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Construction of a Fourier transform based on Haar measure in structure (R^n, +, T)

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Abstract

An important aspects of learning are the theoretical elements of mathematics and the methods of obtaining the function or designbecause it allows for further development of the issues and the ability to apply this problem in practice. The Fourier transform is very useful in science and technology. The article shows how and in what structures is possible to use the Fourier transform in the set R^n. The article describes the project of compact-open topology, which is essential to the construction of the Haar measure and integral. It was also described the concept of the nature of a locally compact topological group, because a group of characters is also essential to the construction of the Fourier transform and inverse Fourier transform. It has been proven that a group of characters is a locally compact topological group, which allows the introduction of the Haar measure. It has given an example application of the theory of Haar integral when calculating the Fourier transform for some of the electronics.

Introduction

The aim of this study is to provide the mathematical basis for the construction of the Fourier transform. The Fourier transform has very wide applications in science. It appeared for the first time in the discussion of the phenomenon of heat flow, now appears in many practical applications of science and technology. The Fourier transform is a basic tool for harmonic analysis and the theory of analysis and signal processing. From the point of view of the theory the Fourier transform is a mathematical tool that occur when one analyses the $L_2(G)$ (with the square integrable functions on a group of alternating G). In applications the Fourier transform is commonly used when $G = R$ (a group of real numbers with addition), $G = [0, 2\pi]$ (group of real numbers from the interval $[0, 2\pi]$, operation is addition modulo 2π) and $G = Z_n$ (group of integers $\{0, 1, ..., n-1\}$ with addition modulo n). It turns out that the integral design of measurement which is used in the transform is based on a complex mathematics. The Haar measure is a standard tool of harmonic analysis on locally compact groups. In this article we considered the commutative group, so just shows the existence of measure is not paying attention to it, whether it is left side or right handed. It is assumed that the reader are familiar with the theory and properties of topological spaces, group theory, basic algebra, theory of integrals and measures, especially the Lebesgue integral.

The compact-open topology

This chapter provides the overall topology of the knowledge needed to deliberations that take place later in the article.

Let X, Y be topological spaces and let

$$C(X, Y) = \{f\colon X \to Y;\ f \text{ is a continuous function}\} \quad (1)$$

For $C(X, Y)$ is defined a compact-open topology by giving subbase. Suppose, that $K \subset X$ be a compact set and $U \subset Y$ be an open set.

Let:

$$P(K,U) = \{f \in C(X,Y);\ f(K) \subset U\} \quad (2)$$

Now we give two definitions related to topological groups which are needed to understand the next chapter:

Definition 1. ***Compact-open topology*** *on the set $C(X, Y)$ is called the topology generated by the base consisting of all sets $P(K_i, U_i)$ where K_i are compact, and U_i are open for $i = 1,2,...k$* [1].

Definition 2. *A topological space X is called* ***locally compact*** *if for every $x \in X$ there exists a neighborhood U, that $\overline{U}$ is a compact set* [2].

There are also two lemmas concerning that represent certain properties of locally compact topological spaces:

Lemma 1. *For each pair X, Z topological spaces and locally compact topological space Y mapping $E: Z^Y \times X^Y \to Z^X$ is a continuous, if the spaces of continuous functions we consider are the compact-open topology* [1].

Lemma 2. *Let X, Y, Z, T be topological spaces and let Y^X is the set of all mappings f such that $f: X \to Y$. Continuous mappings $g: Y \to T$, $h: X \to Z$ define the mappings $\phi: Y^X \to T^X$, $\Psi: Y^Z \to Y^X$ by setting* [1]:

$$\begin{aligned} \phi(f) &= g \circ f \text{ for } f \in Y^X \\ \Psi(f) &= f \circ h \text{ for } f \in Y^Z \end{aligned} \tag{3}$$

Is given a very important theorem on Hausdorff space, which are used to prove Theorem 4 in the next chapter:

Theorem 1. *If Y is a Hausdorff space, then $C(X, Y)$ is also a Hausdorff space* [2].

We have thus obtained the promised reduction:

Lemma 3. *For any topological spaces X, Y a compact set $K \subset X$ and a closed set $U \subset Y$, the set $P(K, U)$ is closed in the space Y^X with the compact-open topology* [1].

Characters of locally compact topological groups

This section focuses on the characters of the group topological $(R^n,+,T)$, where T is the natural topology of Euclidean space R^n. Now we give definitions of the topological group and local compactness and then further properties of locally compact topological groups.

Definition 3. *The structure of $(V,*,T)$ is called a topological group if* [3]:

(a) *$(V,*)$ is a group;*
(b) *(V, T) is a Hausdorff topological space;*
(c) *the action $*$ of a group and the assignment inverse action are continuous.*

Definition 4. *Topological group $(G,*,T)$ is* ***locally campact*** *if topological space (G, T) is locally compact* [2].

Definition 5. *Let $(G,*,T)$ be locally compact topological group.* ***The character*** *of this group is called any continuous homomorphism $h: G \to S^1$, where S^1 is the unit circle* [3].

In further considerations of the S^1, subspace topology will be considered with the natural topology (generated by the Euclidean metric) in R^2.

It takes place the following theorem:

Theorem 2. *There is a character for a group of topological $(R^n,+,T)$.*

Proof. We consider functions $f: R^n \to R$, $g: R \to S^1$, defined by setting:

$$f((x_1,..., x_n)) = \sum_{i=1}^{n} x_i,\ g(y) = e^{iy} \tag{4}$$

There is a clear that the functions f and g are continuous. It is not difficult to note that these two functions are homomorphisms.

Let us now take a new function $h: R^n \to S^1$ given formula:

$$h((x_1,\ldots,x_n)) = g(f(x_1,\ldots,x_n)) = e^{i\sum_{i=1}^{n} x_i} \tag{5}$$

Submission of continuous functions is a continuous function, and so the function h is continuous. Similarly, the submission of two homomorphisms is a homomorphism, so h is a homomorphism, which had to be demonstrated. ■

It is easy to see that:

Remark 1. *In place of the function f in the proof of previous theorem one can put any additive and continuous function.*

Remark 2. *In place of the function h in the proof of the previous theorem one can put any form of an exponential function or function which has values in any circle in R^2 space.*

We denote the set of all characters of a topological group $(R^n,+,T)$ by F. Now it will be presented and proved very important theorem about characters of a topological group $(R^n,+,T)$:

Theorem 3. *The set F has a group structure.*

Proof. Let $j, k, l \in F$ and $x, y \in R^n$. In the set F it can be specify a new action $*: F \times F \to F$, such that $j * k(x) = j(x)k(x)$. Action $*$ is correctly specified. Based on the alternation of the group S^1

$$\begin{aligned} &(j * k)(x + y) = j(x + y)k(x + y) = \\ &= j(x)j(y)k(x)k(y) = j(x)k(x)\, j(y)k(y) = \\ &= (j * k)(x)(j * k)(y) \end{aligned} \tag{6}$$

So $j * k$ is a continuous homomorphism. An inverse to the nature of l can be defined as follows $l^{-1}(x) = (l(x))^{-1}$.

This element is correctly specified, because

$$\begin{aligned} l^{-1}(x + y) &= (l(x + y))^{-1} = l(x)^{-1} l(y)^{-1} = \\ &= l^{-1}(x) l^{-1}(y) \end{aligned} \tag{7}$$

so l^{-1} is a continuous homomorphism.

It is easy to check that the neutral element in the set F is the nature of $F_0 = 1$. ■

It has been shown that F is an Abelian group (because S^1 is an Abelian group) and it is called a group of characters.

From previous lemmas and theorem 1 we have the most important conclusion in this chapter:

Theorem 4. *Let F be the set of characters of topological groups* (R^n,+,T). *In the set* $C((R^n, S^1))$ *is determined the compact-open topology*.

If in the set $F \subset C(R^n, S^1)$ we introduce the subspace topology T', $(F,*,T')$ becomes a ***locally compact topological group***.

Proof. According to the assumptions and Lemma 1 action $*: F \times F \to F$ is a continuous function. From Lemma 2 it follows immediately that the operation assignment inverse in a topological group $(F,*,T')$ is a continuous function.

On the basis of Theorem 1 (F, T') is also a Hausdorff space, because S^1 is a Hausdorff space. It has been proven that $(F,*,T')$ is a topological group. It must be demonstrated even local compactness of the group.

Let $K \subset R^n$ be a compact set, $U \subset S^1$ be an open set. It is obvious that $P(K, U) \subset P(K, \overline{U})$. From Lemma 3 we know that $P(K, \overline{U})$ is a closed set. By definition compact-open topology is known that we can choose any finite open sets to cover $P(K, \overline{U})$, so the set $P(K, \overline{U})$ is compact, qed. ■

The Fourier transform of a topological group (R^n, +, T)

To define the Fourier transform, it must be to provide a measure of structure in the set R^n. Let $A \subset R^n$ be a compact set and $\theta \in \mathrm{Int}(A)$. The existence of such a set follows from the local compactness of the set R^n. Let $U \subset R^n$ be an open set and $\theta \in U$. Then for any compact set $C \subset R^n$ it can be choosen of each cover open sets to choose finite subcover such that: $C \subset \bigcup_{i=1}^{n}(x_i + U)$, where $x_i \in R^n$. Denote by:

$$C:U = \min\left\{n; \mathrm{V}_{x_1,\dots,x_n} C \subset \bigcup_{i=1}^{n}(x_i + U)\right\} \quad (8)$$

Let $\lambda_U : R^n \to Q$ will be defined by:

$$\lambda_U(C) = \frac{C:U}{A:U} \quad (9)$$

Function λ_U satisfies the conditions of measure, and the only accurate to a multiplicative constant positive. Function λ_U will be called the Haar measure. Using a standard structure of the Legesque integral, on the basis of the Haar measure, it can be built the integral as follows:

$$\int_{R^n} f(x) \mathrm{d}\mu(x) \quad (10)$$

which we call the Haar integral (μ is a Haar measure in R^n).

Let now

$$f \in L_1(R^n) = \\ = \left\{f, f: R^n \to R, \int_{R^n} |f(x)| \mathrm{d}\mu(x) < \infty, x \in R^n\right\}.$$

Then the ***Fourier transform*** of the function f, $\tilde{f}: F \to R$, is defined as:

$$\tilde{f}(j) = \int_{R^n} f(x) j(x)^{-1} \mathrm{d}\mu(x), x \in R^n, j \in F \quad (11)$$

The inverse transform $\breve{g}: R^n \to R$ to the function g is defined by:

$$\breve{g}(x) = \int_F g(j) j(x) \mathrm{d}m(j), x \in R^n, j \in F \quad (12)$$

where m is a Haar mesure in F (in a compact – open topology). The next theorem describe the property of an inversion of the Fourier transform:

Theorem 5. (*Fourier transform inversion formula*). *For each of the Haar measure on* R^n *there exists a unique Haar measure on F such that for each function f the continued and integrable on* R^n, *which transform* $\tilde{f} \in L_1(R^n)$ *occurs* [3]

$$f(x) = \int_F \tilde{f}(j) j(x) \mathrm{d}m(j) \quad (13)$$

where $x \in R^n$.

The Haar measure can be restricted to Borel sets in R^n that are measurable in the sense of Lebesque. Then the Haar measure is equal to the extent Lebesque measure. For any continuous function $f: R \to R$, the Lebesque integral is equal to the Riemann integral (if the Riemann integral exists) and then the Fourier transform of a function f has the form:

$$\tilde{f}(j) = \int_{[a,b]} f(x) j(x)^{-1} \mathrm{d}x, x \in R \quad (14)$$

However, when the Riemann integral does not exist, it should be to use the Lebesque integral or the Haar integral. From the properties of Haar integrals and characters of the R^n follows next corollary:

Corollary 1. *Let* $f \in L_1(R^n)$, $j, k \in F$. *Then*

(*i*) $\tilde{f}(j) = \int_{R^n} f(x) j(x)^{-1} \mathrm{d}\mu(x) = \int_{R^n} f(x) \overline{j(x)} \mathrm{d}\mu(x);$

(*ii*) $\left|\tilde{f}(j * k)\right| \le \int_{R^n} |f(x)| \mathrm{d}\mu(x);$

(*iii*) $\left|\tilde{f}(j_1 * j_2 * \ldots * j_n)\right| \le \int_{R^n} |f(x)| \mathrm{d}\mu(x),\ j_i \in F.$

Proof. Item (*i*) follows from properties of complex numbers located on the unit circle. Item (*ii*) can be demonstrated as follows:

$$\left|\tilde{f}(j*k)\right| = \left|\int_{R^n} f(x)(j*k)^{-1}(x)\mathrm{d}\mu(x)\right| \le$$
$$\le \int_{R^n}\left|f(x)(j*k)^{-1}(x)\right|\mathrm{d}\mu(x) =$$
$$= \int_{R^n}\left|f(x)[(j*k)(x)]^{-1}\right|\mathrm{d}\mu(x) =$$
$$= \int_{R^n}|f(x)||j(x)|^{-1}|k(x)|^{-1}\mathrm{d}\mu(x)$$

so

$$\left|\tilde{f}(j*k)\right| \le \left|\int_{R^n} f(x)\mathrm{d}\mu(x)\right|.$$

The proof of item (*iii*) is analogous to the proof of item (*ii*). ■

Based on complex numbers properties it can be demonstrated a transformation, allowing to come to the Fourier transform for some function which have a set of values in a circle with the radius *r*:

Corollary 2. Let $h: R^n \to R^2$ be a continuous function, which has values closed in a circle with the radius *r* and with center *s*. The number *s* can be written as a complex number. Then the formula:

$$\frac{1}{r}\int_{R^n} f(x)h(x)^{-1}\mathrm{d}\mu(x) - \frac{s}{r}\int_{R^n} f(x)\mathrm{d}\mu(x) \quad (15)$$

is a Fourier transform of a function *f*, where $f: R^n \to R,\ f \in L_1(R^n)$.

Proof. It can be apply the transformation $k(x) = (h(x) - s)/r$, and then the function *k* will be a character of a topological group R^n.

Using Haar integral to calculate Fourier Transform

Let a chip that performs the function *f* from two different and independent signals $f_1(x)$ and $f_2(x)$. At the output the function $f(x, y)$ has the form:

$$f(x,y) = \begin{cases} x^2 + \sin(y) & \text{for } x, y \in [0,1]^2 \setminus Q^2 \\ \sqrt{\mathrm{ctg}(x+y)} & \text{for } x, y \in [0,1]^2 \cap Q^2 \end{cases} \quad (16)$$

where *Q* is the set of rational numbers. Let a character of a set R^2 is the formula $j(x, y) = e^{i(x+y)}$. The function *f* is not integrable in the sense of Riemann. To calculate the Fourier transform for such a system, it must be to calculate the Haar integral. Because function *f* is defined on the set R^2, the Haar measure equals the Lebesque measure, then the Haar integral equals the Lebesque integral and

$$\tilde{f}(j) = (L)\int_{[0,1]^2} f(x,y)j(x,y)^{-1}\mathrm{d}\mu(x) \quad (17)$$

The function *f* is not integrable in the sense of Riemann, so be sure to get out of the Lebesque integral. Then there is the following equality

$$\tilde{f}(j) = (L)\int_{[0,1]^2} f(x,y)j(x,y)^{-1}\mathrm{d}\mu(x) =$$
$$= (L)\int_{[0,1]^2\setminus Q^2}\left(x^2 + \sin(y)\right)e^{-i(x+y)}\mathrm{d}\mu(x) +$$
$$+ (L)\int_{[0,1]^2\cap Q^2}\sqrt{\mathrm{ctg}(x+y)}\,e^{-i(x+y)}\mathrm{d}\mu(x) =$$
$$= (R)\int_{[0,1]^2}\left(x^2 + \sin(y)\right)e^{-i(x+y)}\mathrm{d}x\,\mathrm{d}y +$$
$$- (L)\int_{[0,1]^2\cap Q^2}\sqrt{\mathrm{ctg}(x+y)}\,e^{-i(x+y)}\mathrm{d}\mu(x) +$$
$$+ (L)\int_{[0,1]^2\cap Q^2}\sqrt{\mathrm{ctg}(x+y)}\,e^{-i(x+y)}\mathrm{d}\mu(x) =$$
$$= (R)\int_{[0,1]^2}\left(x^2 + \sin(y)\right)e^{-i(x+y)}\mathrm{d}x\,\mathrm{d}y$$

After the transformation we come to the Riemann integral, which is simple to calculate. However, there are continuous functions in R^n, which is not integrable in the sense of Riemann, but these functions are integrate in the sense of Lebesque.

Conclusions

Analyzing the content of the article it is easy to see that the Fourier transform can be used for functions that field is a subset of R^n. It can be notice that due to the structure of Fourier transform on locally compact topological groups, in a simple way, we can determine the inverse transform of the Fourier transform of the function. A function can have an infinite number of transforms, as it is possible the use many characters for any function.

By using the Haar integral, it can be calculate Fourier transform for every function, which is defined on any subset of R^n. Because not every function is integrable in the sense of Riemann, construction of the Fourier transform with using a group of characters on R^n and locally compactness R^n set is a powerful tool.

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