,10}}{+C_{1,5}D_{1,5}+C_{1,6}D_{1,6}+C_{1,7}D_{1,7}+\cdots+C_{1,10}D_{1,10})}, \tag{15}$$

where notation of $C_{i,j}$ and $D_{1,k}$, with respect to the subscripts $i = 1,\ldots,7$ and $j,k = 1,\ldots,10$, is formed by the cells in matrices $\underline{\mathbf{C}}(q^{-1})$ and $\underline{\mathbf{D}}(q^{-1})$, respectively. Mentioned matrices are shown in the Appendix A.

Let us switch now to the another interesting case covering the left-invertible approach. Again, two instances are given subject to the discussed generalized inverses.

3.2. Left-invertible scenario.

Consider a nonsquare polynomial matrix $\underline{\mathbf{B}}_{4\times 3}(q^{-1})$ expressed in the following manner

$$\underline{\mathbf{B}}(q^{-1}) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}. \tag{16}$$

The respective left S-inverse and left σ-inverse for our system form as follows

$$\underline{\mathbf{B}}^{\mathrm{L}}_{S}(q^{-1}) = \underline{\mathbf{V}}^{-1}(q^{-1})\underline{\boldsymbol{\Sigma}}^{\mathrm{L}}(q^{-1})\underline{\mathbf{U}}^{-1}(q^{-1}) = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}^{-1} \begin{bmatrix} 1/s_{11} & 0 & 0 & m_{11} \\ 0 & 1/s_{22} & 0 & m_{21} \\ 0 & 0 & 1/s_{33} & m_{31} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix}^{-1}, \tag{17}$$

and

$$\underline{\mathbf{B}}^{\mathrm{L}}_{\sigma}(q^{-1}) = \left[\underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1})\underline{\mathbf{B}}(q^{-1})\right]^{-1}\underline{\boldsymbol{\beta}}^{\mathrm{T}}(q^{-1}) = \left(\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \\ \beta_{41} & \beta_{42} & \beta_{43} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}\right)^{-1} \cdot \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \\ \beta_{41} & \beta_{42} & \beta_{43} \end{bmatrix}^{\mathrm{T}}. \tag{18}$$

Again, similarly to the relation (12), the operation

$$\underline{\mathbf{B}}^{\mathrm{L}}_{S}(q^{-1}) = \underline{\mathbf{B}}^{\mathrm{L}}_{\sigma}(q^{-1}), \tag{19}$$

returns us

$$\underline{m}_{11}(q^{-1}) = \frac{\mathrm{E}_{2,1}\mathrm{D}_{1,1}+\mathrm{E}_{2,2}\mathrm{D}_{1,2}+\mathrm{E}_{2,3}\mathrm{D}_{1,3}+\mathrm{E}_{2,4}\mathrm{D}_{1,4}}{\mathrm{s}_{11}(\mathrm{E}_{1,1}\mathrm{D}_{1,1}+\mathrm{E}_{1,2}\mathrm{D}_{1,2}+\mathrm{E}_{1,3}\mathrm{D}_{1,3}+\mathrm{E}_{1,4}\mathrm{D}_{1,4})}, \tag{20}$$

$$\underline{m}_{21}(q^{-1}) = \frac{\mathrm{E}_{3,1}\mathrm{D}_{1,1}+\mathrm{E}_{3,2}\mathrm{D}_{1,2}+\mathrm{E}_{3,3}\mathrm{D}_{1,3}+\mathrm{E}_{3,4}\mathrm{D}_{1,4}}{\mathrm{s}_{22}(\mathrm{E}_{1,1}\mathrm{D}_{1,1}+\mathrm{E}_{1,2}\mathrm{D}_{1,2}+\mathrm{E}_{1,3}\mathrm{D}_{1,3}+\mathrm{E}_{1,4}\mathrm{D}_{1,4})}, \tag{21}$$

$$\underline{m}_{31}(q^{-1}) = \frac{\mathrm{E}_{4,1}\mathrm{D}_{1,1}+\mathrm{E}_{4,2}\mathrm{D}_{1,2}+\mathrm{E}_{4,3}\mathrm{D}_{1,3}+\mathrm{E}_{4,4}\mathrm{D}_{1,4}}{\mathrm{s}_{33}(\mathrm{E}_{1,1}\mathrm{D}_{1,1}+\mathrm{E}_{1,2}\mathrm{D}_{1,2}+\mathrm{E}_{1,3}\mathrm{D}_{1,3}+\mathrm{E}_{1,4}\mathrm{D}_{1,4})}, \tag{22}$$

under $\mathrm{E}_{i,j}$ and $\mathrm{D}_{1,k}$, $i,j,k = 1,\ldots 4$, corresponding to the cells of the matrices $\mathbf{E}(q^{-1})$ and $\mathbf{D}(q^{-1})$, respectively. In this case, the mentioned structures are expressed in the Appendix B.

Remark 5. It should be emphasized, that in the contrary to the right-invertible scenario, the left-invertible instance can not be used in, e.g., the inverse model control-oriented approach strictly dedicated to the multivariable plants. Although, the such generalized left inverse seems to be rather impoverished, it can successfully be applied to other tasks, for instance, related to the signal reconstruction process observed in the wireless communication technologies [7].

Having the new issues covering the comparison of two classes of the generalized inverses, let us continue with two algorithms allowing to determine our degrees of freedom. Henceforth, the numerical methods can simplify significantly the calculation process of the complex problem drafted in the manuscript.

4. Mathematical statements

In this section, we introduce the new methodology in order to engage any nonsquare right and left inverses, whatever their sizes and orders. Please note, that the proposed algorithms have been implemented and verified in a number of simulation runs using the Matlab environment.

4.1. Right-invertible scenario. As it has already been mentioned, the results presented in this section are derived from the Eqs. (13)–(15), supported by a plethora of the numerical instances. Naturally, the representative illustration has additionally been given in the previous section. Notwithstanding, due to the complexity of the presented considerations, the formal proof of the generalized formula guaranteeing the unification of S and σ inverses does not exist, so far. Hence, the algorithm to be shown can only be justified on the basis of the new subsequent statement in the form of the following

Conjecture 1. Consider the right-invertible polynomial matrices described in the backward shift operator-oriented q^{-1}-domain. The complete relationship between generalized right S- and σ-inverse of arbitrary matrix $\underline{\mathbf{B}}_{m\times n}(q^{-1})$, as in the Formulas (5) and (7), can be represented in the respective form

$$m_{ij}(q^{-1}) = k\underline{\mathbf{C}}_{l}(q^{-1})\underline{\mathbf{D}}^{\mathrm{T}}(q^{-1})\left(\mathrm{s}_{jj}\underline{\mathbf{C}}_{1}(q^{-1})\underline{\mathbf{D}}^{\mathrm{T}}(q^{-1})\right)^{-1}, \tag{23}$$

where $i \in 1,2,\ldots,(n-m)$, $j \in 1,2,\ldots,m$, $l \in 2,3,\ldots,m(n-m)+1$, whilst k defines the sign determined by the Algorithm 1, whereas the matrices $\underline{\mathbf{C}}(q^{-1})$ and $\underline{\mathbf{D}}(q^{-1})$ are given in the Appendix A.

The procedure allowing to obtain the crucial sign sounds as follows.

Algorithm 1 Algorithm for calculation of the sign k derived from the Eq. (23)

Require: The filled matrix $\underline{\mathbf{M}}(q^{-1})$ without the signs
$r \leftarrow$number of rows of the $\underline{\mathbf{M}}(q^{-1})$
$c \leftarrow$number of columns of the $\underline{\mathbf{M}}(q^{-1})$
$k = 1$
for $j = c, j > 0, j--$ **do**
  for $i = 1, i \leq r, i++$ **do**
    $\underline{\mathbf{M}}_{i,j}(q^{-1}) = k \cdot \underline{\mathbf{M}}_{i,j}(q^{-1})$
  end for
  $k = k \cdot (-1)$
end for
return $\underline{\mathbf{M}}(q^{-1})$ ▷ The final form of the $\underline{\mathbf{M}}(q^{-1})$.

In the next section, the left-invertible instance is proposed. It should be emphasized, that although the nomenclature of the given relation seems to be rather similar to the notation covering the right-invertible solution, the two structures have raised from the different operations.

4.2. Left-invertible scenario. The following conjecture, as before, has based on a number of simulation tests and the expressions as in Eqs. (20) to (22).

Conjecture 2. Consider the left-invertible polynomial matrices described in the backward shift operator-oriented q^{-1}-domain. The complete relationship between generalized left S- and σ-inverse of arbitrary matrix $\underline{\mathbf{B}}_{m\times n}(q^{-1})$, as in the Formulas (6) and (8), can be represented in the respective form

$$m_{ij}(q^{-1}) = k\underline{\mathbf{E}}_{l}(q^{-1})\underline{\mathbf{D}}^{\mathrm{T}}(q^{-1})\left(\mathrm{s}_{jj}\underline{\mathbf{E}}_{1}(q^{-1})\underline{\mathbf{D}}^{\mathrm{T}}(q^{-1})\right)^{-1}, \tag{24}$$

where $i \in 1,2,\ldots,n$, $j \in 1,2,\ldots,(m-n)$, $l \in 2,3,\ldots,n(m-n)+1$, whereas k is calculated according to the Algorithm 2. The appropriate matrices $\underline{\mathbf{E}}(q^{-1})$ and $\underline{\mathbf{D}}(q^{-1})$ are shown in the Appendix B.

Observe, that the crucial operations related to the new approach given in the manuscript are endorsed by the set of the complex peculiarities described in detail in the appendix sections.

Algorithm 2 Algorithm for calculation of the sign k derived from the Eq. (24)

Require: The filled matrix $\underline{\mathbf{M}}(q^{-1})$ without the signs

$r \leftarrow$number of rows of the $\underline{\mathbf{M}}(q^{-1})$
$c \leftarrow$number of columns of the $\underline{\mathbf{M}}(q^{-1})$
$k = 1$
for $i = r, i > 0, i--$ **do**
  for $j = 1, j \leq c, j++$ **do**
    $\underline{\mathbf{M}}_{i,j}(q^{-1}) = k \cdot \underline{\mathbf{M}}_{i,j}(q^{-1})$
  end for
  $k = k \cdot (-1)$
end for
return $\underline{\mathbf{M}}(q^{-1})$ ▷ The final form of the $\underline{\mathbf{M}}(q^{-1})$.

5. Conclusions

The manuscript presents the new analytical results of conducted studies covering the interesting relationship between two S and σ inverses, described in the polynomial domain. The comparison has been prepared in terms of existing degrees of freedom, strictly derived from both right and left generalized inverses. Two new crucial conjectures have been applied, in order to extend the new methodology given in the paper for all class of the discrete-time linear multivariable plants. Henceforth, the new approach can be devoted to any matrix polynomial involving the polynomials of the different orders. The application of the discussed control-oriented issues in the practical engineering tasks is unquestionable. Last, but not least, the generalization of the methods presented in the manuscript is still waiting for further intense research investigation.

APPENDIX

A. Right-invertible case. In this section, the crucial components of our new algorithmic solution related to the right-invertible nonsquare polynomial matrices are indicated. Therefore, in order to obtain the main inverse-oriented structures, the matrix polynomials $\underline{\mathbf{C}}(q^{-1})$ and $\underline{\mathbf{D}}(q^{-1})$, effectively involved in the Formulas (13)–(15), should be constructed. Naturally, the inverse operations are strongly supported by the new Conjecture 1.

At the beginning, the matrix $\underline{\mathbf{C}}(q^{-1})$ as in Eq. (23) should be calculated from the polynomial matrix $\underline{\mathbf{V}}(q^{-1})$. In fact, such scenario constitutes the significant compact extension of the material given in Ref. [16]. Thus, the difference is only that, we calculate the minors of the some analyzed square matrix in the backward shift operator q^{-1}-related domain in different ways. In contrary to the original solution, where the sliding mechanism has appointed the said minors, the minors of the new approach are created on the basis of the combinations without repetitions of columns and rows of $\underline{\mathbf{V}}(q^{-1})$, separately. The new idea covering this essential method is observed in Fig. 1 and supported by the new Algorithm 3.

$\underline{\mathbf{c}}_{1,1}(q^{-1})$ $\underline{\mathbf{c}}_{1,2}(q^{-1})$ $\underline{\mathbf{c}}_{1,3}(q^{-1})$

$$\underline{\mathbf{V}}(q^{-1}) = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{bmatrix}$$

$\underline{\mathbf{c}}_{1,1}(q^{-1})$ $\underline{\mathbf{c}}_{2,1}(q^{-1})$ $\underline{\mathbf{c}}_{3,1}(q^{-1})$

$$\underline{\mathbf{V}}(q^{-1}) = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{bmatrix}$$

Fig. 1. The graphical objective of the Algorithm 3

Algorithm 3 Algorithm for calculation of the matrix $\underline{\mathbf{C}}(q^{-1})$ from the matrix $\underline{\mathbf{V}}(q^{-1})$

Require: The matrix $\underline{\mathbf{V}}(q^{-1})$; The sizes m and n of the matrix $\underline{\mathbf{B}}(q^{-1})$

comb_wr[.] $\leftarrow$ the table of the combinations without repetitions (1:n,m)
number_comb $\leftarrow$ number of the combinations without repetitions providing by the structure *comb_wr*[.]
for $i = 1, i \leq (n-m) \cdot m + 1, i++$ **do**
  for $j = 1, j \leq number\_comb, j++$ **do**
    $\underline{\mathbf{C}}_{i,j}(q^{-1})$= det(*choose_rows*($\underline{\mathbf{V}}(q^{-1})$,...
    comb_wr[.]), *choose_column*($\underline{\mathbf{V}}(q^{-1})$, *comb_wr*[.]))
  end for
end for
return $\underline{\mathbf{C}}(q^{-1})$ ▷ The final form of the $\underline{\mathbf{C}}(q^{-1})$.

Finally, based on the aforementioned considerations, the expected polynomial matrix $\underline{\mathbf{C}}(q^{-1})$ sounds as follows

$$\underline{\mathbf{C}}(q^{-1}) = \begin{bmatrix} \det(\mathbf{c}_{11}) & \det(\mathbf{c}_{12}) & \det(\mathbf{c}_{13}) & \ldots & \det(\mathbf{c}_{1,10}) \\ \det(\mathbf{c}_{21}) & \det(\mathbf{c}_{22}) & \det(\mathbf{c}_{23}) & \ldots & \det(\mathbf{c}_{2,10}) \\ \det(\mathbf{c}_{31}) & \det(\mathbf{c}_{32}) & \det(\mathbf{c}_{33}) & \ldots & \det(\mathbf{c}_{3,10}) \\ \vdots & \vdots & \vdots & & \vdots \\ \det(\mathbf{c}_{71}) & \det(\mathbf{c}_{72}) & \det(\mathbf{c}_{73}) & \ldots & \det(\mathbf{c}_{7,10}) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \\ \vdots \\ \mathbf{C}_7 \end{bmatrix}. \tag{25}$$

Secondly, in the simpler manner, the polynomial matrix $\underline{\mathbf{D}}(q^{-1})$ should be calculated after employing the postulated Algorithm 4 assisted by Fig. 2.

Algorithm 4 Algorithm for calculation of the matrix $\underline{\mathbf{D}}(q^{-1})$ from the matrix $\underline{\boldsymbol{\beta}}(q^{-1})$

Require: The matrix $\underline{\boldsymbol{\beta}}(q^{-1})$; The sizes m and n of the matrix $\underline{\mathbf{B}}(q^{-1})$

1: *comb_wr*[.] $\leftarrow$ the table of the combinations without repetitions (1:n,m)

2: *number_comb* $\leftarrow$ number of the combinations without repetitions providing by the structure *comb_wr*[.]

3: **for** $i = 1, i \leq number\_comb, i{+}{+}$ **do**

4: $\quad \underline{\mathbf{D}}_{1,i}(q^{-1}) = \det(choose\_column(\underline{\boldsymbol{\beta}}(q^{-1}), comb\_wr[i]))$

5: **end for**

6: **return** $\underline{\mathbf{D}}(q^{-1})$ $\qquad \triangleright$ The final form of the $\underline{\mathbf{D}}(q^{-1})$.

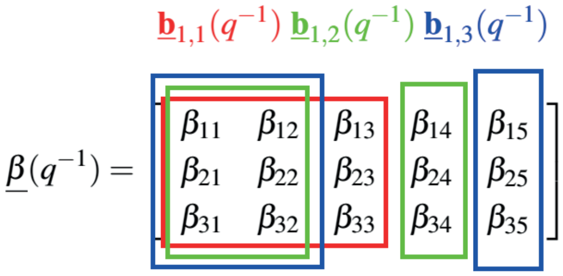

Fig. 2. The graphical objective of the Algorithm 4

In such a case, the matrix polynomial $\underline{\mathbf{D}}(q^{-1})$, associated with the degrees of freedom $\underline{\boldsymbol{\beta}}(q^{-1})$, is formed as

$$\underline{\mathbf{D}}(q^{-1}) = \begin{bmatrix} \det(\mathbf{b}_{11}) & \det(\mathbf{b}_{12}) & \det(\mathbf{b}_{13}) & \ldots & \det(\mathbf{b}_{1,10}) \end{bmatrix}. \tag{26}$$

Remark 6. Note, that the entire machinery presented here is strongly connected with the Section 3.1, where selected system described by the matrix $\underline{\mathbf{B}}_{3\times 5}(q^{-1})$ is considered.

Let us switch now to the second set of left-invertible plants.

B. Left-invertible case. Consequently, in order to receive the solution concerning the left-invertible polynomial matrices, the subsequent procedure should certainly be performed. Through the Formulas (20)–(22), supported by the Conjecture 2, the expected relationship should immediately be obtained. For this reason, the matrix polynomial structures $\underline{\mathbf{E}}(q^{-1})$ and $\underline{\mathbf{D}}(q^{-1})$ have to first be calculated. Therefore, the matrix $\underline{\mathbf{E}}(q^{-1})$ of the relation (24) is derived from the $\underline{\mathbf{U}}(q^{-1})$. Hence, the Fig. 3 and Algorithm 5 reinforce the newly introduced relation-based phenomenon.

In that way, we arrive at the polynomial matrix $\underline{\mathbf{E}}(q^{-1})$ in the following manner

$$\underline{\mathbf{E}}(q^{-1}) = \begin{bmatrix} \det(\mathbf{e}_{11}) & \det(\mathbf{e}_{12}) & \det(\mathbf{e}_{13}) & \det(\mathbf{e}_{14}) \\ \det(\mathbf{e}_{21}) & \det(\mathbf{e}_{22}) & \det(\mathbf{e}_{23}) & \det(\mathbf{e}_{24}) \\ \det(\mathbf{e}_{31}) & \det(\mathbf{e}_{32}) & \det(\mathbf{e}_{33}) & \det(\mathbf{e}_{34}) \\ \det(\mathbf{e}_{41}) & \det(\mathbf{e}_{42}) & \det(\mathbf{e}_{43}) & \det(\mathbf{e}_{44}) \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix}. \tag{27}$$

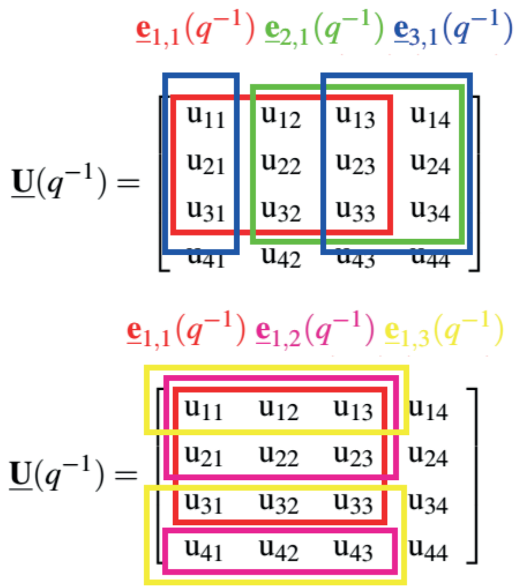

Fig. 3. The graphical objective of the Algorithm 5

Algorithm 5 Algorithm for calculation of the matrix $\underline{\mathbf{E}}(q^{-1})$ from the matrix $\underline{\mathbf{U}}(q^{-1})$

Require: The matrix $\underline{\mathbf{U}}(q^{-1})$; The sizes m and n of the matrix $\underline{\mathbf{B}}(q^{-1})$

1: *comb_wr*[.] $\leftarrow$ the table of the combinations without repetitions (1:m,n)

2: *number_comb* $\leftarrow$ number of combinations without repetitions providing by the structure *comb_wr*[.]

3: **for** $i = 1, i \leq (m-n)\cdot n + 1, i{+}{+}$ **do**

4: $\quad$ **for** $j = 1, j \leq number\_comb, j{+}{+}$ **do**

5: $\qquad \underline{\mathbf{E}}_{i,j}(q^{-1}) = \det(choose\_rows(\underline{\mathbf{U}}(q^{-1}), \ldots$

6: $\qquad comb\_wr[.]), choose\_column(\underline{\mathbf{U}}(q^{-1}), comb\_wr[.]))$

7: $\quad$ **end for**

8: **end for**

9: **return** $\underline{\mathbf{E}}(q^{-1})$ $\qquad \triangleright$ The final form of the $\underline{\mathbf{E}}(q^{-1})$.

In the end, the closing step of our procedure should be formulated. The same like in the right-oriented scenario, the polynomial matrix $\underline{\mathbf{D}}(q^{-1})$ is strictly associated with the matrix polynomial $\underline{\boldsymbol{\beta}}(q^{-1})$. Thus, the Algorithm 6 together with the Fig. 4

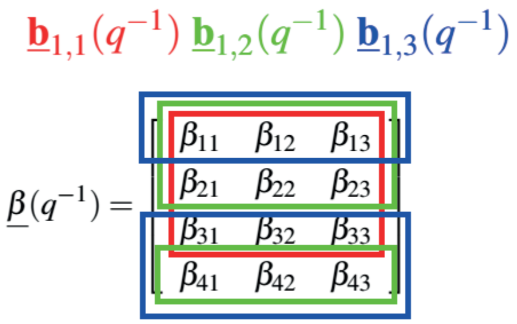

Fig. 4. The graphical objective of the Algorithm 6

Algorithm 6 Algorithm for calculation of the matrix $\underline{\mathbf{D}}(q^{-1})$ from the matrix $\underline{\boldsymbol{\beta}}(q^{-1})$

Require: The matrix $\underline{\boldsymbol{\beta}}(q^{-1})$; The sizes m and n of the matrix $\underline{\mathbf{B}}(q^{-1})$
1: $comb\_wr[.] \leftarrow$ the table of the combinations without repetitions (1:m,n)
2: $number\_comb \leftarrow$ number of the combinations without repetitions providing by the structure $comb\_wr[.]$
3: **for** $i = 1, i \leq number\_comb, i{+}{+}$ **do**
4: $\quad \underline{\mathbf{D}}_{1,i}(q^{-1}) = \det(choose\_rows(\underline{\boldsymbol{\beta}}(q^{-1}), comb\_wr[i]))$
5: **end for**
6: **return** $\underline{\mathbf{D}}(q^{-1})$ ▷ The final form of the $\underline{\mathbf{D}}(q^{-1})$.

end up with the last newly introduced accomplishment of the manuscript related to the Conjecture 2.

The final form of the nonsquare structure $\underline{\mathbf{D}}(q^{-1})$ sounds as follows

$$\underline{\mathbf{D}}(q^{-1}) = \begin{bmatrix} \det(\mathbf{b}_{11}) & \det(\mathbf{b}_{12}) & \det(\mathbf{b}_{13}) & \det(\mathbf{b}_{14}) \end{bmatrix}. \tag{28}$$

Remark 7. Notice, that the above-mentioned issues are associated with the Section 3.2 covering the chosen matrix $\underline{\mathbf{B}}_{4\times 3}(q^{-1})$.

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