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CAUCHY-BINET TYPE FORMULAS FOR FREDHOLM OPERATORS

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Abstract. Suppose $A \in L(Y,Z)$, $B \in L(X,Y)$ are Fredholm operators acting in linear spaces. By referring to the correspondence between Fredholm operators and their determinant systems, we derive the formulas for a determinant system for *AB* which are expressed via determinant systems for *A* and *B*. In our approach, applying results of the theory of determinant systems plays the crucial role and yields Cauchy-Binet type formulas. The formulas are utilized in many branches of applied science and engineering.

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1. Introduction

The purpose of this paper is to exhibit a method of construction of a determinant system for a product of arbitrary linear Fredholm operators acting between linear spaces. The method is based on tools of the determinant theory created by Leżański [1], developed and modified by Sikorski [2-4] and Buraczewski [5, 6].

We address the problem of how to express a determinant system for product *AB* of Fredholm operators $A : Y \to Z$ and $B : X \to Y$, *X*, *Y*, *Z* being linear spaces over the same field (real or complex), in terms of determinant systems for *A* and *B*. In the derivation of the main result we use some ideas presented in [7] for Fredholm endomorphisms and extend them to Fredholm operators acting between arbitrary linear spaces. Since the method proposed in the paper is purely algebraic, we dispense with assumptions related to a topological structure of linear spaces involved. The formulas, obtained as a direct and constructive solution to the above mentioned problem, are generalizations of the classical Cauchy-Binet formula [8-10], which states that if *A* and *B* are two matrices over field *F* of sizes $n \times m$ and

$m\times n$, respectively, with $n\le m$, then $\det(AB)=\sum_p \det(A_p)\det(B^p)$, where the sum is taken over all increasing sequences $p=(p_1,p_2,\dots p_n)$, with $p_i\in\{1,\dots,m\}$, and A_p (B^p) is $n\times n$ submatrix of A (B) obtained by deleting all columns (rows) except these with indices in p. When $n = m$, the formula becomes the well-known product formula $\det(AB)=\det(A)\det(B)$ for determinants. The Cauchy-Binet formula plays an important role in studies of determinants, permanents and other classes of matrix functions. An increasing interest in its applications in many branches of applied science, such as matrix analysis and engineering [11-13], is a motivation of the paper. It is worth emphasizing that, so far, many considerable contributions to generalizing the Cauchy-Binet theorem have been made [14-17]. In our approach, the proposed generalization to Fredholm operators is based on the correspondence between any Fredholm operator and its determinant system. We also make use of analogues of the Laplace expansion formula that are available for terms of determinant systems.

2. Preliminaries

In this section we recall the main notions and facts concerning the determinant systems theory and we fix the notation [3-6, 18-20].

Suppose (Ξ,X), (Ω,Y) and (Λ,Z) are pairs of conjugate linear spaces (over the real or complex field F) with respect to scalar products I on $\Xi\times X$, J on $\Omega\times Y$ and K on $\Lambda\times Z$, respectively, satisfying the cancellation laws [6]. Elements $\xi\in\Xi$ and $x\in X$ are called *orthogonal* if $I(\xi,x)=\xi x=0$; moreover, $X_0^{\perp}=\{\xi\in\Xi:\xi x=0 \text{ for all } x\in X_0\}$ and $\Xi_0^{\perp}=\{x\in X:\xi x=0 \text{ for all } \xi\in\Xi_0\}$ for given subsets $X_0\subset X$ and $\Xi_0\subset\Xi$. Denote by $D\begin{pmatrix}\xi_1,\dots, & \xi_\mu\\ y_1,\dots, & y_m\end{pmatrix}$ the value of a $(\mu+m)$-linear functional $D:\Xi^\mu\times Y^m\to F$ at a point $(\xi_1,\dots, \ \xi_\mu,y_1,\dots, \ y_m)$. D is said to be *bi-skew symmetric* if it is skew symmetric both in variables $\xi_1,\dots,\xi_\mu$ and $y_1,\dots,y_m$; $bss_{\mu,m}(\Xi,Y)$ stands for the set of all bi-skew symmetric functionals on $\Xi^\mu\times Y^m$. We call D an (Ω,X) *- functional on* $\Xi^\mu\times Y^m$ if for arbitrary fixed elements $\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_\mu\in\Xi$ and $y_1,\dots,y_m\in Y$ there exists an element $x_i\in X$ such that $\xi x_i=D\begin{pmatrix}\xi_1,\dots, & \xi_{i-1},\xi,\xi_{i+1},\dots, & \xi_\mu\\ y_1, & \dots, & y_m\end{pmatrix}$ for every $\xi\in\Xi$ $(i=1,\dots,\mu)$ and for arbitrary fixed elements $\xi_1,\dots,\xi_\mu\in\Xi$, $y_1,\dots,y_{j-1},y_{j+1},\dots,y_m\in Y$ there exists an element $\omega_j\in\Omega$ such that

$\omega_j y = D\begin{pmatrix} \xi_1, & \ldots, & \xi_\mu \\ y_1,\ldots, & y_{j-1}, y, y_{j+1},\ldots, & y_m \end{pmatrix}$ for every $y \in Y$ $(j=1,\ldots,m)$. $L_{\mu,m}(\Xi,Y)$ is identified with the set of all (Ω, X) - functionals on $\Xi^\mu \times Y^m$. A bilinear (Ω, X) - functional D on $\Xi \times Y$ is said to be an *operator on* $\Xi \times Y$ and ξDy stands for its value at (ξ, y). We denote by $op(\Xi \to \Omega, Y \to X)$ the set of all (Ω, X) - operators on $\Xi \times Y$. Each $D \in op(\Xi \to \Omega, Y \to X)$ can be simultaneously interpreted as a linear mapping $D: \Xi \to \Omega$ and as a linear mapping $D: Y \to X$. Thus $\xi Dy = (\xi D)y = \xi(Dy)$ for $(\xi, y) \in \Xi \times Y$. The operator $x_0 \cdot \omega_0 \in op(\Xi \to \Omega, Y \to X)$, $x_0 \in X$, $\omega_0 \in \Omega$ being fixed non-zero elements, defined by $\xi(x_0 \cdot \omega_0)y = \xi x_0 \cdot \omega_0 y$ for $(\xi, y) \in \Xi \times Y$, is called *one-dimensional*.

For $A \in op(\Omega \to \Xi, X \to Y)$ let $N(A) = \{x \in X : Ax = 0\}$, $R(A) = \{Ax : x \in X\}$, $\mathcal{N}(A) = \{\omega \in \Omega : \omega A = 0\}$, $\mathcal{R}(A) = \{\omega A : \omega \in \Omega\}$. A is said to be a *Fredholm operator on* $\Omega \times X$ *of order* $r(A) = \min\{n', m'\}$ *and index* $d(A) = n' - m'$, if $\dim N(A) = n' < \infty$, $\dim \mathcal{N}(A) = m' < \infty$, $R(A) = \mathcal{N}(A)^\perp$ and $\mathcal{R}(A) = N(A)^\perp$ [5, 21]. An operator $B \in op(\Xi \to \Omega, Y \to X)$ satisfying identities $ABA = A$, $BAB = B$ is called a *reflexive generalized inverse of* A [22]. A sequence $(D_n)_{n \in N_0}$ is said to be a *determinant system for A* if $D_n \in bss_{\mu_n, m_n}(\Xi, Y) \cap L_{\mu_n, m_n}(\Xi, Y)$, with $\mu_n, m_n \in N_0$, $\mu_n = \mu_0 + n$, $m_n = m_0 + n$, $\min(\mu_0, m_0) = 0$, and the generalized Laplace expansion formulas hold

$$D_{n+1}\begin{pmatrix} \xi_0, & \ldots, & \xi_{\mu_n} \\ Ax, & y_1,\ldots, & y_{m_n} \end{pmatrix} = \sum_{i=0}^{\mu_n} (-1)^i \xi_i x \cdot D_n \begin{pmatrix} \xi_0,\ldots, & \xi_{i-1}, \xi_{i+1},\ldots, & \xi_{\mu_n} \\ y_1, & \ldots, & y_{m_n} \end{pmatrix},$$

$$D_{n+1}\begin{pmatrix} \omega A, & \xi_1,\ldots, & \xi_{\mu_n} \\ y_0, & \ldots, & y_{m_n} \end{pmatrix} = \sum_{j=0}^{m_n} (-1)^j \omega y_j \cdot D_n \begin{pmatrix} \xi_1, & \ldots, & \xi_{\mu_n} \\ y_0,\ldots, & y_{j-1}, y_{j+1},\ldots, & y_{m_n} \end{pmatrix},$$

where $x \in X, \omega \in \Omega$, $\xi_i \in \Xi, y_j \in Y$, $i = 1,\ldots,\mu_n, j = 1,\ldots,m_n$. The least $r \in N_0$, such that $D_r \neq 0$, and the difference $\mu_0 - m_0$ are called the *order* and the *index of* $(D_n)_{n \in N_0}$, respectively.

As well-known [3, 5], an operator $A \in op(\Omega \to \Xi, X \to Y)$ has a determinant system $(D_n)_{n \in N_0}$ if and only if A is Fredholm; the orders (the indices) of A and $(D_n)_{n \in N_0}$ are the same. Moreover, if A is Fredholm, $B \in op(\Xi \to \Omega, Y \to X)$ is its reflexive generalized inverse and $\{z_1,\ldots,z_{n'}\}$, $\{\varsigma_1,\ldots,\varsigma_{m'}\}$ $(\min\{n', m'\} = r)$

form complete systems of solutions of the homogenous equations $Ax = 0$ and $\omega A = 0$, respectively, then $(D_n)_{n\in N_0}$ defined by the formula

$$D_n\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}=\begin{vmatrix}\xi_1 By_1 & \cdots & \xi_1 By_{n+m'-r} & \xi_1 z_1 & \cdots & \xi_1 z_{n'}\\ \vdots & & \vdots & \vdots & & \vdots\\ \xi_{n+n'-r}By_1 & \cdots & \xi_{n+n'-r}By_{n+m'-r} & \xi_{n+n'-r}z_1 & \cdots & \xi_{n+n'-r}z_{n'}\\ \varsigma_1 y_1 & \cdots & \varsigma_1 y_{n+m'-r} & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & & \vdots\\ \varsigma_{m'}y_1 & \cdots & \varsigma_{m'}y_{n+m'-r} & 0 & \cdots & 0\end{vmatrix} \quad (1)$$

for $\xi_i \in \Xi$ $(i = 1,\ldots,n + n' - r)$, $y_j \in Y$ $(j = 1,\ldots,n + m' - r)$, is a determinant system for A.

3. Main result

In this section we examine Fredholm operators acting from one linear space into another one. We provide a construction of a determinant system for a product of two fixed Fredholm operators. For the sake of completeness, we start by quoting some auxiliary results concerning reflexive generalized inverses of Fredholm operators, which are necessary for the proof of the main theorem of the paper.

In what follows, $A_1 \in op(\Omega \to \Xi, X \to Y)$, $A_2 \in op(\Lambda \to \Omega, Y \to Z)$ denote Fredholm operators of orders $r(A_1) = r' = min\{n', m'\}$, $r(A_2) = r'' = min\{n'', m''\}$ and indices $d(A_1) = d' = n' - m'$, $d(A_2) = d'' = n'' - m''$, respectively. Let $(x'_1,\ldots,x'_{n'})$, $(\omega'_1,\ldots,\omega'_{m'})$, $(y''_1,\ldots,y''_{n''})$ and $(\lambda''_1,\ldots,\lambda''_{m''})$ be bases of $N(A_1)$, $\mathcal{N}(A_1)$, $N(A_2)$ and $\mathcal{N}(A_2)$, respectively. The following direct sum decompositions hold: $Y = R(A_1) \oplus Y'$, $\Omega = \mathcal{R}(A_2) \oplus \Omega''$, $Y' \subseteq Y, \Omega'' \subseteq \Omega$ being subspaces such as $\dim Y' = m'$ and $\dim \Omega'' = n''$. Moreover, denoting $Y_1 = N(A_2) \cap R(A_1)$, $\Omega_1 = \mathcal{N}(A_1) \cap \mathcal{R}(A_2)$, we also obtain $N(A_2) = Y_1 \oplus Y_2$, $\mathcal{N}(A_1) = \Omega_1 \oplus \Omega_2$, where $Y_2 = N(A_2) \cap Y'$, $\Omega_2 = \mathcal{N}(A_1) \cap \Omega''$ and $\dim Y_2 = \dim \Omega_2 = t$. Let $(y''_1,\ldots,y''_{n''-t})$, $(\omega'_1,\ldots,\omega'_{m'-t})$ be bases of subspaces Y_1, Ω_1, respectively, and $Y_2 = \mathrm{span}(y_1,\ldots,y_t)$, $\Omega_2 = \mathrm{span}(\omega_1,\ldots,\omega_t)$, where $\omega_i y_j = \delta_{ij}$ $(i, j = 1,\ldots,t)$, δ_{ij} being the Kronecker symbol. Furthermore, $Y' = Y_2 \oplus Y_3$, $\Omega'' = \Omega_2 \oplus \Omega_3$, $Y_3 \subseteq Y, \Omega_3 \subseteq \Omega$ being subspaces of dimensions $m' - t$, $n'' - t$, respectively.

Under the above given assumptions we recall [20] the following two results.

Lemma 3.1. *If* $B_1 \in op(\Xi \to \Omega, Y \to X)$, $B_2 \in op(\Omega \to \Lambda, Z \to Y)$ *are arbitrary reflexive generalized inverses of Fredholm operators* A_1, A_2, *respectively, then* $(x'_1,\dots,x'_{n'}, B_1 y''_1,\dots,B_1 y''_{n''-t})$ *and* $(\lambda''_1,\dots,\lambda''_{m''},\omega'_1 B_2,\dots,\omega'_{m'-t}B_2)$ *are bases of* $N(A_2A_1)$ *and* $\mathcal{N}(A_2A_1)$, *respectively.*

Lemma 3.2. *We assume that:*

(i) $\left(D_n^{(1)}\right)_{n\in N_0}$, $\left(D_n^{(2)}\right)_{n\in N_0}$ *are fixed determinant systems for Fredholm operators* A_1, A_2, *respectively;*

(ii) $\xi'_1,\dots,\xi'_{n'} \in \Xi$ $y'_1,\dots,y'_{m'-t} \in Y_3$ *are such elements that:*

$$\delta' = D_{r'}^{(1)}\begin{pmatrix} \xi'_1, & \dots, & \xi'_{n'} \\ y'_1,\dots, & y'_{m'-t}, y_1,\dots, & y_t \end{pmatrix} \neq 0;$$

(iii) $z''_1,\dots,z''_{m''} \in Z$, $\omega''_1,\dots,\omega''_{n''-t} \in \Omega_3$ *are such elements that:*

$$\delta'' = D_{r''}^{(2)}\begin{pmatrix} \omega''_1,\dots, & \omega''_{n''-t},\omega_1,\dots, & \omega_t \\ z''_1, & \dots, & z''_{m''} \end{pmatrix} \neq 0;$$

(iv) $B_1 \in op(\Xi \to \Omega, Y \to X)$ *is a reflexive generalized inverse of* A_1 *defined by the formula*

$$\xi B_1 y = \frac{1}{\delta'} D_{r'+1}^{(1)}\begin{pmatrix} \xi, & \xi'_1, & \dots, & \xi'_{n'} \\ y, & y'_1,\dots, & y'_{m'-t}, y_1,\dots, & y_t \end{pmatrix} \text{ for } (\xi, y) \in \Xi \times Y; \tag{2}$$

(v) $B_2 \in op(\Omega \to \Lambda, Z \to Y)$ *is a reflexive generalized inverse of* A_2 *defined by the formula*

$$\omega B_2 z = \frac{1}{\delta''} D_{r''+1}^{(2)}\begin{pmatrix} \omega, & \omega''_1,\dots, & \omega''_{n''-t},\omega_1,\dots, & \omega_t \\ z, & z''_1, & \dots, & z''_{m''} \end{pmatrix} \text{ for } (\omega, z) \in \Omega \times Z. \tag{3}$$

Then the operator $B_1B_2 \in op(\Xi \to \Lambda, Z \to X)$ *is a reflexive generalized inverse of* $A_2A_1 \in op(\Lambda \to \Xi, X \to Z)$.

The following lemma plays an essential role in the sequel. It describes the connection between two arbitrary reflexive generalized inverses of a fixed Fredholm operator.

Lemma 3.3. *If* $B, C \in op(\Xi \to \Omega, Y \to X)$ *are reflexive generalized inverses of a Fredholm operator* $A \in op(\Omega \to \Xi, X \to Y)$ *and* $(\tilde{x}_1,\dots,\tilde{x}_n)$, $(\tilde{\omega}_1,\dots,\tilde{\omega}_m)$ *are bases of* $N(A)$, $\mathcal{N}(A)$, *respectively, then there exist elements* $\eta_i \in \mathcal{R}(B)$ $(i = 1,\dots,n)$, $u_j \in R(B)$ $(j = 1,\dots,m)$ *such that*

$$C = B + \sum_{i=1}^{n} \tilde{x}_i \cdot \eta_i + \sum_{j=1}^{m} u_j \cdot \tilde{\omega}_j + \sum_{i=1}^{n}\sum_{j=1}^{m} (\eta_i A u_j)\tilde{x}_i \cdot \tilde{\omega}_j \,. \tag{4}$$

Proof. By the relationship between A and its reflexive generalized inverse B [5], there exist elements $\tilde{\xi}'_1,\dots,\tilde{\xi}'_n \in \Xi$, $\tilde{y}'_1,\dots,\tilde{y}'_m \in Y$ such that $\tilde{\xi}'_i \tilde{x}_j = \delta_{ij}$ $(i,j=1,\dots,n)$, $\tilde{\omega}_i \tilde{y}'_j = \delta_{ij}$ $(i,j=1,\dots,m)$ and the following identities hold:

$$BA = I - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}'_i \,,\quad AB = J - \sum_{j=1}^{m} \tilde{y}'_j \cdot \tilde{\omega}_j \,. \tag{5}$$

Similarly, in view of the relationship between A and C,

$$CA = I - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i \,,\quad AC = J - \sum_{j=1}^{m} \tilde{y}''_j \cdot \tilde{\omega}_j \,, \tag{6}$$

where $\tilde{\xi}''_1,\dots,\tilde{\xi}''_n \in \Xi$, $\tilde{y}''_1,\dots,\tilde{y}''_m \in Y$ are elements satisfying conditions: $\tilde{\xi}''_i \tilde{x}_j = \delta_{ij}$ $(i,j=1,\dots,n)$, $\tilde{\omega}_i \tilde{y}''_j = \delta_{ij}$ $(i,j=1,\dots,m)$. By (5) and (6), bearing in mind that $\tilde{\xi}'_i B = 0 \quad (i=1,\dots,n)$, $CAB - BAB = \left(\sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}'_i - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i \right) B = -\sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i B$. Consequently, $C\left(J - \sum_{j=1}^{m} \tilde{y}'_j \cdot \tilde{\omega}_j \right) - B = -\sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i B$, which implies

$$C = B - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i B + \sum_{j=1}^{m} C\tilde{y}'_j \cdot \tilde{\omega}_j \,. \tag{7}$$

It follows from (7) that

$$CAC = \left(BA - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i BA + \sum_{j=1}^{m} C\tilde{y}'_j \cdot \tilde{\omega}_j A \right)\left(B - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i B + \sum_{j=1}^{m} C\tilde{y}'_j \cdot \tilde{\omega}_j \right). \tag{8}$$

Since $\tilde{\omega}_i \in \mathcal{N}(A)$ $(i=1,\dots,n)$, we transform the right-hand side of (8) into the form

$$BAB - \sum_{i=1}^{n} \tilde{x}_i \cdot \tilde{\xi}''_i BAB - \sum_{i=1}^{n} BA\tilde{x}_i \cdot \tilde{\xi}''_i B + \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\tilde{\xi}''_i BA\tilde{x}_k\right)\tilde{x}_i \cdot \tilde{\xi}''_k B + \tag{9}$$

$$+ \sum_{j=1}^{m} BAC\tilde{y}'_j \cdot \tilde{\omega}_j - \sum_{i=1}^{n}\sum_{j=1}^{m} \left(\tilde{\xi}''_i BAC\tilde{y}'_j\right)\tilde{x}_i \cdot \tilde{\omega}_j \,.$$

Furthermore, remembering that $\widetilde{x}_i \in N(A)$ $(i = 1,\ldots,n)$, we express (9) by

$$BAB - \sum_{i=1}^{n} \widetilde{x}_i \cdot \widetilde{\xi}_i'' BAB + \sum_{j=1}^{m} BAC\widetilde{y}_j' \cdot \widetilde{\omega}_j - \sum_{i=1}^{n}\sum_{j=1}^{m} \left(\widetilde{\xi}_i'' BAC\widetilde{y}_j'\right) \widetilde{x}_i \cdot \widetilde{\omega}_j . \qquad (10)$$

Hence, the identities $BAB = B,\ CAC = C$, combined with (8) and (10), lead to

$$C = B - \sum_{i=1}^{n} \widetilde{x}_i \cdot \widetilde{\xi}_i'' B + \sum_{j=1}^{m} BAC\widetilde{y}_j' \cdot \widetilde{\omega}_j - \sum_{i=1}^{n}\sum_{j=1}^{m} \left[\left(\widetilde{\xi}_i'' B\right) A \left(BAC\widetilde{y}_j'\right)\right] \widetilde{x}_i \cdot \widetilde{\omega}_j . \qquad (11)$$

Finally, by putting $\eta_i = -\widetilde{\xi}_i'' B$ $(i = 1,\ldots,n)$ and $u_j = BAC\widetilde{y}_j'$ $(j = 1,\ldots,m)$ in (11), we arrive at (4), which is the required result.

Having established Lemmas 3.1-3.3, we are now in a position to state and prove the main result of the paper.

Theorem 3.4. *Let* $\left(D^{(1)}{}^{n}_{m}\right)_{n,m\in N_0}$ $(n - m = d')$, $\left(D^{(2)}{}^{n}_{m}\right)_{n,m\in N_0}$ $(n - m = d'')$ *be fixed determinant systems for Fredholm operators* A_1, A_2, *respectively. If* $C_1 \in op(\Xi \to \Omega, Y \to X)$, $C_2 \in op(\Omega \to \Lambda, Z \to Y)$ *are arbitrary reflexive generalized inverses of* A_1, A_2, *respectively, then the sequence* $\left(D^n_m\right)$ $\left(n \geq \max\{d' + d'', 0\},\ n - m = d' + d''\right)$ *defined by the formula:*

$$D^n_m\begin{pmatrix} \xi_1,\ldots, & \xi_n \\ z_1,\ldots, & z_m \end{pmatrix} = \sum_{p,q} \operatorname{sgn} p \operatorname{sgn} q\ D^{(2)}{}^{n-n'+t}_{m-m'+t}\begin{pmatrix} \xi_{p_1}C_1,\ldots, & \xi_{p_{n-n'}}C_1, \omega_1,\ldots, & \omega_t \\ z_{q_1}, & \ldots, & z_{q_{m-m'+t}} \end{pmatrix} \times$$

$$\times D^{(1)}{}^{n'}_{m'}\begin{pmatrix} \xi_{p_{n-n'+1}}, & \ldots, & \xi_{p_n} \\ C_2 z_{q_{m-m'+t+1}},\ldots, & C_2 z_{q_m}, y_1,\ldots, & y_t \end{pmatrix}, \qquad (12)$$

for $\xi_i \in \Xi, z_j \in Z$ $(i = 1,\ldots,n,\ j = 1,\ldots,m)$, *where* $p = (p_1,\ldots,p_n)$, $q = (q_1,\ldots,q_m)$ *are arbitrary permutations of integers* $1,\ldots,n$ *and* $1,\ldots,m$, *respectively, such that*

$$p_1 < \ldots < p_{n-n'}, \quad p_{n-n'+1} < \ldots < p_n, \quad q_1 < \ldots < q_{m-m'+t}, \quad q_{m-m'+t+1} < \ldots < q_m,$$

is a determinant system for $A_2A_1 \in op(\Lambda \to \Xi, X \to Z)$.

Proof. Let B_1, B_2 be reflexive generalized inverses of A_1, A_2, respectively, defined by formulas (2), (3). According to Lemma 3.2, B_1B_2 is a reflexive generalized inverse of operator A_2A_1. It follows from (1), in view of Lemma 3.1, that the sequence $\left(\widetilde{D}^n_m\right)$ defined by

$$\widetilde{D}_m^n\begin{pmatrix}\xi_1,\ldots, & \xi_n\\ z_1,\ldots, & z_m\end{pmatrix}= \tag{13}$$

$$=\begin{vmatrix}
\xi_1 B_1 B_2 z_1 & \ldots & \xi_1 B_1 B_2 z_m & \xi_1 B_1 y_1'' & \ldots & \xi_1 B_1 y_{n''-t}'' & \xi_1 x_1' & \ldots & \xi_1 x_{n'}'\\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
\xi_n B_1 B_2 z_1 & \ldots & \xi_n B_1 B_2 z_m & \xi_n B_1 y_1'' & \ldots & \xi_n B_1 y_{n''-t}'' & \xi_n x_1' & \ldots & \xi_n x_{n'}'\\
\lambda_1'' z_1 & \ldots & \lambda_1'' z_m & 0 & \ldots & 0 & 0 & \ldots & 0\\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
\lambda_{m''}'' z_1 & \ldots & \lambda_{m''}'' z_m & 0 & \ldots & 0 & 0 & \ldots & 0\\
\omega_1' B_2 z_1 & \ldots & \omega_1' B_2 z_m & 0 & \ldots & 0 & 0 & \ldots & 0\\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
\omega_{m'-t}' B_2 z_1 & \ldots & \omega_{m'-t}' B_2 z_m & 0 & \ldots & 0 & 0 & \ldots & 0
\end{vmatrix}$$

$\left(n \geq \max\{d'+d'',0\}, n-m=d'+d''\right)$ for $\xi_i \in \Xi,\ z_j \in Z\ \ (i=1,\ldots,n,\ \ j=1,\ldots,m)$, is a determinant system for $A_2 A_1$. The order and the index of $\left(\widetilde{D}_m^n\right)$ are equal to $r=\min\{n'+n''-t, m'+m''-t\}$ and $d=d'+d''$, respectively. Let $\widetilde{C}_2$ denote the reflexive generalized inverse of A_2 expressed by

$$\widetilde{C}_2 = B_2 + \sum_{i=1}^{n''-t} y_i'' \cdot \eta_i'' + \sum_{j=1}^{m''} u_j'' \cdot \lambda_j'' + \sum_{i=1}^{n''-t}\sum_{j=1}^{m''}\left(\eta_i'' A_2 u_j''\right) y_i'' \cdot \lambda_j'', \tag{14}$$

where $\eta_i'' \in \mathcal{R}(B_2)(i=1,\ldots,n''-t),\ u_j'' \in R(B_2)(j=1,\ldots,m'')$. Since $\omega_k \in R(A_1)^{\perp}$ and $\omega_k B_2 = 0\ \ (k=1,\ldots,t),\ \ \omega_k \widetilde{C}_2 = 0$. Let C_1 be an arbitrary fixed reflexive generalized inverse of A_1. Hence, by Lemma 3.3,

$$C_1 = B_1 + \sum_{i=1}^{n'} x_i' \cdot \eta_i' + \sum_{j=1}^{m'} u_j' \cdot \omega_j' + \sum_{i=1}^{n'}\sum_{j=1}^{m'}\left(\eta_i' A_1 u_j'\right) x_i' \cdot \omega_j', \tag{15}$$

for some $\eta_i' \in \mathcal{R}(B_1)(i=1,\ldots,n'),\ u_j' \in R(B_1)(j=1,\ldots,m')$. Assume $y_{n''-t+i}'' = y_i$, $\omega_{m'-t+i}' = \omega_i\ (i=1,\ldots,t)$. The orthogonality of $y_1'',\ldots,y_{n''-t}''$ to all $\omega_1',\ldots,\omega_{m'}'$ and the orthogonality of $\omega_1',\ldots,\omega_{m'-t}'$ to all $y_1'',\ldots,y_{n''}''$, combined with (14), (15), lead to $C_1\widetilde{C}_2 = B_1 B_2 + S$, where $S \in op(\Xi \to \Lambda, Z \to X)$ is a finitely dimensional operator of the form

$$S=\sum_{i=1}^{n''-t}B_1y_i''\cdot\lambda_i+\sum_{i=1}^{n'}x_i'\cdot\lambda_{n''-t+i}+\sum_{i=1}^{m'-t}x_i\cdot\omega_i'B_2+\sum_{i=1}^{m''}x_{m'-t+i}\cdot\lambda_i'',$$

for some $\lambda_i\in\Lambda\left(i=1,\ldots,n'+n''-t\right)$, $x_j\in X\left(j=1,\ldots,m'+m''-t\right)$. Moreover, by virtue of (15), for any $\xi_i\in\Xi\left(i=1,\ldots,n\right)$

$$\xi_iC_1y_j''=\xi_i\left(B_1+\sum_{k=1}^{n'}x_k'\cdot\eta_k'\right)y_j''\quad\left(j=1,\ldots,n''-t\right).$$

Similarly, by (14), for any $z_j\in Z\left(j=1,\ldots,m\right)$

$$\omega_i'\widetilde{C}_2z_j=\omega_i'\left(B_2+\sum_{k=1}^{m''}u_k''\cdot\lambda_k''\right)z_j\quad\left(i=1,\ldots,m'-t\right).$$

Replacing B_1 by C_1 and B_2 by $\widetilde{C}_2$ in (13), we conclude that the right-hand side of (13) remains unchanged. Furthermore, applying the formula for the generalized expansion of a determinant, we transform the determinant in (13) into the sum

$$\sum_p\operatorname{sgn}p\begin{vmatrix}\lambda_1''z_1&\cdots&\lambda_1''z_m&0&\cdots&0\\\vdots&&\vdots&\vdots&&\vdots\\\lambda_{m''}''z_1&\cdots&\lambda_{m''}''z_m&0&\cdots&0\\\omega_1'\widetilde{C}_2z_1&\cdots&\omega_1'\widetilde{C}_2z_m&0&\cdots&0\\\vdots&&\vdots&\vdots&&\vdots\\\omega_{m'-t}'\widetilde{C}_2z_1&\cdots&\omega_{m'-t}'\widetilde{C}_2z_m&0&\cdots&0\\\xi_{p_1}C_1\widetilde{C}_2z_1&\cdots&\xi_{p_1}C_1\widetilde{C}_2z_m&\xi_{p_1}C_1y_1''&\cdots&\xi_{p_1}C_1y_{n''-t}''\\\vdots&&\vdots&\vdots&&\vdots\\\xi_{p_{n-n'}}C_1\widetilde{C}_2z_1&\cdots&\xi_{p_{n-n'}}C_1\widetilde{C}_2z_m&\xi_{p_{n-n'}}C_1y_1''&\cdots&\xi_{p_{n-n'}}C_1y_{n''-t}''\end{vmatrix}\begin{vmatrix}\xi_{p_{n-n'+1}}x_1'&\cdots&\xi_{p_{n-n'+1}}x_{n'}'\\\vdots&&\vdots\\\xi_{p_n}x_1'&\cdots&\xi_{p_n}x_{n'}'\end{vmatrix}\qquad(16)$$

multiplied by $\left(-1\right)^{n\left(m'+m''-t\right)}$, where $p=\left(p_1,\ldots,p_n\right)$ is a permutation of integers $1,\ldots,n$ fulfilling the condition $p_1<\ldots<p_{n-n'}$, $p_{n-n'+1}<\ldots<p_n$. Moreover, denoting by q any permutation of integers $1,\ldots,m$ such that $q_1<\ldots<q_{m-m'+t}$, $q_{m-m'+t+1}<\ldots<q_m$ and making use of well-known properties of classical determinants, in view of (16), the right-hand side of (13) is equal, up to a sign, to the sum

$$\sum_{p,q}\operatorname{sgn}p\operatorname{sgn}q\begin{vmatrix}\xi_{p_1}C_1y''_1 & \dots & \xi_{p_1}C_1y''_{n''-t} & \xi_{p_1}C_1\widetilde{C}_2z_{q_1} & \dots & \xi_{p_1}C_1\widetilde{C}_2z_{q_{m-m'+t}}\\ \vdots & & \vdots & \vdots & & \vdots\\ \xi_{p_{n-n'}}C_1y''_1 & \dots & \xi_{p_{n-n'}}C_1y''_{n''-t} & \xi_{p_{n-n'}}C_1\widetilde{C}_2z_{q_1} & \dots & \xi_{p_{n-n'}}C_1\widetilde{C}_2z_{q_{m-m'+t}}\\ 0 & \dots & 0 & \lambda''_1z_{q_1} & \dots & \lambda''_1z_{q_{m-m'+t}}\\ \vdots & & \vdots & \vdots & & \vdots\\ 0 & \dots & 0 & \lambda''_{m''}z_{q_1} & \dots & \lambda''_{m''}z_{q_{m-m'+t}}\end{vmatrix}\times \tag{17}$$

$$\times\begin{vmatrix}\omega'_1\widetilde{C}_2z_{q_{m-m'+t+1}} & \dots & \omega'_1\widetilde{C}_2z_{q_m}\\ \vdots & & \vdots\\ \omega'_{m'-t}\widetilde{C}_2z_{q_{m-m'+t+1}} & \dots & \omega'_{m'-t}\widetilde{C}_2z_{q_m}\end{vmatrix}\begin{vmatrix}\xi_{p_{n-n'+1}}x'_1 & \dots & \xi_{p_{n-n'+1}}x'_{n'}\\ \vdots & & \vdots\\ \xi_{p_n}x'_1 & \dots & \xi_{p_n}x'_{n'}\end{vmatrix}.$$

Next, taking into account the identities $\omega_i y''_j = 0 \quad (i = 1,\dots,t,\ j = 1,\dots,n''-t)$, $\omega'_i y_j = 0 \quad (i = 1,\dots,m'-t,\ j = 1,\dots,t)$, $\omega_i\widetilde{C}_2 = 0\,(i = 1,\dots,t)$ and $\det\lfloor\omega_i y_j\rfloor_{1\le i,j\le t} = \det\lfloor\delta_{ij}\rfloor_{1\le i,j\le t} = 1$, the sum (17) can be expressed (up to a sign) by

$$\sum_{p,q}\operatorname{sgn}p\operatorname{sgn}q\begin{vmatrix}\xi_{p_1}C_1\widetilde{C}_2z_{q_1} & \dots & \xi_{p_1}C_1\widetilde{C}_2z_{q_{m-m'+t}} & \xi_{p_1}C_1y''_1 & \dots & \xi_{p_1}C_1y''_{n''-t} & \xi_{p_1}C_1y_1 & \dots & \xi_{p_1}C_1y_t\\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \xi_{p_{n-n'}}C_1\widetilde{C}_2z_{q_1} & \dots & \xi_{p_{n-n'}}C_1\widetilde{C}_2z_{q_{m-m'+t}} & \xi_{p_{n-n'}}C_1y''_1 & \dots & \xi_{p_{n-n'}}C_1y''_{n''-t} & \xi_{p_{n-n'}}C_1y_1 & \dots & \xi_{p_{n-n'}}C_1y_t\\ \omega_1\widetilde{C}_2z_{q_1} & \dots & \omega_1\widetilde{C}_2z_{q_{m-m'+t}} & \omega_1y''_1 & \dots & \omega_1y''_{n''-t} & \omega_1y_1 & \dots & \omega_1y_t\\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \omega_t\widetilde{C}_2z_{q_1} & \dots & \omega_t\widetilde{C}_2z_{q_{m-m'+t}} & \omega_ty''_1 & \dots & \omega_ty''_{n''-t} & \omega_ty_1 & \dots & \omega_ty_t\\ \lambda''_1z_{q_1} & \dots & \lambda''_1z_{q_{m-m'+t}} & 0 & \dots & 0 & 0 & \dots & 0\\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \lambda''_{m''}z_{q_1} & \dots & \lambda''_{m''}z_{q_{m-m'+t}} & 0 & \dots & 0 & 0 & \dots & 0\end{vmatrix}\times$$

$$\times\begin{vmatrix}\omega'_1\widetilde{C}_2z_{q_{m-m'+t+1}} & \dots & \omega'_1\widetilde{C}_2z_{q_m} & \omega'_1y_1 & \dots & \omega'_1y_t\\ \vdots & & \vdots & \vdots & & \vdots\\ \omega'_{m'-t}\widetilde{C}_2z_{q_{m-m'+t+1}} & \dots & \omega'_{m'-t}\widetilde{C}_2z_{q_m} & \omega'_{m'-t}y_1 & \dots & \omega'_{m'-t}y_t\\ \omega_1\widetilde{C}_2z_{q_{m-m'+t+1}} & \dots & \omega_1\widetilde{C}_2z_{q_m} & \omega_1y_1 & \dots & \omega_1y_t\\ \vdots & & \vdots & \vdots & & \vdots\\ \omega_t\widetilde{C}_2z_{q_{m-m'+t+1}} & \dots & \omega_t\widetilde{C}_2z_{q_m} & \omega_ty_1 & \dots & \omega_ty_t\end{vmatrix}\begin{vmatrix}\xi_{p_{n-n'+1}}x'_1 & \dots & \xi_{p_{n-n'+1}}x'_{n'}\\ \vdots & & \vdots\\ \xi_{p_n}x'_1 & \dots & \xi_{p_n}x'_{n'}\end{vmatrix}. \tag{18}$$

It follows from (18), bearing in mind the definition of $\left(D^{(2)}{}^n_m\right)_{n,m\in N_0}$ and relying on properties of partitioned matrices, that the value $\widetilde{D}^n_m\begin{pmatrix}\xi_1,\dots, & \xi_n\\ z_1,\dots, & z_m\end{pmatrix}$ is equal, up to a factor of 1 or -1, to

$$\sum_{p,q} \operatorname{sgn} p \operatorname{sgn} q D^{(2)}{}^{n-n'+t}_{m-m'+t}\begin{pmatrix} \xi_{p_1}C_1,\ldots, & \xi_{p_{n-n'}}C_1,\omega_1,\ldots, & \omega_t \\ z_{q_1}, & \ldots, & z_{q_{m-m'+t}} \end{pmatrix}\times \tag{19}$$

$$\times\begin{vmatrix} \xi_{p_{n-n'+1}}C_1\widetilde{C}_2 z_{q_{m-m'+t+1}} & \cdots & \xi_{p_{n-n'+1}}C_1\widetilde{C}_2 z_{q_m} & \xi_{p_{n-n'+1}}C_1 y_1 & \cdots & \xi_{p_{n-n'+1}}C_1 y_t & \xi_{p_{n-n'+1}}x'_1 & \cdots & \xi_{p_{n-n'+1}}x'_{n'} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \xi_{p_n}C_1\widetilde{C}_2 z_{q_{m-m'+t+1}} & \cdots & \xi_{p_n}C_1\widetilde{C}_2 z_{q_m} & \xi_{p_n}C_1 y_1 & \cdots & \xi_{p_n}C_1 y_t & \xi_{p_n}x'_1 & \cdots & \xi_{p_n}x'_{n'} \\ \omega'_1\widetilde{C}_2 z_{q_{m-m'+t+1}} & \cdots & \omega'_1\widetilde{C}_2 z_{q_m} & \omega'_1 y_1 & \cdots & \omega'_1 y_t & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \omega'_{m'-t}\widetilde{C}_2 z_{q_{m-m'+t+1}} & \cdots & \omega'_{m'-t}\widetilde{C}_2 z_{q_m} & \omega'_{m'-t} y_1 & \cdots & \omega'_{m'-t} y_t & 0 & \cdots & 0 \\ \omega_1\widetilde{C}_2 z_{q_{m-m'+t+1}} & \cdots & \omega_1\widetilde{C}_2 z_{q_m} & \omega_1 y_1 & \cdots & \omega_1 y_t & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \omega_t\widetilde{C}_2 z_{q_{m-m'+t+1}} & \cdots & \omega_t\widetilde{C}_2 z_{q_m} & \omega_t y_1 & \cdots & \omega_t y_t & 0 & \cdots & 0 \end{vmatrix}.$$

By combining (18), (19) with the relationship between A_1 and its determinant system, we give rise to the identity

$$\widetilde{D}^n_m\begin{pmatrix} \xi_1,\ldots, & \xi_n \\ z_1,\ldots, & z_m \end{pmatrix} = k\sum_{p,q} \operatorname{sgn} p \operatorname{sgn} q D^{(2)}{}^{n-n'+t}_{m-m'+t}\begin{pmatrix} \xi_{p_1}C_1,\ldots, & \xi_{p_{n-n'}}C_1,\omega_1,\ldots, & \omega_t \\ z_{q_1}, & \ldots, & z_{q_{m-m'+t}} \end{pmatrix}\times$$

$$\times D^{(1)}{}^{n'}_{m'}\begin{pmatrix} \xi_{p_{n-n'+1}}, & \ldots, & \xi_{p_n} \\ \widetilde{C}_2 z_{q_{m-m'+t+1}},\ldots, & \widetilde{C}_2 z_{q_m}, y_1,\ldots, & y_t \end{pmatrix}, \tag{20}$$

where $k=1$ or $k=-1$. In view of (14), Lemma 3.3 implies that

$$C_2 = \widetilde{C}_2 + \sum_{i=1}^{t} y_i \cdot \eta''_{n''-t+i} + \sum_{i=1}^{t}\sum_{j=1}^{m''}\left(\eta''_{n''-t+i}A_2 u''_j\right) y_i \cdot \lambda''_j,$$

for some $\eta''_{n''-t+i} \in \mathcal{R}(B_2)\,(i=1,\ldots,t)$, is an arbitrary reflexive generalized inverse of A_2. Bearing in mind the bi-skew symmetry of $D^{(1)}{}^{n'}_{m'}$, we can substitute C_2 for $\widetilde{C}_2$ in (20). Since a determinant system for the fixed Fredholm operator is determined up to a constant (non-zero) factor, the sequence $\left(D^n_m\right)$ defined by (12) is a determinant system for A_2A_1. This completes the proof.

As a direct consequence of Theorem 3.4, we obtain the following result.

Corollary 3.5. *Under the assumptions of Theorem 3.4, with* $d'=d''=0$, the *formula* (12) *is of Cauchy-Binet type.*

4. Conclusions

In the paper, products of Fredholm operators acting between arbitrary linear spaces were considered. By exploiting terms of determinant systems for operators

A and *B*, with *AB* well-defined, we provided a direct construction of a determinant system for *AB*. The obtained result leads to a generalization of the Cauchy-Binet formula to Fredholm operators and yields an important tool for solutions of problems in various branches of applied science and engineering.

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