TYPE-REDUCTION OF THE DISCRETISED INTERVAL TYPE-2 FUZZY SET: APPROACHING THE CONTINUOUS CASE THROUGH PROGRESSIVELY FINER DISCRETISATION

Sarah Greenfield and Francisco Chiclana
Centre for Computational Intelligence
Faculty of Technology, De Montfort University
Leicester, LE1 9BH, UK

Abstract

The defuzzification of a type-2 fuzzy set is a two stage process consisting of firstly *type-reduction*, and secondly defuzzification of the resultant type-1 set. This paper considers three approaches to discrete interval type-reduction: 1. The exhaustive method which produces the Type-Reduced Set, 2. the Greenfield-Chiclana Collapsing Defuzzifier which gives rise to the Representative Embedded Set Approximation, and 3. the Nie-Tan Method from which the Nie-Tan Set is derived. In the discrete case these three type-1 sets are distinct. The behavior of the three sets under fine discretisation is investigated experimentally, in order to shed light on the relationships between the continuous versions of these type-1 sets.

1 Introduction

The final stage of a Fuzzy Inferencing System (FIS) is defuzzification. In a type-2 FIS (Figure 1.) defuzzification consists of two stages 1. *type-reduction*, the procedure by which a type-2 set is converted to a type-1 set, and 2. defuzzification proper in which this type-1 set is defuzzified to give a crisp number [8].

This paper concerns three type-reduction strategies for interval type-2 fuzzy sets. These techniques apply to *discrete* sets i.e. ones that have been discretised through a process of slicing. The rationale for discretisation is that a computer can process a finite number of slices, whilst it is unable to process the continuous fuzzy sets from which the slices are taken. By exploring the effect of making the domain discretisation finer, light is shed on the continuous cases of the resultant type-1 sets, which are:

1. The Type-Reduced Set (TRS) as derived through exhaustive defuzzification,
2. the Representative Embedded Set Approximation (RESA), created by the Greenfield-Chiclana Collapsing Defuzzifier (GCCD), and
3. the Nie-Tan Set (NTS), produced by the Nie-Tan Method.

This paper is structured as follows: The next section introduces the concepts fundamental to type-2 fuzzy logic that are used in the rest of the paper. Section 3 presents exhaustive defuzzification, Section 4 the Greenfield-Chiclana Collapsing Defuzzifier, and Section 5 the Nie-Tan Method; these are all type-reduction strategies applicable to interval type-2 fuzzy sets. In Section 6 the experiments concerning fine discretisation that form the core of this paper are described. The results are tabulated and conclusions drawn. Lastly, in Section 7, further work resulting from this piece of research is discussed.

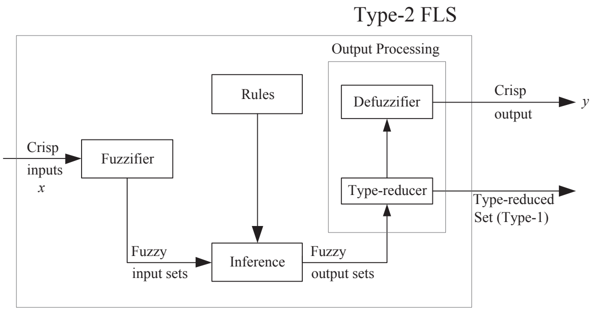

Figure 1. Type-2 FIS (from Mendel [8])

2 Preliminaries

To make the paper self-contained, the main concepts that will be used are introduced here.

2.1 Mathematical Definition of a Type-2 Fuzzy Set

The Type-1 Fuzzy Set Let X be a universe of discourse. A fuzzy set A on X is characterised by a membership function $\mu_A : X \to [0,1]$ and can be represented as follows [10]:

$$A = \{(x, \mu_A(x));\ \mu_A(x) \in [0,1]\, \forall x \in X\}. \quad (1)$$

An alternative notation for a fuzzy set A with continuous universe of discourse is

$$A = \int_{x \in X} \mu_A(x)/x. \quad (2)$$

When the universe of discourse is discrete the fuzzy set A is represented as

$$A = \sum_{x \in X} \mu_A(x)/x. \quad (3)$$

Note that the membership grades of A are crisp numbers. This sort of fuzzy set is known as a *type-1 fuzzy set*.

The Type-2 Fuzzy Set In the following we will use the notation $U = [0,1]$. Let $\tilde{P}(U)$ be the set of fuzzy sets in U. A type-2 fuzzy set $\tilde{A}$ in X is a fuzzy set whose membership grades are themselves fuzzy [11, 12, 13]. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in U for all x, i.e. $\mu_{\tilde{A}} : X \to \tilde{P}(U)$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x));\ \mu_{\tilde{A}}(x) \in \tilde{P}(U)\, \forall x \in X\}. \quad (4)$$

It follows that $\forall x \in X\, \exists J_x \subseteq U$ such that $\mu_{\tilde{A}}(x) : J_x \to U$. Applying (1), we have:

$$\mu_A(x) = \{(u, \mu_{\tilde{A}}(x)(u)) | \mu_{\tilde{A}}(x)(u) \in U \\ \forall u \in J_x \subseteq U\}. \quad (5)$$

X is called the primary domain and J_x the primary membership of x while U is known as the secondary domain and $\mu_{\tilde{A}}(x)$ the secondary membership of x.

Putting (4) and (5) together we obtain:

Definition 1 (Type-2 Fuzzy Set)

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u))) | \mu_{\tilde{A}}(x)(u) \in U \forall x \in X \\ \wedge \forall u \in J_x \subseteq U\}. \quad (6)$$

This ‘vertical representation’ of a type-2 fuzzy set is used to define the concept of an *embedded set* (Subsection 2.3) of a type-2 fuzzy set, which is fundamental to the definition of the *centroid* of a type-2 fuzzy set.

When the secondary membership grades of a type-2 fuzzy set are all 1, the set is known as an *interval type-2 fuzzy set*.

Definition 2 (Interval Type-2 Fuzzy Set) *An interval type-2 fuzzy set is a type-2 fuzzy set with a constant secondary membership grade of* 1*, i.e.* $\mu_{\tilde{A}}(x)(u) = 1, \forall u \in J_x$.

2.2 Centroid of a Type-2 Fuzzy Set

Centroid of a Type-1 Fuzzy Set There are several techniques for defuzzifying a type-1 set [6, pages 336–338]. However in this paper we only use the *centroid* method. The definition of the centroid of a type-1 fuzzy set A in X, (also referred to as the centre of gravity or centre of mass,) requires the universe of discourse to be a subset of the set of real numbers. Therefore, from now on we will assume that the domain of the type-1 fuzzy set is of such type.

For a continuous universe of discourse the centroid of fuzzy set A in X is defined as

$$C_A = \frac{\int_x x \cdot \mu_A(x) dx}{\int_x \mu_A(x_i)}. \quad (7)$$

The centroid when the domain X is discretised into n points is

$$C_A = \frac{\sum_{i=1}^{n} x_i \cdot \mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}. \quad (8)$$

Centroid of a Type-2 Fuzzy Set By applying Zadeh's Extension Principle [11], Karnik and Mendel [5, page 198] define the centroid of a type-2 fuzzy set:

Definition 3 (Centroid of a Type-2 Fuzzy Set) *The centroid of a type-2 fuzzy set $\tilde{A}$ with domain X discretised into n points $x_1, \ldots, x_n$ with $x_1 < x_2 < \ldots < x_n$. is*

$$C_{\tilde{A}} = \int_{u_1 \in J_{x_1}} \cdots \int_{u_n \in J_{x_n}} [\mu_{\tilde{A}}(x_1)(u_1) * \ldots * \mu_{\tilde{A}}(x_n)(u_n)] \Big/ \frac{\sum_{i=1}^{n} x_i \cdot u_i}{\sum_{i=1}^{n} u_i}. \quad (9)$$

Note that the centroid is a *type-1 fuzzy set* in U. In practice its computation requires the secondary domain U to be discretised as well as the primary domain. Obviously this definition is meaningful only when X is numeric in nature.

Centroid of an Interval Type-2 Fuzzy Set When $\tilde{A}$ is an interval type-2 fuzzy set, i.e. $\mu_{\tilde{A}}(x)(u) = 1 \; \forall x, u$, it follows that the centroid is the crisp set

$$C_{\tilde{A}} = \int_{u_1 \in J_{x_1}} \cdots \int_{u_n \in J_{x_n}} 1 \Big/ \frac{\sum_{i=1}^{n} x_i \cdot u_i}{\sum_{i=1}^{n} u_i}. \quad (10)$$

2.3 Embedded Sets

An *embedded type-2 set* (or 'embedded set' for short), is a special kind of type-2 fuzzy set, relating to the type-2 fuzzy set in which it is embedded in this way: For every primary domain value, x, there is a unique secondary domain value, u, plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)$.

Definition 4 (Embedded Set) *Let $\tilde{A}$ be a type-2 fuzzy set in X. For discrete universes of discourse X and U,* an embedded type-2 set *$\tilde{A}_e$ of $\tilde{A}$ is defined as the following type-2 fuzzy set*

$$\tilde{A}_e = \sum_{i=1}^{N} [\mu_{\tilde{A}}(x_i)(u_i)/u_i] \Big/ x_i \quad u_i \in J_{x_i} \subseteq U \;\wedge\; x_i \in X.$$

$\tilde{A}_e$ contains exactly one element from J_{x_1}, J_{x_2}, ..., J_{x_N}, namely u_1, u_2, ..., u_N, each with its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_1), \mu_{\tilde{A}}(x_2)(u_2), \ldots, \mu_{\tilde{A}}(x_N)(u_N)$. [8, page 98, Definition 3-10]

Definition 5 (Degree of Discretisation) *The* degree of discretisation *of a discretised fuzzy set is the separation of the slices.*

Definition 6 (Scalar Cardinality) *The* scalar cardinality *of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A. Thus,*

$$|A| = \sum_{x \in X} \mu_A(x). \text{ [7, page 17]}$$

To distinguish scalar cardinality from cardinality in the classical sense, we adopt the '$\| \; \|$' symbol for scalar cardinality.

3 Exhaustive Defuzzification

The final defuzzification stage of an FIS consists of two parts. Firstly, through a procedure known as *type-reduction*, a type-1 set termed the *Type-Reduced Set (TRS)* is derived. The TRS is identified with the centroid of the type-2 set (Subsection 2.2). Defuzzifying the TRS is straightforward, and this is the second stage of type-2 defuzzification. The challenging and complex step when deriving the defuzzified value of a type-2 fuzzy set is the computation of the TRS.

Algorithm 1, the type-reduction algorithm, was originally described by Mendel [8, pages 248–252]. It relies on the concept of an embedded set (Subsection 2.3). This stratagem has become known as the *exhaustive method*, as it requires that every embedded set be processed [4].

Stage 2 of Algorithm 1 requires the application of a t-norm ($*$) to the secondary membership degrees. In our work we use the minimum t-norm, but other choices are available [6, page 63]. However in the context of type-reduction, the product t-norm is to be avoided as it does not produce meaningful results for type-2 fuzzy sets with general secondary membership functions [5, pages 200–201].

Algorithm: Type-reduction of a discretised type-2 fuzzy set to a type-1 fuzzy set

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set

1 **forall the** *embedded sets* **do**

2 find the minimum secondary membership grade {T-norms other than minimum may be employed};

3 calculate the primary domain (x) value of the type-1 centroid of the type-2 embedded set;

4 pair the secondary grade with the x-value to give set of ordered pairs (x,z) { some values of x may correspond to more than 1 value of z };

5 **end**

6 **forall the**

7 select the maximum secondary grade { make each x correspond to a unique secondary domain value };

8 **end**

Embedded sets are very numerous, often totalling many trillions. Though individually easily processed, embedded sets in their totality give rise to a processing bottleneck simply by virtue of their high cardinality. Consequently, exhaustive defuzzification is to be seen as a theoretical approach rather than a practical technique. At coarse discretisations this strategy may be implemented but is extremely slow; at finer discretisations the issues of memory space and representation of very large numbers make implementation impossible. Despite its practical shortcomings, we regard exhaustive defuzzification as the standard by which the *accuracy* of other type-2 defuzzification algorithms is to be evaluated.

In the interval case all secondary grades are 1, which means that the minimum secondary grade is bound to be 1. Steps 2 and 7 may therefore be eliminated from the interval algorithm. All the ordered pairs of the TRS will be of the form $(x,1)$; graphically they lie on a horizontal line.

4 The Greenfield-Chiclana Collapsing Defuzzifier

A computationally simple alternative to the exhaustive method is the *Greenfield-Chiclana Collapsing Defuzzifier* [2]. This technique converts an interval type-2 fuzzy set into a type-1 fuzzy set which approximates to the *Representative Embedded Set (RES)*, whose defuzzified value is by definition equal to that of the original type-2 set (Figure 2.). We term this type-1 set the *Representative Embedded Set Approximation (RESA)*. As a type-1 set, the RESA may then be defuzzified straightforwardly. Hence the collapsing process reduces the computational complexity of type-2 defuzzification.

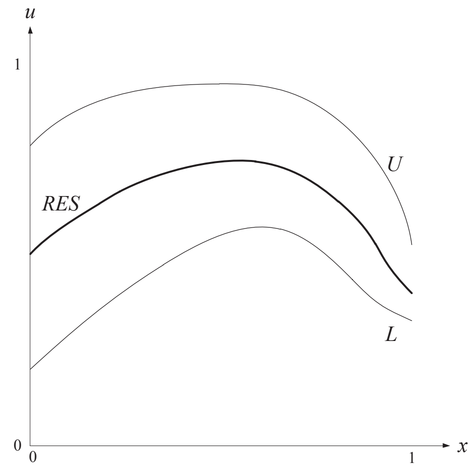

Figure 2. A Representative Embedded Set (continuous case)

Full details of the collapsing algorithm may be found at [2]. We formally state the Simple[1] Representative Embedded Set Approximation:

[1] In [2], we used the term 'simple' to describe an interval set in which each vertical slice consists of only two points, corresponding to L and U. The term is redundant in the context of this paper.

Theorem 1 (Simple Rep. Embedded Set Approx.) *The membership function of the* embedded set *R derived by dynamically collapsing slices of a discretised type-2 interval fuzzy set* $\tilde{F}$*, having lower membership function L, and upper membership function U, is:*

$$\mu_R(x_i) = \mu_L(x_i) + r_i \tag{11}$$

with

$$r_i = \frac{\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) b_i}{2\left(\|L\| + \sum_{j=1}^{i-1} r_j\right) + b_i}, \tag{12}$$

and $b_i = \mu_U(x_i) - \mu_L(x_i),\ r_0 = 0$.

This is an iterative formula. Collapsing proceeds vertical slice by vertical slice. The first slice is collapsed, the first y-value of the RESA calculated, the next slice is collapsed and the second y-value of the RESA calculated, and so on until all the slices have been collapsed. In this formula b_i is the blur for vertical slice i, i.e. the difference between the upper membership function and the lower membership function for slice i. r_i is the amount by which the y-value of L must be increased to give the y-value of the RESA R.

There are many variants of the collapsing strategy, as slice collapse may proceed in any slice order. The different variants give rise to slightly different defuzzified values [3].

During the collapsing process the RESA approaches the NTS (Section 5; Figures 3 and 4). We explain this with reference to *collapsing forward*, the variant in which collapse takes place in order of increasing domain value: As the collapse proceeds, $\sum_{j=1}^{j=i-1} r_j$ in the collapsing formula (Equation 12) increases, making the expression $\|L\| + \sum_{j=1}^{j=i-1} r_j$ also increase. r_i therefore increases, with $\frac{b_i}{2}$ as its upper bound. This argument is not dependent on the order of slice collapse, applying equally to any variant of collapsing.

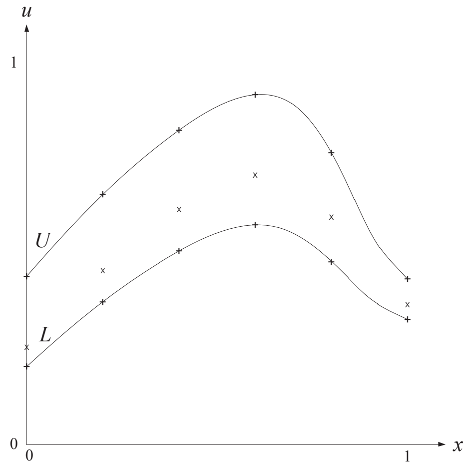

Figure 3. The RESA with 6 vertical slices (degree of discretisation 0.2). As collapsing proceeds from left to right, the RESA values (marked by crosses) approach the midpoint of L and U from below

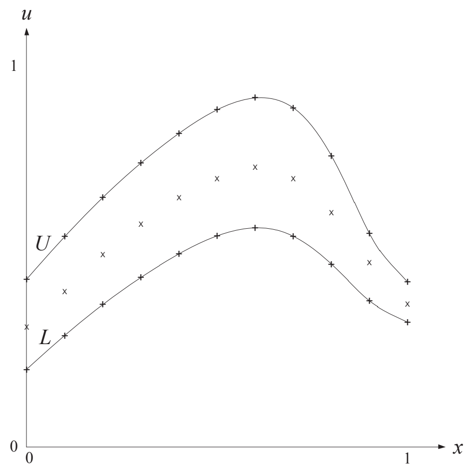

Figure 4. The RESA with 11 vertical slices (degree of discretisation 0.1). As collapsing proceeds from left to right, the RESA values (marked by crosses) approach the midpoint of L and U from below

5 The Nie-Tan Method

Nie and Tan [9] describe a computationally simple, efficient, approximate type-reduction method for interval sets, which involves taking the mean of the lower and upper membership functions of the interval set, so creating a type-1 fuzzy set. Symbolically, $\mu_N(x_i) = \frac{1}{2}(\mu_L(x_i) + \mu_U(x_i))$, where N is the resultant type-1 set, termed the *Nie-Tab Set (NTS)*. Figures 5 and 6 depict two NTSs.

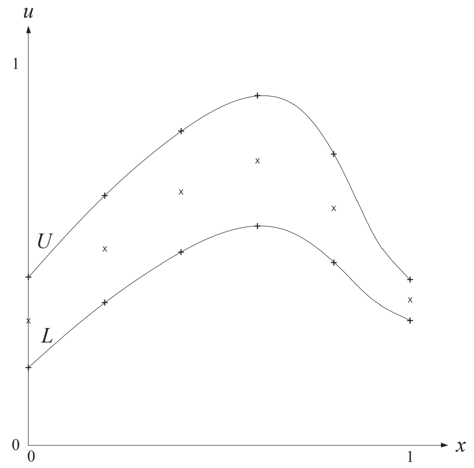

Figure 5. The NTS with 6 vertical slices (degree of discretisation 0.2). The NTS values (marked by crosses) are at the midpoint of L and U

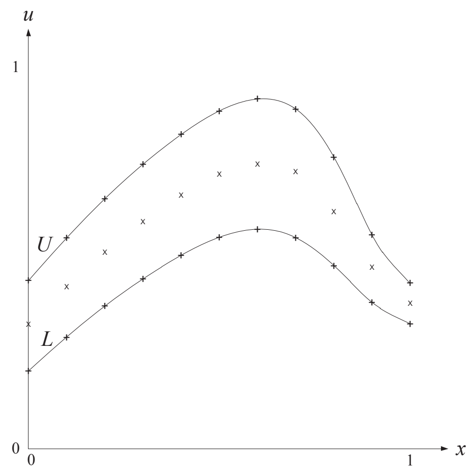

Figure 6. The NTS with 11 vertical slices (degree of discretisation 0.1). The NTS values (marked by crosses) are at the midpoint of L and U

Comparisons have been made in relation to speed and accuracy for the GCCD and Nie-Tan Method using the extremely slow exhaustive method [4] as a benchmark for accuracy. However the concern of this paper is not in making comparisons but rather on the continuous forms of the type-1 sets resulting from the three type-reduction strategies.

6 What Happens as Domain Discretisation Becomes Finer?

Experimental evidence [4] strongly suggested that as the domain discretisation is made finer, the defuzzified values of both the RESA and the NTS approach the defuzzified value of the TRS. The remainder of this paper is concerned with further experiments which were conducted to investigate this phenomenon.

6.1 Experimental Methodology

Three asymmetric test sets were formed with their primary domains and secondary domains scaled from 0 to 1. Each set was created in six versions, reflecting different degrees of discretisation of the domain, the coarsest employing 3 slices, the finest 21 slices[2].

Asymmetric Gaussian Test Set I This test set (Figure 7) consists of Gaussian lower and upper membership functions placed in such a way as to give an FOU lacking in symmetry. There is no way of knowing its defuzzified value other than by exhaustive defuzzification (Subsection 3).

Asymmetric Gaussian Test Set II This set (Figure 8) is constructed similarly to the Asymmetric Gaussian Test Set I; its defuzzified value is only revealed by exhaustive defuzzification.

Asymmetric Gaussian Test Set III This set (Figure 9) is also constructed along the same lines as the Asymmetric Gaussian Test Set I. As with test sets I and II, its defuzzified value may only be determined by exhaustive defuzzification.

[2]For the exhaustive method using more than 21 slices generates so many embedded sets that the computation breaks down.

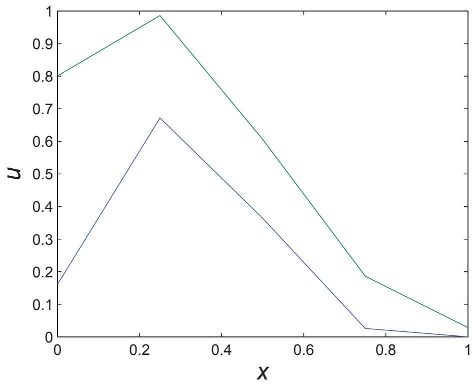

Figure 7. Asymmetric Test Set I, discretised into 5 vertical slices (degree of discretisation 0.25)

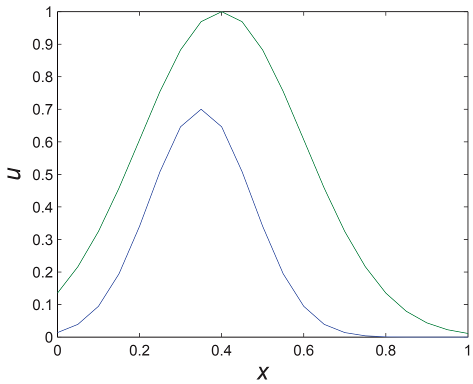

Figure 8. Asymmetric Test Set II, discretised into 21 vertical slices (degree of discretisation 0.05)

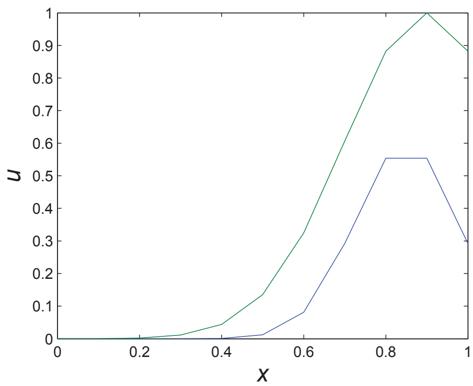

Figure 9. Asymmetric Test Set III, discretised into 11 vertical slices (degree of discretisation 0.1)

Each version of each test set was defuzzified using Matlab^{TM} implementations of the exhaustive, collapsing[3] and Nie-Tan methods, and the defuzzified values recorded. In order to exhibit the convergence trends, it was necessary to record results to a much greater degree of accuracy than would be appropriate within a practical application. The code for the tests may be accessed on [1].

6.2 Results and Conclusions

The results of the test runs are shown in Tables 1 to 3. Figures 10 to 12 chart the number of slices against the defuzzified value for each method for the three test sets.

A clear pattern of convergence for the three defuzzification methods is revealed. For test sets I and III the convergence starts from the outset i.e. from 3 slices, but for test set II it starts at 5 slices. Specifically, this set of experiments shows that:

1. As discretisation becomes finer, the exhaustive defuzzified value converges.
2. As discretisation becomes finer, the collapsing defuzzified value converges.
3. As discretisation becomes finer, the Nie-Tan defuzzified value converges.
4. All three methods' defuzzified values converge to the same value as discretisation becomes finer[4].
5. In every test run the collapsing defuzzified value is closer to the exhaustive defuzzified value than the Nie-Tan defuzzified value demonstrating that the GCCD is more accurate than the Nie-Tan Method.

It has been demonstrated [4] that in the continuous case the RESA and NTS are identical: The Nie-Tan method computes $\mu_N(x_i) = \frac{1}{2}(\mu_L(x_i) + \mu_U(x_i))$. As the degree of discretisation becomes finer, $||L||$ in the collapsing formula (Equation 12) tends to infinity, making the expression $\|L\| + \sum_{j=1}^{j=i-1} r_j$ also tend to infinity. r_i therefore increases, with $\frac{b_i}{2}$ as its upper bound. Thus in the continuous case the collapsing defuzzifier computes $\mu_R(x_i) = \mu_L(x_i) +$

[3]The collapsing variant employed for the tests was the *outward right-left*, which has been shown experimentally to be the most accurate form of the collapsing algorithm [3].

[4]Indeed the convergence of the three methods is so marked that for finer discretisations the symbols in Charts 10 to 12 tend to obscure one another.

Deg. of Discretisation	No. of Vertical Slices	Exhaustive Defuzzified Value	DV Collapsing Method	Error Collapsing Method	DV Nie-Tan Method	Error Nie-Tan Method
0.5	3	0.2899142309	0.2887656256	-0.0011486053	0.2621675145	-0.0277467164
0.25	5	0.2906756945	0.2901943927	-0.0004813018	0.2838845979	-0.0067910966
0.125	9	0.3043413255	0.3041596296	-0.0001816959	0.3017531920	-0.0025881335
0.1	11	0.3074987724	0.3073656467	-0.0001331257	0.3055512196	-0.0019475528
0.0625	17	0.3125118626	0.3124408475	-0.0000710151	0.3114135349	-0.0010983277
0.05	21	0.3142610070	0.3142075422	-0.0000534648	0.3134149648	-0.0008460422

Table 1. Defuzzified Values and Errors Obtained for Asymmetric Gaussian Test Set I

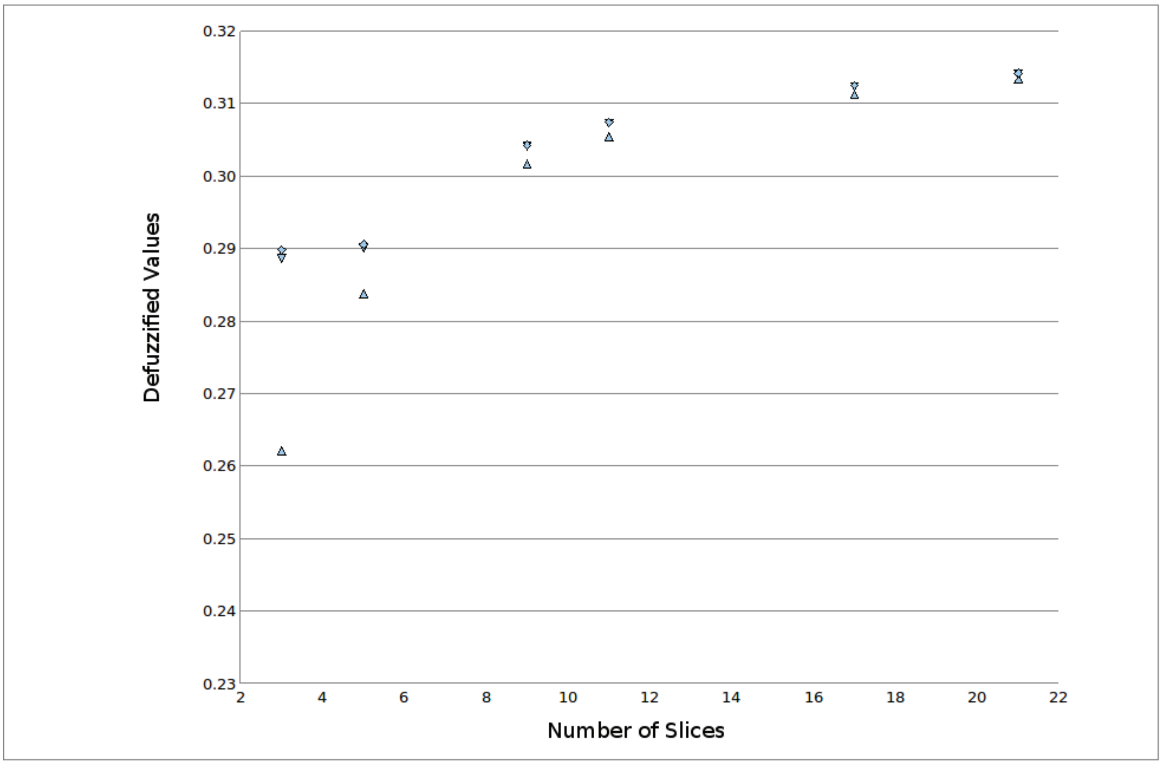

Figure 10. Relationship between the number of slices and the defuzzified value for Test Set I. Key: *diamonds* signify the exhaustive value, *triangles down* the collapsing value, and *triangles up* the value obtained by the Nie-Tan method

Deg. of Discretisation	No. of Vertical Slices	Exhaustive Defuzzified Value	DV Collapsing Method	Error Collapsing Method	DV Nie-Tan Method	Error Nie-Tan Method
0.5	3	0.4463569414	0.4448341702	-0.0015227712	0.4500873236	0.0037303822
0.25	5	0.3801544893	0.3790081321	-0.0011463572	0.3849969594	0.0048424701
0.125	9	0.3858091233	0.3850411480	-0.0007679753	0.3881115147	0.0023023914
0.1	11	0.3869140095	0.3862692345	-0.0006447750	0.3887383533	0.0018243438
0.0625	17	0.3886718708	0.3882407109	-0.0004311599	0.3897953470	0.0011234762
0.05	21	0.3892851035	0.3889328008	-0.0003523027	0.3901792082	0.0008941050

Table 2. Defuzzified Values and Errors Obtained for Asymmetric Gaussian Test Set II

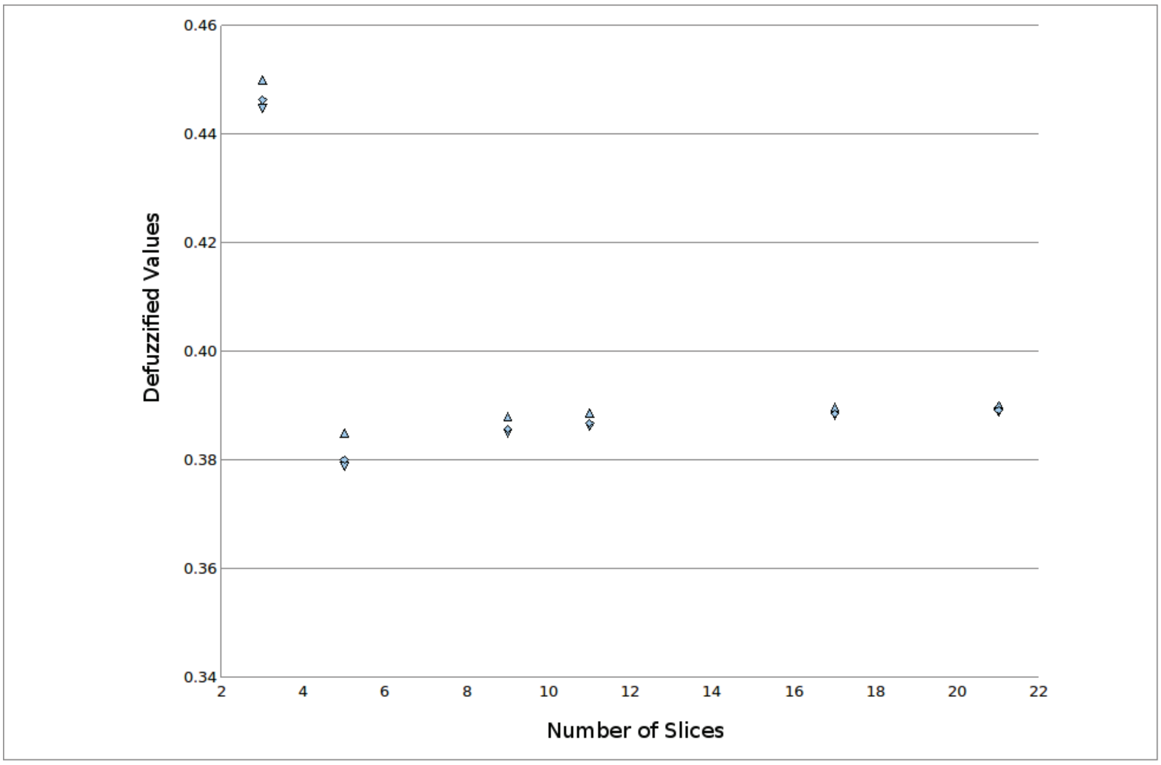

Figure 11. Relationship between the number of slices and the defuzzified value for Test Set I. Key: *diamonds* signify the exhaustive value, *triangles down* the collapsing value, and *triangles up* the value obtained by the Nie-Tan method

Deg. of Discretisation	No. of Vertical Slices	Exhaustive Defuzzified Value	DV Collapsing Method	Error Collapsing Method	DV Nie-Tan Method	Error Nie-Tan Method
0.5	3	0.9372015952	0.9387776322	0.0015760370	0.9442741258	0.0070725306
0.25	5	0.8448008416	0.8456145928	0.0008137512	0.8509954859	0.0061946443
0.125	9	0.8292260864	0.8296816205	0.0004555341	0.8310512070	0.0018251206
0.1	11	0.8251133032	0.8254723401	0.0003590369	0.8264315382	0.0013182350
0.0625	17	0.8184951716	0.8187130364	0.0002178648	0.8191875153	0.0006923437
0.05	21	0.8161608127	0.8163331340	0.0001723213	0.8166788889	0.0005180762

Table 3. Defuzzified Values and Errors Obtained for Asymmetric Gaussian Test Set III

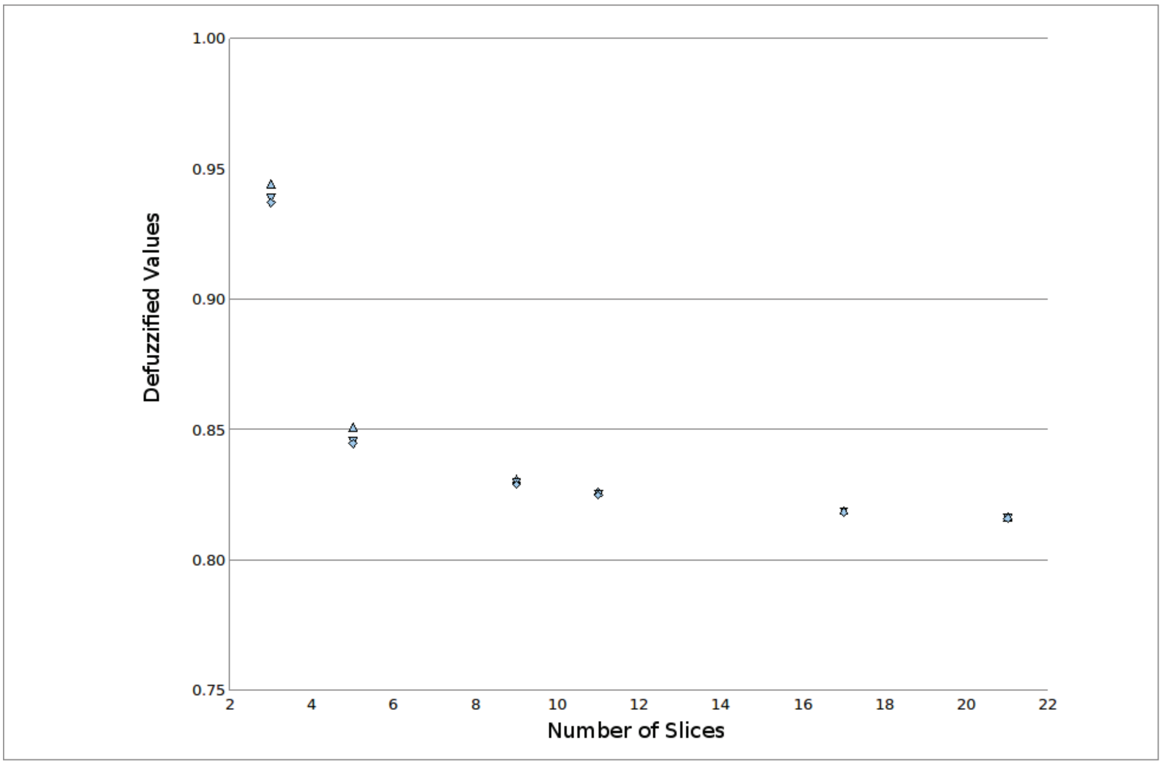

Figure 12. Relationship between the number of slices and the defuzzified value for Test Set III. Key: *diamonds* signify the exhaustive value, *triangles down* the collapsing value, and *triangles up* the value obtained by the Nie-Tan method

$\frac{1}{2}(\mu_U(x_i) - \mu_L(x_i)) = \mu_L(x_i) + \frac{1}{2}\mu_U(x_i) - \frac{1}{2}\mu_L(x_i) = \frac{1}{2}(\mu_L(x_i) + \mu_U(x_i))$. Therefore in the continuous case the collapsing and Nie-Tan methods are equivalent.

The experiments show that though the continuous TRS is distinct from the continuous RESA/continuous NTS, the three type-1 sets all have the same defuzzified value. In conclusion,

- the continuous RESA is identical with the continuous NTS, and
- the continuous TRS, the continuous RESA and the continuous NTS all produce the same defuzzified value.

7 Further Work

1. We would like to provide a mathematical proof that the continuous NTS defuzzifies to the same value as the continuous TRS.
2. We believe the continuous RESA to be the RES [4]. We have already shown that the continuous RESA is the same as the continuous NTS. To prove this conjecture, it would be sufficient to prove that the continuous NTS has the same defuzzified value as the continuous TRS.
3. *Continuous* fuzzy inferencing would be an interesting topic for future research.

References

[1] Sarah Greenfield. Matlab code for interval type-2 defuzzification, February 2011. http://www.tech.dmu.ac.uk/~sarahg/.

[2] Sarah Greenfield, Francisco Chiclana, Simon Coupland, and Robert I. John. The collapsing method of defuzzification for discretised interval type-2 fuzzy sets. *Information Sciences: Special Section on Higher Order Fuzzy Sets*, 179(13):2055–2069, June 2008.

[3] Sarah Greenfield, Francisco Chiclana, and Robert I. John. The collapsing method: Does the direction of collapse affect accuracy? In *Proceedings of IFSA-EUSFLAT 2009*, pages 980–985, Lisbon, Portugal, July 2009.

[4] Sarah Greenfield, Francisco Chiclana, and Robert I. John. Type-reduction of the discretised interval type-2 fuzzy set. In *Proceedings of FUZZ-IEEE 2009*, pages 738–743, Jeju Island, Korea, August 2009.

[5] Nilesh N. Karnik and Jerry M. Mendel. Centroid of a type-2 fuzzy set. *Information Sciences*, 132:195–220, 2001.

[6] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic*. Prentice-Hall P T R, 1995.

[7] George J. Klir and Tina A. Folger. *Fuzzy sets, Uncertainty, and Information*. Prentice-Hall International, 1992.

[8] Jerry M. Mendel. *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Prentice-Hall PTR, 2001.

[9] Maowen Nie and Woei Wan Tan. Towards an efficient type-reduction method for interval type-2 fuzzy logic systems. In *Proceedings of FUZZ-IEEE 2008*, pages 1425–1432, Hong Kong, June 2008.

[10] Lotfi Zadeh. Fuzzy Sets. In *Information and Control 8, pp. 338–353*. Elsevier Inc., 1965.

[11] Lotfi A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences*, 8:199–249, 1975.

[12] Lotfi A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning – II. *Information Sciences*, 8:301–357, 1975.

[13] Lotfi A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning – III. *Information Sciences*, 9:43–80, 1975.