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Stability and positivity with respect to part of the variables for positive Markovian jump systems

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Abstract. The stability and positivity of linear positive Markovian jump systems with respect to part of the variables is considered. The methodologies of stability of positive systems with known transition probabilities based on common linear copositive Lyapunov function and stability of linear systems with respect to part of the variables are combined to find sufficient conditions of the stochastic stability and positivity of Markovian jump systems with respect to part of the variables. The results are extended for a class of nonlinear positive Markovian jump systems with respect to part of the variables. An example is given to illustrate the obtained results.

Key words: hybrid system, Lyapunov method, Markov jump system, stability and positivity with respect to the part of variables.

1. Introduction

Positive systems is a wide class of mathematical models used in description of many processes in the real world. The theory of these systems was presented, for instance in [1, 4, 9, 22]. Such models were used for physical, chemical, biological, economical, social processes, for instance, absolute temperature, concentrations of chemicals, number of cells, stock prices, mortality rates, electrical ladder network. The problem of the stability and stabilizability analysis of linear positive systems is connected with the spectral analysis of Metzler matrices. It has been studied by some authors, see for instance [16–18]. Another wide class of mathematical models used in description of many processes are Markovian jump systems (see for example, [3,30]). Usually, they are described by a set of differential (difference) equations and a switching rule in the form of a right-continuous Markov chain. One of the basic problems is the stability and stabilizability of these systems [5, 6, 28, 31, 33].

To describe positive processes with possible uncertainties usually random processes are used, in particular Markov jump systems. The theoretical study of Markovian jump systems was presented in many papers and books, for instance [3, 24, 26]. A particular class of these systems are positive systems with Markovian jump parameters. Many applications of these systems were reviewed in [29]. At the same time the analysis of switching systems used in control theory [12] was applied to the study of a particular class called positive linear switched systems [7, 15]. One of the most important problems in the study of these systems is the stability problem of positive solutions. The problem of stability and stabilizability of linear positive systems with a completely known Markovian switching process was considered in [2, 20–22, 29] and with a partially known Markovian switching process in [19, 32].

In the study of large scale systems it was often difficult to find the sufficient conditions of stability and at the same time in real systems we are only interested in the qualitative analysis of some variables. It motivated researchers to investigate stability problems with respect to part of the variables (partial stability). This idea is very useful in the study of some mechanical holonomic and nonholonomic systems and systems describing the dynamics of controlled solid. Many other examples and applications and theoretical results in this field one can find in the Vorotnikov's book [27].

Since the stability criteria obtained for non-hybrid system can not be directly applied to hybrid systems it is necessary to consider separately in details every class of hybrid systems. A generalization of the partial stability of nonlinear stochastic systems with a completely known Markovian switching process was proposed in [10, 13, 14] and with a partially known Markovian switching process in [25].

The qualitative analysis of dynamic systems with respect to part of the variables was also used in the study of stabilizability [23] and controllability [11]. In the case of dynamic non–hybrid systems described by linear differential equations researchers proposed to transform the original system to an auxiliary system of linear differential equations of lower order and then to look for stability or stabilizability conditions for the auxiliary system and next to find sufficient conditions of stabilizability or controllability for the corresponding part of the variables.

In this paper we consider the problem of the stability and positivity of positive Markovian jump systems with respect to part of variables. We use the methodologies of stability of positive

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systems with known transition probabilities based on common linear copositive Lyapunov function and stability of linear systems with respect to part of the variables and we combine them to find sufficient conditions of the stochastic stability and positivity of Markovian jump systems with respect to part of the variables.

The obtained results are extended for a class of nonlinear positive Markovian jump systems with respect to part of the variables. To the best of our knowledge this generalization is new. The obtained results are illustrated by an example.

2. Problem formulation and mathematical preliminaries

Throughout this paper we use the following notation. $\mathscr{R}_+ = [0,\infty)$, $\mathbb{T} = [t_0,\infty)$, $t_0 \geq 0$. Let $|\cdot|$ and $<\cdot>$ be the Euclidean norm and the inner product in $\mathscr{R}^n$, respectively. $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$, $\mathbf{x} = [x_1,\ldots,x_n]^T$. By $(\mathbf{A})^T$, $(\mathbf{A})^{-1}$ and $\lambda(\mathbf{A})$ we denote the transposition, the inverse and the eigenvalue of the matrix $\mathbf{A}$, respectively; the symbol $\mathbf{1}_n$ denotes the n-dimensional vector with all entries equal to 1; $\mathbf{I}_n$ is the $n\times$ n identity matrix. For a vector $\mathbf{v} \in \mathscr{R}^n$, we introduce a notation $\mathbf{v} \succ \mathbf{0}$ and $\mathbf{v} \succeq \mathbf{0}$, if it implies for all $i = 1,\ldots,n$, $v_i > 0$ and $v_i \geq 0$, respectively. For a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ the notations $\mathbf{A} \succ \mathbf{0}$ or $\mathbf{A} \succeq \mathbf{0}$ imply for all $i,j = 1,\ldots,n$, $a_{ij} > 0$ or $a_{ij} \geq 0$, respectively. A matrix $\mathbf{A}$ is said to be a Metzler matrix if its off-diagonal elements are all nonnegative real numbers.

We consider the linear system with Markovian jump parameters described by

$$\dot{\mathbf{x}}(t) = \mathscr{A}(r(t))\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \qquad r(t_0) = r_0, \tag{1}$$

where $\mathbf{x} \in \mathscr{R}^n$ is the system state, $\mathbf{x}_0 \in \mathscr{R}^n$ and r_0 are initial conditions. $\mathscr{A}(l) \in \mathscr{R}^{n\times n}$ for all $l \in \mathscr{S}$ are assumed to be Metzler matrices for all $l \in \mathscr{S}$.

Let $\{r(t), t \in \mathbb{T}\}$ represent the switching process in the form of a right-continuous Markov chain $\{r(t),\ t \geq 0\}$ on the probability space taking values in a finite state space $\mathscr{S} = \{1,\ldots,N\}$ with the generator $\Gamma = [\gamma_{lj}]_{N\times N}$, i.e.

$$\mathbb{P}\{r(t+\delta) = j|r(t) = l\} = \begin{cases} \gamma_{lj}\delta + o(\delta), & \text{if } l \neq j, \\ 1+\gamma_{ll}\delta + o(\delta), & \text{if } l = j, \end{cases} \tag{2}$$

where $\delta > 0$, $\gamma_{lj} \geq 0$ is the transition rate from l to j if $l \neq j$, $\gamma_{ll} = -\sum_{l\neq j}\gamma_{lj}$. We assume that the Markov chain is irreducible i.e. $\mathrm{rank}(\Gamma) = N-1$, and has a unique stationary distribution $\mathscr{P} = [p_1, p_2, \ldots, p_N]^T \in \mathscr{R}^N$ which can be determined by solving

$$\begin{cases} \Gamma^T\mathscr{P} = \mathbf{0} \\ \text{subject to } \sum_{l=1}^{N} p_l = 1 \text{ and } p_l > 0 \text{ for all } l \in \mathscr{S}. \end{cases} \tag{3}$$

We quote some useful definitions and lemmas.

Definition 1. We consider a linear deterministic vector differential equation

$$\dot{\mathbf{x}}(t) = \mathscr{A}\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{4}$$

where $\mathbf{x} \in \mathscr{R}^n$, $\mathscr{A} \in \mathscr{R}^{n\times n}$. We call system (4) to be positive if for any initial condition $\mathbf{x}(t_0) \succeq \mathbf{0}$ the solution of equation (4) satisfies the inequality $\mathbf{x}(t) \succeq \mathbf{0}$ for all $t \in \mathbb{T}$.

Lemma 1. [9] System (4) is positive if and only if $\mathscr{A}$ is a Metzler matrix.

Lemma 2. [9] Consider positive system (4), then the following conditions are equivalent
(a) $\mathscr{A}$ is a Hurwitz matrix;
(b) There exists a vector $\mathbf{v} \succ \mathbf{0}$ such that $\mathscr{A}^T\mathbf{v} \prec \mathbf{0}$.

Definition 2. If there exists a vector $\mathbf{v} \in \mathscr{R}^n$, $\mathbf{v} \succ \mathbf{0}$ such that $\mathscr{A}^T\mathbf{v} \prec \mathbf{0}$, then the function

$$V(\mathbf{x}(t)) = \mathbf{x}(t)^T\mathbf{v} \tag{5}$$

is said to be a linear copositive Lyapunov function of system (4).

Lemma 3. [29] $\mathscr{A} \in \mathscr{R}^{n\times n}$ is a Metzler matrix if and only if there exists a positive constant ρ such that $\mathscr{A} + \rho\mathbf{I}_n \succeq \mathbf{0}$.

For simplicity we denote the solution $\mathbf{x}(t,\mathbf{x}_0,r_0)$ of equation (1) with $r_0 \in \mathscr{S}$ by $\mathbf{x}(t)$.

Definition 3. [29] Solution $x(t) \equiv 0$ of system (1) is said to be stochastically stable, if for any initial condition $\mathbf{x}_0 \in \mathscr{R}^n$ the solution of (1) satisfies the following condition

$$E\left[\int_0^\infty \|\mathbf{x}(t)\|dt\right] < \infty. \tag{6}$$

Definition 4a. [2] For any integer number $p > 0$, solution $x(t) \equiv 0$ of system (1) is said to be p – moment exponentially stable, if there exist positive constants α, c such that for any initial condition $\mathbf{x}_0 \succ \mathbf{0}$, the following inequality is satisfied

$$E[|\mathbf{x}(t,\, \mathbf{x}_0,\, t_0)|^p] \leq cE|\mathbf{x}_0|^p\exp\{-\alpha(t-t_0)\},\ \ t \geq t_0 \tag{7}$$

Definition 4b. [2] Solution $x(t) \equiv 0$ of system (1) is said to be exponentially mean stable, if there exist positive constants α, c such that for any initial condition $\mathbf{x}_0 \succ \mathbf{0}$, the following inequality is satisfied

$$E[\mathbf{x}(t,\, \mathbf{x}_0,\, t_0)] \leq cE|\mathbf{x}_0|\exp\{-\alpha(t-t_0)\}\mathbf{1}_n, \quad t \geq t_0. \tag{8}$$

For $p = 1$ and $p = 2$ the above definition is called exponential mean stability and exponential mean–square stability, respectively.

It was shown in [2] that the 1 – moment exponential stability and exponential mean stability are equivalent.

We choose linear copositive Lyapunov functions for system (1)

$$V(\mathbf{x}(t), l) = \mathbf{x}(t)^T\mathbf{v}(l), \qquad l \in \mathscr{S}, \tag{9}$$

where $\mathbf{x}$, $\mathbf{v}(l) \in \mathscr{R}^n$, $\mathbf{v}(l) \succ \mathbf{0}$. Since the process $\{(\mathbf{x}(t), r(t))\}$ defined in system (1) is a Markov process with an initial state $(\mathbf{x}_0, r_0)$ (see [8]) its weak infinitesimal generator $\mathscr{L}$ acting on function (9) is defined for all $l \in \mathscr{S}$ by

$$\mathscr{L}V(\mathbf{x}(t), l) = \lim_{\delta \to 0^+} \frac{1}{\delta}\big[E[V(\mathbf{x}(t+\delta), r(t+\delta)) - V(\mathbf{x}(t), l)]\big]. \tag{10}$$

Now we recall two basic theorems that establish the sufficient conditions of stability of positive Markovian jump linear systems (1).

Theorem 1. [29] System (1) for the matrix Γ given by (2), (3) is positive and stochastically stable if there exists a constant $\rho > 0$ and a set of vectors $\mathbf{v}(l) \in \mathscr{R}^n$ for each $l \in \mathscr{S}$ such that

$$\mathbf{v}(l) \succ \mathbf{0}, \tag{11}$$

$$(\mathscr{A}(l)^T + \rho \mathbf{I}_n)\mathbf{v}(l) + \sum_{j=1}^{N} \gamma_{lj}\mathbf{v}(j) \prec \mathbf{0}. \tag{12}$$

Theorem 2. [2] System (1) for the matrix Γ given by (2, 3) is positive and mean exponentially stable if and only if there exists a set of vectors $\mathbf{v}(l) \in \mathscr{R}^n$ for each $l \in \mathscr{S}$ such that

$$\mathbf{v}(l) \succ \mathbf{0}, \tag{13}$$

$$\mathscr{A}(l)^T\mathbf{v}(l) + \sum_{j=1}^{N} \gamma_{lj}\mathbf{v}(j) \prec \mathbf{0}. \tag{14}$$

We assume that the vector $\mathbf{x}$ in equation (4) can be splitted into two vectors $\mathbf{y}$ and $\mathbf{z}$, i.e.

$$\mathbf{x}^T = [y_1, \ldots, y_m, z_1, \ldots, z_r] = [\mathbf{y}^T, \mathbf{z}^T], \quad m > 0, \quad r \geq 0, \quad n = m + r. \tag{15}$$

We propose the following definition of the positivity of the part of the variables:

Definition 5. We consider again linear deterministic vector differential equation (4) and we call system (4) to be $\mathbf{y}$-positive (or partial positive) if for any initial condition $\mathbf{x}(t_0) \succeq \mathbf{0}$ the $\mathbf{y}$ part of the vector $\mathbf{x}$ of the solution of equation (4) satisfies the inequality $\mathbf{y}(t) \succeq \mathbf{0}$ for all $t \in \mathbb{T}$.

Using Definition 3 and 4 and the definition of stochastic stability with respect to part of the variables presented in [27] we propose the following definitions of stochastic stability and exponential mean stability with respect to part of the variables.

Definition 6. System (1) is said to be $\mathbf{y}$-stochastically stable($\mathbf{y}$-SS) (stochastically stable with respect to part of the variables), for any initial condition $\mathbf{x}(t_0) \succeq \mathbf{0}$ the vector $\mathbf{y}$ of the solution of (1) satisfies the following condition

$$E\left[\int_0^\infty \|\mathbf{y}(t)\| dt\right] < \infty. \tag{16}$$

Definition 7. [27] System (1) is said to be $\mathbf{y}$-exponentially mean stable ($\mathbf{y}$-EM stable) if there exist positive constants α, c such that for any $|\mathbf{x}_0|$ the following inequality is satisfied

$$\mathbb{E}\big[|\mathbf{y}(t, \mathbf{x}_0, t_0)|\big] \leq c\mathbb{E}|\mathbf{x}_0| \exp\{-\alpha(t - t_0)\}, \quad t \geq t_0. \tag{17}$$

3. Main results

3.1. Construction of an auxiliary μ-system. We consider again linear system (4) and we assume that it is the positive system, where the vector $\mathbf{x}$ is defined by (15). Then, it can be represented in the following vector form

$$\begin{aligned} \dot{\mathbf{y}}(t) &= [\mathbf{A}\mathbf{y}(t) + \mathbf{C}\mathbf{z}(t)], \quad & \mathbf{y}(t_0) = \mathbf{y}_0, \\ \dot{\mathbf{z}}(t) &= [\mathbf{P}\mathbf{y}(t) + \mathbf{Q}\mathbf{z}(t)], \quad & \mathbf{z}(t_0) = \mathbf{z}_0, \end{aligned} \tag{18}$$

where $\mathbf{A} \in \mathscr{R}^{m \times m}$, $\mathbf{C} \in \mathscr{R}^{m \times r}$, $\mathbf{P} \in \mathscr{R}^{r \times m}$, $\mathbf{Q} \in \mathscr{R}^{r \times r}$, are constant matrices.

Following Vorotnikov [27] we construct an auxiliary μ-system. We define new variables μ_i, $1 \leq i \leq p$ from the equality $\mu = \mathbf{C}\mathbf{z}$ and also from the variables

$$\begin{aligned} &\mu^{(1)} = \mathbf{C}\dot{\mathbf{z}} = \mathbf{C}\mathbf{Q}\mathbf{z}, \quad \mu^{(2)} = \mathbf{C}\ddot{\mathbf{z}} = \mathbf{C}\mathbf{Q}^{\mathbf{2}}\mathbf{z}, \quad \cdots, \\ &\mu^{(k)} = \mathbf{C}\mathbf{z}^{(k)} = \mathbf{C}\mathbf{Q}^{\mathbf{k}}\mathbf{z}, \quad 1 \leq k \leq p-1. \end{aligned} \tag{19}$$

To determine a number of variables required to form an auxiliary μ-system we introduce auxiliary matrices

$$\mathbf{K}_0 = \mathbf{C}^T, \qquad \mathbf{K}_i = \mathbf{Q}^T\mathbf{K}_{i-1}, \quad 1 \leq i \leq r \tag{20}$$

and

$$\mathscr{K}_p = [\mathbf{K}_0, \ \mathbf{K}_1, \ \cdots, \ \mathbf{K}_{p-1}]. \tag{21}$$

Lemma 4. [27] The μ-system is $m+h$ dimensional iff the $rank\,\mathscr{K}_p = h$.

We assume that s is the minimal number satisfying the condition $rank\ \mathscr{K}_{s-1} = rank\ \mathscr{K}_s$. We define the matrices L_i, $i = 1, \ldots 5$ as follows:

(i) The rows of $h \times r$ matrix L_1 are linearly independent columns of matrix $\mathscr{K}_{s-1}$
(ii) The columns of $h \times h$ matrix L_2 are the first h columns of matrix L_1
(iii) The first h rows of $r \times h$ matrix $\mathbf{L}_3$ are the rows of matrix $\mathbf{L}_2^{-1}$. The remaining rows of matrix $\mathbf{L}_3$ are zero row-vectors.
(iv) The matrices $\mathbf{L}_4$ and $\mathbf{L}_5$ are defined by

$$\mathbf{L}_4 = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_1 \end{bmatrix}, \qquad \mathbf{L}_5 = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_3 \end{bmatrix}, \tag{22}$$

where $\mathbf{I}_m$ is the identity matrix of order m.

We denote the state vector of the auxiliary μ-system by

$$\mathscr{X} = [y_1, \ldots, y_m, \mu_1, \ldots, \mu_h]^T. \tag{23}$$

Vorotnikov [27] has shown that the state vector of the auxiliary μ-system satisfies linear differential equation

$$\dot{\mathscr{X}}(t) = \mathbf{L_4}\mathscr{A}\mathbf{L_5}\mathscr{X}(t), \qquad \mathscr{X}(t_0) = \mathscr{X}_0. \tag{24}$$

3.2. Partial stability and partial positivity of linear systems. Vorotnikov [27] has also shown the auxiliary μ-system can be used to find sufficient conditions of exponential stability with respect to part of the variables.

Theorem 3. Assume that the null solution $\mathscr{X}(t) \equiv 0$ of the auxiliary μ-system differential equation (24) is exponentially stable with respect to the all variables.

Then the null solution $\mathbf{x}(t) \equiv 0$ of the original system (4) is exponentially stable with respect to the variables $y_1, \ldots, y_m$.

Theorem 4. Assume that the auxiliary μ-system of differential equation (24) is positive defined with respect to the all variables, then the original system (4) is positive defined with respect to the variables $y_1, \ldots, y_m$.

Corollary 1. The auxiliary μ-system of differential equation (24) is exponentially stable iff all real parts of eigenvalues of the matrix $\mathbf{L_4}\mathscr{A}\mathbf{L_5}$ are negative.

Corollary 2. If we assume that system (4) represented in form (18) satisfies conditions

$\mathbf{A}$ and $\mathbf{Q}$ are Metzler matrices and $\mathbf{C} \succeq \mathbf{0}, \mathbf{P} \succeq \mathbf{0}$,

then the matrix $\mathscr{A}$ is a Metzler matrix and system (4) is positive.

Corollary 3. If we assume that system (4) represented in form (18) satisfies conditions

$\mathscr{A}$ and $\mathbf{L}_1\mathbf{Q}\mathbf{L}_3$ are Metzler matrices and $\mathbf{C}\mathbf{L}_3 \succeq \mathbf{0}$, $\mathbf{L}_1\mathbf{P} \succeq \mathbf{0}$, then the matrix $\mathbf{L}_4\mathscr{A}\mathbf{L}_5$ is a Metzler matrix and auxiliary μ-system (24) is positive.

Since the original system (18) can be stable or unstable and positive or not positive, then this gives us four cases to consider.

Theorem 5.

(i) Assume that the auxiliary μ-system of differential equation (24) is unstable and is not positive with respect to the all variables, then the original system (18) is unstable with respect to the variables $y_1, \ldots, y_m$ and is not positive with respect to the variables $y_1, \ldots, y_m$.

(ii) Assume that the auxiliary μ-system of differential equation (24) is exponentially stable and is not positive with respect to the all variables, then the original system (18) is exponentially stable with respect to the variables $y_1, \ldots, y_m$ and is not positive with respect to the variables $y_1, \ldots, y_m$.

(iii) Assume that the auxiliary μ-system of differential equation (24) is exponentially unstable and is positive with respect to the all variables, then the original system (18) is exponentially unstable with respect to the variables $y_1, \ldots, y_m$ and is positive with respect to the variables $y_1, \ldots, y_m$.

(iv) Assume that the auxiliary μ-system of differential equation (24) is exponentially stable and is positive with respect to the all variables, then the original system (18) is exponentially stable with respect to the variables $y_1, \ldots, y_m$ and is positive with respect to the variables $y_1, \ldots, y_m$.

The proof of this theorem follows from the proof of Theorem 1.1.1 [27] and the properties of the Metzler matrices.

Corollary 4. The stability criteria in Theorem 5 were given in the sense of Definition 5. In a similar way one can formulate the criteria in the sense of Definition 6 and Definition 7.

4. Partial stability and positivity of linear systems with Markovian switching

We consider system (1) and we assume that it can be represented in the form (18) for all $l \in \mathscr{S}$ such that the matrices $\mathbf{A} = \mathbf{A}(l)$, $\mathbf{C} = \mathbf{C}(l)$, $\mathbf{P} = \mathbf{P}(l)$, $\mathbf{Q} = \mathbf{Q}(l)$ are constant. Repeating considerations given in Section 3.1 we find that the state vector of the auxiliary μ–subsystems satisfy linear differential equations

$$\dot{\mathscr{X}}(t) = \mathbf{L_4}\mathscr{A}(l)\mathbf{L_5}\mathscr{X}(t), \qquad \mathscr{X}(t_0) = \mathscr{X}_0, \quad l \in \mathscr{S}. \tag{25}$$

We note that the auxiliary μ–subsystems of differential equations (25) are deterministic equations for each $l \in \mathscr{S}$. If we add the Markovian switching rule defined by the matrix Γ given by (2), (3), then we obtain a stochastic equation called a Markovian jump system. In further consideration we may use the methodology proposed for instance in [3, 30].

Now we extend Theorem 5 to the case of positive linear systems with Markovian switching.

Theorem 6. We consider system (1) and the matrix Γ given by (2), (3) and assume that

(i) the auxiliary μ-systems of differential equations $\dot{\mathscr{X}}(t) = \bar{\mathscr{A}}(r(t))\mathscr{X}(t)$ is stochastically unstable and the matrices $\bar{\mathscr{A}}(l) = \mathbf{L_4}\mathscr{A}(l)\mathbf{L_5}$ are not Metzler matrices for all $l \in \mathscr{S}$, then the original system (1) is stochastically unstable and is not positive with respect to the variables $y_1, \ldots, y_m$.

(ii) for the auxiliary μ-systems of differential equations $\dot{\mathscr{X}}(t) = \bar{\mathscr{A}}(r(t))\mathscr{X}(t)$ there exist a constant $\rho > 0$ and a set of vectors $\upsilon(l) \in \mathscr{R}^{m+h}$ for each $l \in \mathscr{S}$ such that

$$\upsilon(l) \succ \mathbf{0}, \tag{26}$$

$$\left(\bar{\mathscr{A}}(l)^T + \rho \mathbf{I}_{m+h}\right)\upsilon(l) + \sum_{j=1}^{N} \gamma_{lj}\upsilon(j) \prec \mathbf{0} \tag{27}$$

and the matrices $\bar{\mathscr{A}}(l) = \mathbf{L_4}\mathscr{A}(l)\mathbf{L_5}$ are not Metzler matrices for all $l \in \mathscr{S}$, then the original system (1) is stochastically stable and is not positive with respect to the variables $y_1, \ldots, y_m$.

(iii) the auxiliary μ-system of differential equations $\dot{\mathscr{X}}(t) = \bar{\mathscr{A}}(r(t))\mathscr{X}(t)$ is stochastically unstable and the matrices $\bar{\mathscr{A}}(l) = \mathbf{L_4}\mathscr{A}(l)\mathbf{L_5}$ are Metzler matrices for all $l \in \mathscr{S}$, then the original system (1) is stochastically unstable and is positive with respect to the variables $y_1, \ldots, y_m$.

(iv) for the auxiliary μ-systems of differential equations $\dot{\mathscr{X}}(t) = \bar{\mathscr{A}}(r(t))\mathscr{X}(t)$ there exist a constant $\rho > 0$ and a set of vectors $\upsilon(l) \in \mathscr{R}^{m+h}$ for each $l \in \mathscr{S}$ such that conditions (26) and (27) are satisfied and the matrices $\bar{\mathscr{A}}(l) = \mathbf{L_4}\mathscr{A}(l)\mathbf{L_5}$ are Metzler matrices for all $l \in \mathscr{S}$, then the original system (1) is stochastically stable and is positive with respect to the variables $y_1, \ldots, y_m$.

Proof. Since the proofs of all cases are similar we show a sketch of proof of the case (iv). For simplicity we assume that $t_0 = 0$ and we propose for system $\dot{\mathscr{X}}(t) = \bar{\mathscr{A}}(r(t))\mathscr{X}(t)$ a stochastic linear copositive Lyapunov function

$$V(\mathscr{X}(t), l) = \mathscr{X}(t)^T \upsilon(l), \quad l \in \mathscr{S}, \tag{28}$$

where $\upsilon(l) \in \mathscr{R}^{m+h}$, $\upsilon(l) \succ \mathbf{0}$. Applying operator $\mathscr{L}$ defined by (10) to Lyapunov function (28) we find

$$\mathscr{L}V(\mathscr{X}(t), l) = \mathscr{X}(t)^T \left(\mathbf{L_5}^T \mathscr{A}^T(l) \mathbf{L_4}^T \upsilon(l) + \sum_{j=1}^{N} \gamma_{lj} \upsilon(j) \right). \tag{29}$$

Hence and from conditions (26) and (27) it follows

$$\mathscr{L}V(\mathscr{X}(t), l) \leq -\rho V(\mathscr{X}(t), l), \quad l \in \mathscr{S}. \tag{30}$$

Further repeating the proof of Theorem 1 [29] we find that the auxiliary μ-systems of differential equations $\dot{\mathscr{X}}(t) = \bar{\mathscr{A}}(r(t))\mathscr{X}(t)$ is positive and stochastically stable. Hence, it follows that the original system (1) is stochastically stable and is positive with respect to the variables $y_1, \ldots, y_m$. □

The proof of Theorem 6 follows from Theorem 5 and Theorem 2 [2].

Corollary 5. If instead of condition (26) and (27) in Theorem 6 we assume that there exist a set of vectors $\upsilon(l) \in \mathscr{R}^{m+h}$ for each $l \in \mathscr{S}$ such that

$$\upsilon(l) \succ \mathbf{0}, \tag{31}$$

$$\bar{\mathscr{A}}(l)^T \upsilon(l) + \sum_{j=1}^{N} \gamma_{lj} \upsilon(j) \prec \mathbf{0}, \tag{32}$$

then in the thesis of the case (ii) and (iv) in Theorem 6 the stochastic stability is replaced by the exponential mean stability with respect to the variables $y_1, \ldots, y_m$.

The proof of Corollary 5 follows from the proof of the case (iv) in Theorem 6 and theorem 2 [2].

Remark 1. From considerations given in [29] and [2] it follows that if matrices $\mathscr{A}(l)$ are Metzler matrices for all subsystems $l \in \mathscr{S}$ then hybrid system (1) is positive. Similarly if matrices of μ–auxiliary systems $\bar{\mathscr{A}}(l)$, for all subsystems $l \in \mathscr{S}$ are Metzler matrices then hybrid system $\dot{\mathscr{X}} = \bar{\mathscr{A}}(r(t))\mathscr{X}$ is also positive.

Example. Consider a linear stochastic hybrid system consisting with two subsystems $l = 1, 2$ having the same structure with different parameters described by

$$\dot{\mathbf{x}}(t) = \mathscr{A}(l)\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \quad r(t_0) = r_0, \tag{33}$$

where $\mathbf{x}(t) = [y_1(t), y_2(t), z_1(t), z_2(t)]^T$

$$\mathscr{A}(l) = \begin{bmatrix} -a(l) & 1 & 0 & 0 \\ -b(l) & -c(l) & \alpha_1(l) & \alpha_2(l) \\ 0 & 0 & -\beta_{11}(l) & -\beta_{12}(l) \\ 0 & 0 & 0 & \beta_{22}(l) \end{bmatrix} \tag{34}$$

with Markovian switching process $r(t)$ defined by (2), (3), where all coefficients $a(l)$, $b(l)$, $c(l)$, $\alpha_i(l)$, $\delta(l)$, $i, j = 1, 2$, $l = 1, 2$ are assumed to be real and positive.

We introduce the new auxiliary variable μ defined by

$$\mu = \alpha_1(l)z_1 + \alpha_2(l)z_2 \tag{35}$$

and we assume that the coefficients $\alpha_i(l)$ and $\beta_{ij}(l)$, $i, j, l = 1, 2$ satisfy the following condition

$$\alpha_2(l) = \frac{\beta_{12}(l)\alpha_1(l)}{\beta_{11}(l) + \beta_{22}(l)}. \tag{36}$$

The new auxiliary hybrid μ-system defined for the vector $\xi = [y_1, y_2, \mu]^T$ has the form

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \mu(t) \end{bmatrix} = \begin{bmatrix} -a(l) & 1 & 0 \\ -b(l) & -c(l) & 1 \\ 0 & 0 & -\beta_{11}(l) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \mu(t) \end{bmatrix} \tag{37}$$

with Markovian switching process $r(t)$ defined by (2), (3).

The Lyapunov function $V(\xi, l)$ for both subsystems has the linear form of the vector $\xi = [y_1, y_2, \mu]^T$

$$V(\xi, l) = \xi(t)^T \mathbf{v}(l) = v_1(l)y_1 + v_2(l)y_2 + v_3(l)\mu, \tag{38}$$

where all coefficients $v_i(l)$, $i = 1, 2, 3$, $l = 1, 2$ are assumed to be real and positive.

Now, we calculate $\mathscr{L}V(\xi, l)$ for $l = 1, 2$

$$\begin{aligned} \mathscr{L}^{(37)}V(\xi, 1) &= v_1(1)(-a(1)y_1 + y_2) \\ &+ v_2(1)(-b(1)y_1 - c(1)y_2 + \mu) + v_3(1)(-\beta_{11}(l)\mu) \\ &+ \gamma_{11}V(\xi, 1) + \gamma_{12}V(\xi, 2). \end{aligned} \tag{39}$$

Hence

$$\begin{aligned} \mathscr{L}^{(37)}V(\xi, 2) &= v_1(2)(-a(2)y_1 + y_2) \\ &+ v_2(2)(-b(2)y_1 - c(2)y_2 + \mu) + v_3(1)(-\beta_{11}(l)\mu) \\ &+ \gamma_{22}V(\xi, 2) + \gamma_{21}V(\xi, 1). \end{aligned} \tag{40}$$

We define the first and second subsystem by the following coefficients

For first subsystem $\quad l = 1$

$a(1) = 3, \quad b(1) = -1, \quad c(1) = -2, \quad \beta_{11}(1) = 1.$

For second subsystem $\quad l = 2$

$a(2) = 3, \quad b(2) = -1, \quad c(1) = 8, \quad \beta_{11}(2) = 1.$

Then conditions (39) and (40) will take the form

$$\begin{aligned}\mathscr{L}^{(37)}V(\xi,1) &= v_1(1)(-3y_1+y_2)+v_2(1)(y_1+2y_2+\mu)\\ &\quad -v_3(1)\mu+\gamma_{11}\big(v_1(1)y_1+v_2(1)y_2+v_3(1)\mu\big)\\ &\quad +\gamma_{12}\big(v_1(2)y_1+v_2(2)y_2+v_3(2)\mu\big)\\ &= y_1\big(-3v_1(1)+v_2(1)+\gamma_{11}v_1(1)+\gamma_{12}v_1(2)\big)\\ &\quad +y_2\big(v_1(1)+2v_2(1)+\gamma_{11}v_2(1)+\gamma_{12}v_2(2)\big)\\ &\quad +\mu\big(v_2(1)-v_3(1)+\gamma_{11}v_3(1)+\gamma_{12}v_3(2)\big),\end{aligned} \tag{41}$$

$$\begin{aligned}\mathscr{L}^{(37)}V(\xi,2) &= v_1(2)(-3y_1+y_2)+v_2(2)(y_1-8y_2+\mu)\\ &\quad -v_3(2)\mu+\gamma_{22}\big(v_1(2)y_1+v_2(2)y_2+v_3(2)\mu\big)\\ &\quad +\gamma_{21}\big(v_1(1)y_1+v_2(1)y_2+v_3(1)\mu\big)\\ &= y_1\big(-3v_1(2)+v_2(2)+\gamma_{22}v_1(2)+\gamma_{21}v_1(1)\big)\\ &\quad +y_2\big(v_1(2)-8v_2(2)+\gamma_{22}v_2(2)+\gamma_{21}v_2(1)\big)\\ &\quad +\mu\big(v_2(2)-v_3(2)+\gamma_{22}v_3(2)+\gamma_{21}v_3(1)\big).\end{aligned} \tag{42}$$

If we assume that the positive coordinates $v_i(l)$, $i=1,2,3$, $l=1,2$ and the elements of the matrix Γ are equal
$v_1(1)=v_1(2)=0.75$, $v_2(1)=0.75$, $v_2(2)=0.5$,
$v_3(1)=v_3(2)=1$, $\gamma_{11}=\gamma_{22}=-10$, $\gamma_{12}=\gamma_{21}=10$,
then conditions (39) and (40) will take the form

$$\begin{aligned}\mathscr{L}^{(37)}V(\xi,1) &= -1.5y_1-0.25y_2-0.25\mu\\ &\le -0.2(0.75y_1+0.75y_2+\mu)=-0.2V(\xi,1),\end{aligned} \tag{43}$$

$$\begin{aligned}\mathscr{L}^{(37)}V(\xi,2) &= -1.75y_1-0.75y_2-0.5\mu\\ &\le -0.2(0.75y_1+0.5y_2+\mu)=-0.2V(\xi,2).\end{aligned} \tag{44}$$

Hence it follows that $\mathscr{L}^{(37)}V(\xi,l)\le -0.2V(\xi,l)$, $l=1,2$. From Theorem 6 (case (iv)) and Corollary 5 it follows that system (34) is stable in the sense of both Definitions 3 and 4 with respect to the variables y_1,y_2.

4.1. A class of positive Markovian jump nonlinear systems.

We consider a nonlinear stochastic system with Markovian jump parameters described by

$$\begin{aligned}&\dot{\mathbf{x}}(t)=\mathscr{A}(r(t))\mathbf{x}(t)+\mathbf{b}(r(t))\phi(y),\\ &\mathbf{x}(t_0)=\mathbf{x}_0, \qquad r(t_0)=r_0,\end{aligned} \tag{45}$$

where the notation and all assumptions are the same as for system (1) and $\mathbf{b}(l)\in\mathscr{R}^n$, $\mathbf{b}(l)\succeq\mathbf{0}$;
$\phi(y)$ is a scalar nonlinear function, $\phi:\mathscr{R}_+\to\mathscr{R}_+$ such that

$$0\le\phi(y)\le My, \qquad y=\mathbf{c}(r(t))^T\mathbf{x}, \tag{46}$$

where M is a positive constant $M>0$, y is a nonnegative scalar variable, $y\ge 0$, $\mathbf{c}(l)\in\mathscr{R}^n$, $\mathbf{c}(l)\succeq\mathbf{0}$, for all $l\in\mathscr{S}$.

Since $\mathscr{A}(l)+M\mathbf{b}(l)\mathbf{c}(l)^T$ is also a Metzler matrix for all $l\in\mathscr{S}$ then it is simple to show that the solution of each system (45) is positive for all $l\in\mathscr{S}$.

We consider again positive system (45) for the matrix Γ given by (2), (3).

Theorem 7. System (45) with a transition rate matrix Γ given by (2), (3) is positive and stochastically stable if there exists a constant $\rho>0$ and a set of vectors $\mathbf{v}(l)\in\mathscr{R}^n$ for each $l\in\mathscr{S}$ such that

$$\mathbf{v}(l)\succ\mathbf{0}, \tag{47}$$

$$(\mathscr{A}(l)^T+M\mathbf{b}(l)\mathbf{c}(l)^T+\rho\mathbf{I}_n)\mathbf{v}(l)+\sum_{j=1}^{N}\gamma_{lj}\mathbf{v}(j)\prec\mathbf{0}. \tag{48}$$

Now, we consider a linear system

$$\dot{\mathbf{x}}(t)=\mathscr{A}_M(r(t))\mathbf{x}(t), \qquad \mathbf{x}(t_0)=\mathbf{x}_0, \quad r(t_0)=r_0, \tag{49}$$

where the notation and all assumptions are the same as for system (45) and $\mathbf{b}(l),\mathbf{c}(l)\in\mathscr{R}^n$, $\mathbf{b}(l),\mathbf{c}(l)\succeq\mathbf{0}$, $\mathscr{A}_M^T(l)=\mathscr{A}(l)^T+M\mathbf{b}(l)\mathbf{c}(l)^T$.

Similar to considerations given in previous section we construct an auxiliary μ-system defined by

$$\dot{\mathscr{X}}_M(t)=\mathbf{L_4}\mathscr{A}_M\mathbf{L_5}\mathscr{X}_M(t), \qquad \mathscr{X}_M(t_0)=\mathscr{X}_{M0}, \tag{50}$$

where $\mathscr{X}_M$ is the state vector of the auxiliary μ-system (50)

$$\mathscr{X}_M=[y_1,\ldots,y_m,\mu_1,\ldots,\mu_h]^T. \tag{51}$$

Repeating considerations from previous section one can prove similar criteria of positivity and stability with respect to part of the variables for a class of positive Markovian jump nonlinear systems (45).

5. Conclusion

In this paper we have considered the problem of the stability and positivity of linear positive Markov jump systems with respect to part of the variables. The main idea was to extend the methodology of stability of positive Markov jump systems with known transition probabilities based on common linear copositive Lyapunov function to the case of positive Markov jump systems with respect to the part of variables. It was important that also the property of positivity of was considered for Markov jump systems with respect to the part of variables. The obtained results have been extended for a class of nonlinear positive Markovian jump systems with respect to part of the variables. They can be treated as a generalization of the criteria obtained in [29] and [2].

Also other stability criteria of positive Markov jump systems given in [2] can be adopted to stability and positivity analysis with respect to the part of variables, for instance, the criteria of exponential almost sure stability.

When there are no switchings $r(t)\equiv 0$ in considered nonlinear and linear systems, then the obtained criteria can be reduced to the corresponding criteria given in [27].

At the end, we admit possible future generalization of the class of considered hybrid systems. We may consider a positive linear feedback control system in the form $\dot{\mathbf{x}}(t)=\mathscr{A}(r(t))\mathbf{x}(t)+$

$\mathscr{B}(r(t))\mathbf{u}(t)$ and to look for criteria of stabilizability and controllability with respect to part of the variables of non-hybrid and hybrid systems.

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References

[1] A. Berman, M. Neuman, and R.J. Stern, *Nonnegative Matrices in Dynamic Systems*, Wiley, New York, 1989.
[2] P. Bolzern, P. Colaneri ,and G. De Nicolau, "Stochastic stability of positive Markov jump linear systems", *Automatica* 50, 1181– 1187 (2014).
[3] E.K. Boukas, *Stochastic Hybrid Systems: Analysis and Design*, Birkhäuser, Boston, 2005.
[4] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley and Sons, Inc., New York, 2000.
[5] Y. Guo and F. Zhu, "New results on stability and stabilization of Markovian jump systems with partly known transition probabilities", *Math. Problems in Engng*. 2012, ID 869842, 11 p. (2012).
[6] Y. Guo, "Stabilization of positive Markov jump systems", *J. Franklin Inst*. 353, 3428–3440, (2016).
[7] L. Gurvitz, R. Shorten, and O. Mason, "On the stability of switched positive systems", *IEEE Trans. Autom. Cont*. 52, 1099–1103, (2007).
[8] Y. Ji, H.J. Chizeck, "Controllability, stabilizability, and continuous- time Markovian jump linear quadratic control", *IEEE Trans. Autom. Cont*. 35, 777–788, (1990).
[9] T. Kaczorek, Positive *1-D and 2-D Systems*, Springer-Verlag, Berlin, 2002.
[10] Y. Kao, C. Wang, F. Zha, and H. Cao, "Stability in mean of partial variables for stochastic reaction–diffusion systems with Markovian switching", *J. Franklin Inst*. 351, 500–512, (2014).
[11] A.M. Kovalev, "The controllability of dynamical systems with respect to part of the variables", *J. Appl. Math. Mech*. 57, 995– 1004, (1993).
[12] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, Boston, Basel, Berlin, 2003.
[13] D. Liu, W. Wang, O. Ignatyev, and W. Zhang, "Partial stochastic asymptotic stability of neutral stochastic functional differential equations with Markovian switching by boundary condition", *Adv. Differ. Equat*. 220, 1–8, (2013).
[14] D. Liu and W. Wang, "On the partial stochastic stability of stochastic differential delay equations with Markovian switching", *Proceedings of the 2nd International Conference on Systems Engineering and Modeling (ICSEM-13)* 636–639, (2013).
[15] X. Liu, "Stability analysis of switched positive systems: a switched linear copositive Lyapunov function method", *IEEE Trans. Circuits Syst. II: Express Briefs* 56, (2009), 414–418, (2009).
[16] H. Minc, *Nonnegative Matrices*, J. Wiley, New York, 1988.
[17] W. Mitkowski, "Remarks on stability of positive linear systems", *Control and Cybernetics* 29, 295–304,(2000).
[18] W. Mitkowski, "Dynamical properties of Metzler systems", *Bull. Pol. Ac.: Tech*. 56, 309–312, (2008).
[19] W. Qi and X. Gao, "L1 Control for positive Markovian jump systems with partly known transition rates", *Int. J. Cont. Autom. Syst*. 15, 274–280, (2017).
[20] W. Qi, Ju.H. Park, G. Song, and J.Cheng, "L1 finite time stabilization for positive semi-Markovian switching systems", *Information Sciences* 447, 321–333, (2019).
[21] W. Qi, Ju.H. Park, J. Cheng, Y. Kao, and X. Gao, "Exponential stability and L1-gain analysis for positive time-delay Markovian jump systems with switching transition rates subject to average dwell time", *Information Sciences* 424, 224–234, (2018).
[22] M.A. Rami, "Solvability of static output-feedback stabilization for LTI positive systems", *Syst. Cont. Let*., 60, 704–708, (2011).
[23] V.V. Rumyantsev and A.S. Oziraner, "Partial stability and stabilization of motion", *Vestnik Moscow Univ., Ser. Math. Mech*. 4, 9–16, (1957) (in Russian).
[24] P. Shi and F. Li, "ASurvey on Markovian jump systems: modeling and design", *Int. J. Cont. Autom. Syst*. 13, 1–16, (2015).
[25] L. Socha and Q. Zhu, "Exponential stability with respect to part of the variables for a class of nonlinear stochastic systems with Markovian switchings", *Math. Comput. Simul*. 155, 2–14, (2019).
[26] A. Teel, A. Subbaraman, and A.Sferlazza, "Stability analysis for stochastic hybrid systems: A Survey", *Automatica* 50, 2435–2456, (2014).
[27] V.I. Vorotnikov, *Partial Stability and Control*, Birkhäuser, Boston, 1998.
[28] B. Wang and Q. Zhu, "Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems", *Syst. & Control Lett*. 105, 55–61, (2017).
[29] J. Zhang, Z. Han, and F. Zhu, "Stochastic stability and stabilization of positive systems with Markovian jump parameters", *Nonlinear Anal.: Hybrid Syst*. 12, 147–155, (2014).
[30] L. Zhang and E.K. Boukas, "Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities", *Automatica* 45, 463–468, (2009).
[31] Y. Zhang, Y. He, M.Wu, and J. Zhang, "Stabilization for Markov jump systems with partial information on transition probability based on free-connection weighting matrices", *Automatica* 47, 79–84, (2011).
[32] D. Zhang, Q. Zhang, and B. Du, "Positivity and stability of positive singular Markovian jump time-delay systems with partially unknown transition rates", *J. Franklin Inst. Engng. & Appl. Math*. 354, 627–649, (2017).
[33] Q. Zhu, "Razumikhin-type theorem for stochastic functional differential equations with Levy noise and Markov switching", *Int. J. Control* 90, 1703–1712, (2017).