

## COMPACT GENERALIZED WEIGHTED COMPOSITION OPERATORS ON THE BERGMAN SPACE

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**Abstract.** We characterize the compactness of the generalized weighted composition operators acting on the Bergman space.

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### 1. INTRODUCTION

Let  $\mathbb{D}$  be the unit disk in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  be the class of functions analytic on  $\mathbb{D}$ . Let  $dA(z)$  be the normalized area measure on  $\mathbb{D}$  so that the area of  $\mathbb{D}$  is 1. For  $\alpha > -1$ , an  $f \in H(\mathbb{D})$  is said to belong to the weighted Bergman space, denoted by  $A_\alpha^2 = A_\alpha^2(\mathbb{D})$ , if (see [21])

$$\|f\|_{A_\alpha^2} = \left[ (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) \right]^{\frac{1}{2}} < \infty.$$

When  $\alpha = 0$ , we simply denote  $A_0^2$  by  $A^2$  and the norm  $\|\cdot\|_{A_0^2}$  by  $\|\cdot\|$ , respectively.  $A^2$  is a Hilbert space with the following inner product:

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z).$$

For  $w \in \mathbb{D}$ , the reproducing kernel function  $K_w$  in  $A^2$  is given by (see [21])

$$K_w(z) = \frac{1}{(1 - \bar{w}z)^2}.$$

It is easy to see that  $\|K_w\| = \frac{1}{1 - |w|^2}$ .

An  $f \in H(\mathbb{D})$  is said to belong to the Dirichlet space, denoted by  $\mathcal{D}$ , if

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . The composition operator  $C_\varphi$  is defined by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$ . The weighted composition operator, denoted by  $uC_\varphi$ , is defined as follows:

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D}.$$

We denote the set of nonnegative integers by  $\mathbb{N}_0$ . Let  $n \in \mathbb{N}_0$  and  $f^{(n)}$  denote the  $n$ -th derivative of  $f$ . A linear operator  $D_{\varphi,u}^n$  is defined by

$$(D_{\varphi,u}^n f)(z) = u(z)f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}).$$

When  $n = 0$  and  $u(z) = 1$ , then  $D_{\varphi,u}^n$  is just the composition operator  $C_\varphi$ . See [1,21] for more information about the theory of composition operators. If  $n = 0$ , then  $D_{\varphi,u}^n$  is just the weighted composition operator. When  $u(z) = 1$ ,  $D_{\varphi,u}^n = C_\varphi D^n$ . See, for example, [3,6–9,11,14,15,20] for the study of the operator  $C_\varphi D^n$ . See [5,10,16–19,22,23] and the references therein for the study of the operator  $D_{\varphi,u}^n$ .

In [12], MacCluer and Shapiro showed that  $C_\varphi : A^2 \rightarrow A^2$  is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

In [13], under the condition  $u \in H^\infty$ , Moorhouse showed that  $uC_\varphi : A^2 \rightarrow A^2$  is compact if and only if

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0. \quad (1.1)$$

Karthikeyan in [4] also gave some conditions for  $uC_\varphi : A^2 \rightarrow A^2$  to be compact if and only if (1.1) holds.

Motivated by these results, in this work we study the generalized weighted composition operator  $D_{\varphi,u}^n : A^2 \rightarrow A^2$ . Under some conditions, we show that  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact if and only if

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

Throughout this paper, we say that  $A \lesssim B$  if there exists a constant  $C$  such that  $A \leq CB$ . The symbol  $A \approx B$  means that  $A \lesssim B \lesssim A$ .

2. MAIN RESULTS

In this section, we state and prove our main results in this paper. Hence, we first state some lemmas which will be used in the proofs of the main results.

**Lemma 2.1** ([21]). *For every  $f \in A^2$ ,*

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) \leq 2[|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^2 dA(z)].$$

**Lemma 2.2** ([21]). *For every  $f \in A^2$ ,*

$$\lim_{|z| \rightarrow 1} |f(z)|(1 - |z|^2) = 0.$$

**Lemma 2.3.** *Let  $n \in \mathbb{N}_0$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$ . If  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is bounded, then*

$$(D_{\varphi,u}^n)^* K_w = \overline{u(w)} K_{\varphi(w)}^{(n)}.$$

*Proof.* Let  $f$  be in  $A^2$ . Since  $f^{(n)}(w) = \langle f, K_w^{(n)} \rangle$  (see Theorem 2.16 of [1]), we have

$$\begin{aligned} \langle f, (D_{\varphi,u}^n)^* K_w \rangle &= \langle D_{\varphi,u}^n f, K_w \rangle = \langle u(f^{(n)} \circ \varphi), K_w \rangle \\ &= u(w) f^{(n)}(\varphi(w)) = u(w) \langle f, K_{\varphi(w)}^{(n)} \rangle = \langle f, \overline{u(w)} K_{\varphi(w)}^{(n)} \rangle. \end{aligned}$$

Since  $\langle f, (D_{\varphi,u}^n)^* K_w \rangle = \langle f, \overline{u(w)} K_{\varphi(w)}^{(n)} \rangle$  for all  $f$ , we get the desired result. □

**Lemma 2.4** ([2]). *Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . The operator  $C_\varphi$  is Hilbert-Schmidt on the Dirichlet space  $\mathcal{D}$  if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty.$$

After a calculation, we can obtain the following result (see, e.g., [22]).

**Lemma 2.5.** *Let  $f \in A^2$ . Then*

$$|f^{(n)}(w)| \lesssim \frac{\|f\|}{(1 - |w|)^{n+1}}.$$

To study the compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

**Lemma 2.6.** *Let  $n \in \mathbb{N}_0$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$ . If  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is bounded, then  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact if and only if whenever  $\{f_j\}$  is bounded in  $A^2$  and  $f_j \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ ,  $\lim_{j \rightarrow \infty} \|D_{\varphi,u}^n f_j\| = 0$ .*

Now we are in a position to state and prove the main results in this paper.

**Theorem 2.7.** *Let  $n \in \mathbb{N}_0$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in A^2$ . If  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact, then*

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

*Proof.* For every  $z \in \mathbb{D}$ , let  $f_z = \frac{K_z}{\|K_z\|}$ . Then

$$\|(D_{\varphi,u}^n)^* f_z\| = \|(D_{\varphi,u}^n)^* \frac{K_z}{\|K_z\|}\| = |u(z)| \frac{\|K_{\varphi(z)}^{(n)}\|}{\|K_z\|} \approx |\varphi(z)|^n |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}}.$$

Here we used the fact that

$$\|K_w^{(n)}\| = \left[ \int_{\mathbb{D}} |K_w^{(n)}(z)|^2 dA(z) \right]^{\frac{1}{2}} \approx |w|^n \frac{1}{(1 - |w|^2)^{n+1}}.$$

Since  $D_{\varphi,u}^n$  is compact,  $(D_{\varphi,u}^n)^*$  is also compact. By Theorem 2.17 of [1], we know that  $\frac{K_z}{\|K_z\|}$  tend to 0 weakly as  $|z|$  tends to 1. Therefore,  $\|(D_{\varphi,u}^n)^* f_z\| \rightarrow 0$  as  $|z| \rightarrow 1$ . Hence

$$\lim_{|z| \rightarrow 1} |\varphi(z)|^n |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

Let  $\delta > 0$ . If  $|\varphi(z)| \leq \delta$ , by the fact that  $u \in A^2$  and Lemma 2.2 we have

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} \leq \lim_{|z| \rightarrow 1} \frac{|u(z)|(1 - |z|^2)}{(1 - \delta^2)^{n+1}} = 0.$$

If  $|\varphi(z)| > \delta$ , then

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} \lesssim \lim_{|z| \rightarrow 1} |\varphi(z)|^n |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

Thus we immediately get the desired result. □

**Theorem 2.8.** *Let  $n \in \mathbb{N}_0$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$  such that  $D_{\varphi,u}^n$  is bounded on  $A^2$ . Suppose that  $C_\varphi$  is Hilbert-Schmidt on the Dirichlet space  $\mathcal{D}$  and*

$$\lim_{|\varphi(z)| \rightarrow 1} |u'(z)|(1 - |z|^2)^{1-n} = 0.$$

*Then  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact if and only if*

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

*Proof.* It is easy to see that  $u \in A^2$ , when  $D_{\varphi,u}^n$  is bounded on  $A^2$ . Hence from Theorem 2.7 we only need to prove the sufficiency.

Suppose  $\{f_j\}$  is a bounded sequence which converges to zero uniformly on compact subsets of  $\mathbb{D}$ . Without loss of generality assume that  $\|f_j\| \leq 1$ . We need to prove  $\|D_{\varphi,u}^n f_j\| \rightarrow 0$  as  $j \rightarrow \infty$ . Since

$$\begin{aligned} |[u(f_j^{(n)} \circ \varphi)]'(z)|^2 &= |u'(z)f_j^{(n)}(\varphi(z)) + u(z)f_j^{(n+1)}(\varphi(z))\varphi'(z)|^2 \\ &\leq 2|u'(z)|^2|f_j^{(n)}(\varphi(z))|^2 + 2|u(z)|^2|f_j^{(n+1)}(\varphi(z))|^2|\varphi'(z)|^2, \end{aligned}$$

by Lemma 2.1 we get

$$\begin{aligned} \|D_{\varphi,u}^n f_j\|^2 &\leq 2[|u(0)f_j^{(n)}(\varphi(0))|^2 + \int_{\mathbb{D}} |[u(f_j^{(n)} \circ \varphi)]'(z)|^2(1 - |z|^2)^2 dA(z)] \\ &\leq 2|u(0)|^2|f_j^{(n)}(\varphi(0))|^2 + 4 \int_{\mathbb{D}} |u'(z)|^2|f_j^{(n)}(\varphi(z))|^2(1 - |z|^2)^2 dA(z) \quad (2.1) \\ &\quad + 4 \int_{\mathbb{D}} |u(z)|^2|f_j^{(n+1)}(\varphi(z))|^2|\varphi'(z)|^2(1 - |z|^2)^2 dA(z). \end{aligned}$$

For  $f \in A^2$ , we know that  $f^{(n)} \in A_{2n}^2$  by [21]. Since  $C_\varphi$  is bounded on  $A_{2n}^2$  for any  $\varphi$ , we get

$$M_1 := \int_{\mathbb{D}} |f_j^{(n)}(\varphi(z))|^2(1 - |z|^2)^{2n} dA(z) < \infty. \quad (2.2)$$

Let  $\varepsilon > 0$  be given. Since

$$\lim_{|\varphi(z)| \rightarrow 1} |u'(z)|(1 - |z|^2)^{1-n} = 0,$$

there exists  $r_1 \in (0, 1)$  such that

$$\frac{|u'(z)|^2}{(1 - |z|^2)^{2n-2}} < \frac{\varepsilon}{M_1} \quad (2.3)$$

for  $r_1 < |\varphi(z)| < 1$ . Since  $C_\varphi$  is Hilbert-Schmidt on the Dirichlet space  $\mathcal{D}$ , by Lemma 2.4,

$$M := \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty. \quad (2.4)$$

Also, since

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0,$$

there exists  $r_2 \in (0, 1)$  such that

$$|u(z)|^2 \frac{(1 - |z|^2)^2}{(1 - |\varphi(z)|^2)^{2n+2}} < \frac{\varepsilon}{M} \quad (2.5)$$

for  $r_2 < |\varphi(z)| < 1$ .

Let  $r = \max\{r_1, r_2\}$ . Then by (2.1),

$$\|D_{\varphi, u}^n f_j\|^2 \leq 2I_1 + 4I_2 + 4I_3 + 4I_4 + 4I_5,$$

where

$$\begin{aligned} I_1 &= |u(0)|^2 |f_j^{(n)}(\varphi(0))|^2, \\ I_2 &= \int_{|\varphi(z)| \leq r} |u'(z)|^2 |f_j^{(n)}(\varphi(z))|^2 (1 - |z|^2)^2 dA(z), \\ I_3 &= \int_{|\varphi(z)| > r} |u'(z)|^2 |f_j^{(n)}(\varphi(z))|^2 (1 - |z|^2)^2 dA(z), \\ I_4 &= \int_{|\varphi(z)| \leq r} |u(z)|^2 |f_j^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |z|^2)^2 dA(z), \\ I_5 &= \int_{|\varphi(z)| > r} |u(z)|^2 |f_j^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |z|^2)^2 dA(z). \end{aligned}$$

Since  $\{\varphi(0)\}$  is a compact set, there exists a positive integer  $j_1$  such that

$$I_1 = |u(0)|^2 |f_j^{(n)}(\varphi(0))|^2 < \varepsilon$$

for  $j \geq j_1$ . Since  $D_{\varphi, u}^n$  is bounded on  $A^2$ , applying the bounded operator  $D_{\varphi, u}^n$  to  $z^n$  and  $z^{n+1}$  in  $A^2$ , we easily get that  $u, u\varphi \in A^2$ . Thus

$$M_2 := \int_{|\varphi(z)| \leq r} |u'(z)|^2 (1 - |z|^2)^2 dA(z) < \infty$$

and

$$M_3 := \int_{|\varphi(z)| \leq r} |u(z)|^2 |\varphi'(z)|^2 (1 - |z|^2)^2 dA(z) < \infty.$$

Since  $\{f_j^{(n)}\}$  and  $\{f_j^{(n+1)}\}$  converge to zero uniformly on compact subsets of  $\mathbb{D}$  as  $j \rightarrow \infty$ , there exist positive integers  $j_2$  and  $j_3$  such that for every  $|\varphi(z)| \leq r$ ,

$$|f_j^{(n)}(\varphi(z))|^2 < \frac{\varepsilon}{M_2}$$

for  $j \geq j_2$  and

$$|f_j^{(n+1)}(\varphi(z))|^2 < \frac{\varepsilon}{M_3}$$

for  $j \geq j_3$ . Let  $m = \max\{j_1, j_2, j_3\}$ . Then for  $j \geq m$ ,

$$I_1 < \varepsilon, \quad I_2 \leq \frac{\varepsilon}{M_2} M_2 = \varepsilon, \quad I_4 \leq \frac{\varepsilon}{M_3} M_3 = \varepsilon.$$

From inequalities (2.2) and (2.3), we conclude that  $I_3 < \varepsilon$ .

By Lemma 2.5, (2.4) and (2.5) we have

$$I_5 \lesssim \frac{\varepsilon}{M} \int_{|\varphi(z)| > r} (1 - |\varphi(z)|)^{2n+2} \frac{\|f_j\|^2}{(1 - |\varphi(z)|)^{2n+4}} |\varphi'(z)|^2 dA(z) \lesssim \frac{\varepsilon}{M} M = \varepsilon.$$

From the above estimate we obtain that  $\|D_{\varphi,u}^n f_j\| \lesssim \varepsilon$  when  $j \geq m$ . Since  $\varepsilon$  is an arbitrary positive number, we conclude that  $\|D_{\varphi,u}^n f_j\| \rightarrow 0$  as  $j \rightarrow \infty$ . Therefore  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact. The proof is complete.  $\square$

**Theorem 2.9.** *Let  $n \in \mathbb{N}_0$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$  such that  $D_{\varphi,u}^n$  is bounded on  $A^2$ . Suppose that  $C_\varphi$  is Hilbert-Schmidt on the Dirichlet space  $\mathcal{D}$ ,  $C_\varphi$  is compact on  $A^2$  and*

$$M_4 := \int_{\mathbb{D}} \frac{|u'(z)|^2}{(1 - |\varphi(z)|^2)^{2n}} dA(z) < \infty. \tag{2.6}$$

Then  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact if and only if

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

*Proof.* We also only need to prove the sufficiency.

Suppose  $\{f_j\}$  is a bounded sequence which converges to zero uniformly on compact subsets of  $\mathbb{D}$ . Without loss of generality assume that  $\|f_j\| \leq 1$ . We need to prove that  $\|D_{\varphi,u}^n f_j\| \rightarrow 0$  as  $j \rightarrow \infty$ .

Let  $\varepsilon > 0$  be given. Since  $C_\varphi$  is compact, we have (see [21])

$$\lim_{|z| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0,$$

which implies that

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Then there exists  $\rho_1 \in (0, 1)$  such that

$$\frac{(1 - |z|^2)^2}{(1 - |\varphi(z)|^2)^2} < \frac{\varepsilon}{M_4} \tag{2.7}$$

for  $\rho_1 < |\varphi(z)| < 1$ . Since  $C_\varphi$  is Hilbert-Schmidt on the Dirichlet space  $\mathcal{D}$ , (2.4) holds. Also, since

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0,$$

there exists  $\rho_2 \in (0, 1)$  such that

$$|u(z)|^2 \frac{(1 - |z|^2)^2}{(1 - |\varphi(z)|^2)^{2n+2}} < \frac{\varepsilon}{M}$$

for  $\rho_2 < |\varphi(z)| < 1$ .

Let  $\rho = \max\{\rho_1, \rho_2\}$ . Then

$$\|D_{\varphi,u}^n f_j\|^2 \leq 2J_1 + 4J_2 + 4J_3 + 4J_4 + 4J_5,$$

where

$$\begin{aligned} J_1 &= |u(0)|^2 |f_j^{(n)}(\varphi(0))|^2, \\ J_2 &= \int_{|\varphi(z)| \leq \rho} |u'(z)|^2 |f_j^{(n)}(\varphi(z))|^2 (1 - |z|^2)^2 dA(z), \\ J_3 &= \int_{|\varphi(z)| > \rho} |u'(z)|^2 |f_j^{(n)}(\varphi(z))|^2 (1 - |z|^2)^2 dA(z), \\ J_4 &= \int_{|\varphi(z)| \leq \rho} |u(z)|^2 |f_j^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |z|^2)^2 dA(z) \end{aligned}$$

and

$$J_5 = \int_{|\varphi(z)| > \rho} |u(z)|^2 |f_j^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |z|^2)^2 dA(z).$$

Similar to the proof of Theorem 2.8, we see that there exists a  $k$  such that

$$J_1 < \varepsilon, \quad J_2 \leq \varepsilon, \quad J_4 \leq \varepsilon, \quad J_5 \lesssim \varepsilon.$$

when  $j \geq k$ .

By Lemma 2.5, (2.6) and (2.7), we have

$$J_3 \lesssim \int_{|\varphi(z)| > \rho} |u'(z)|^2 \frac{\|f_j\|^2}{(1 - |\varphi(z)|^2)^{2n+2}} (1 - |z|^2)^2 dA(z) < \frac{\varepsilon}{M_4} M_4 = \varepsilon.$$

From the above estimate we can say that  $\|D_{\varphi,u}^n f_j\| \lesssim \varepsilon$  when  $j$  is large enough. Since  $\varepsilon$  is an arbitrary positive number, we conclude that  $\|D_{\varphi,u}^n f_j\| \rightarrow 0$  as  $j \rightarrow \infty$ . Therefore  $D_{\varphi,u}^n$  is a compact operator on  $A^2$ .  $\square$

Similar to the proof of Theorem 2.9, we immediately get the following result.

**Theorem 2.10.** *Let  $n \in \mathbb{N}_0$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $u \in H(\mathbb{D})$  such that  $D_{\varphi,u}^n$  is bounded on  $A^2$ . Suppose that  $C_\varphi$  is Hilbert-Schmidt on the Dirichlet space  $\mathcal{D}$  and*

$$\lim_{t \rightarrow 1} \int_{|\varphi(z)| > t} \frac{|u'(z)|^2 (1 - |z|^2)^2}{(1 - |\varphi(z)|^2)^{2n+2}} dA(z) = 0.$$

Then  $D_{\varphi,u}^n : A^2 \rightarrow A^2$  is compact if and only if

$$\lim_{|z| \rightarrow 1} |u(z)| \frac{1 - |z|^2}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$



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