

Eigenvalues assignment in uncontrollable linear systems

 Tadeusz KACZOREK *

Białystok University of Technology, ul. Wiejska 45A, 15-351 Białystok, Poland

Abstract. It is shown that in an uncontrollable linear system $\dot{x} = Ax + Bu$ it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedbacks $u = Kx$, $K \in \mathfrak{R}^{n \times n}$ if $\text{rank}[A \ B] = n$. The design procedure consists of two steps. In step 1, a nonsingular matrix $M \in \mathfrak{R}^{n \times n}$ is chosen so that the pair (MA, MB) is controllable. In step 2, the feedback matrix K is chosen so that the closed-loop matrix $A_c = A - BK$ has the desired eigenvalues. The procedure is illustrated by a simple example.

Key words: controllability; eigenvalues; assignment; linear system; feedback; procedure component.

1. INTRODUCTION

The concepts of controllability and observability introduced by Kalman [1,2] have been the basic notions of the modern control theory. It is well-known that if the linear system is controllable then, by the use of state feedback, it is possible to modify the dynamical properties of the closed-loop systems [1–12]. If the linear system is observable, then it is possible to design an observer which reconstructs the state vector of the system [1–12]. Descriptor systems of integer and fractional order have been analyzed in [6, 11]. The stabilization of positive descriptor fractional linear systems with two different fractional orders by decentralized controller has been investigated in [11].

In this paper, it will be shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedback if $\text{rank}[A \ B] = n$. In Section 2 it will be shown that if $\text{rank}[A \ B] = n$, then there exists a nonsingular matrix $M \in \mathfrak{R}^{n \times n}$ such that the pair (MA, MB) is controllable. Two procedures for the computation of the matrix $M \in \mathfrak{R}^{n \times n}$ will be proposed and illustrated by simple numerical examples in Section 3. Concluding remarks will be given in Section 4.

The following notation will be used: \mathfrak{R} – the set of real numbers, $\mathfrak{R}^{n \times m}$ – the set of $n \times m$ real matrices, I_n – the $n \times n$ identity matrix.

2. CONTROLLABILITY OF LINEAR SYSTEMS

Consider the linear continuous-time system

$$\dot{x} = Ax + Bu, \quad (1a)$$

$$y = Cx, \quad (1b)$$

where $x = x(t) \in \mathfrak{R}^n$, $u = u(t) \in \mathfrak{R}^m$, $y = y(t) \in \mathfrak{R}^p$ are the state, input, and output vectors and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $C \in \mathfrak{R}^{p \times n}$.

*e-mail: t.kaczorek@pb.edu.pl

Manuscript submitted 2022-06-21, revised 2022-06-21, initially accepted for publication 2022-08-03, published in December 2022.

Definition 1. [4,5,7,9,10,13] The system (1) (the pair (A, B)) is called controllable if there exists an input $u(t) \in \mathfrak{R}^m$, $t \in [0, t_f]$ which steers the state of the system from the initial state $x(0) \in \mathfrak{R}^n$ to the given final state $x_f = x(t_f)$.

Theorem 1. The system (1a) (the pair (A, B)) is controllable if and only if one of the following conditions is satisfied:

1. (Kalman condition)

$$\text{rank} [B \ AB \ \dots \ A^{n-1}B] = n, \quad (2a)$$

2. (Hautus condition)

$$\text{rank} [I_n s - A \ B] = n, \quad (2b)$$

for $s \in C$ (the field of complex numbers). In the proof of the main result of this section, the following theorem will be used.

Theorem 2. (Kronecker–Capelly theorem, [13]) The equation

$$Px = Q, \quad P \in \mathfrak{R}^{n \times p}, \quad Q \in \mathfrak{R}^{n \times q}, \quad n, p, q \geq 1 \quad (3)$$

has a solution $x \in \mathfrak{R}^{p \times q}$ if and only if

$$\text{rank} [P \ Q] = \text{rank}[P]. \quad (4)$$

Theorem 3. If the pair (A, B) is uncontrollable but satisfies the condition

$$\text{rank} [A \ B] = n, \quad A \in \mathfrak{R}^{n \times n}, \quad B \in \mathfrak{R}^{n \times m}, \quad (5)$$

then there exists a nonsingular matrix $M \in \mathfrak{R}^{n \times n}$ such that the pair

$$(\bar{A}, \bar{B}), \quad \bar{A} = MA, \quad \bar{B} = MB \quad (6)$$

is controllable.

Proof. From (6) we have

$$M[A \ B] = [\bar{A} \ \bar{B}], \quad (7)$$

where the pair (\bar{A}, \bar{B}) is controllable.

From Theorem 2 applied to equation (7), it follows that there exists a nonsingular matrix M satisfying (7) if the condition (5) holds. \square

To compute the desired matrix M the following procedures can be recommended.

Procedure 1. By choosing the controllable pair (\bar{A}, \bar{B}) and post-multiplying equation (7) by the transposed matrix $[A \ B]^T$, we obtain

$$M[AA^T \ BB^T] = \bar{A}A^T + \bar{B}B^T. \quad (8)$$

The matrix

$$[A \ B] \begin{bmatrix} A^T \\ B^T \end{bmatrix} = AA^T + BB^T \quad (9)$$

is nonsingular since $\text{rank}[A \ B] = n$. From (8) we have the desired matrix

$$M = (\bar{A}A^T + \bar{B}B^T) (AA^T + BB^T)^{-1}. \quad (10)$$

To find the desired matrix M the following procedure can be also applied.

Procedure 2. Choose a nonsingular matrix $M \in \mathfrak{R}^{n \times n}$ and compute the matrix $[\bar{A} \ \bar{B}]$. Check if the pair $[A \ B]^T$ is controllable if it is not the case then repeat the procedure for a new matrix M .

Example 1. Consider the uncontrollable system (1) with the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (11)$$

From (11) it follows that it is impossible to stabilize the system by state feedback $u = Kx$, $K = [k_1 \ k_2]$. The system with (11) satisfies condition (5) since

$$\text{rank} \begin{bmatrix} A & B \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} = 2 = n. \quad (12)$$

According to Procedure 1, we choose the controllable pair in the form

$$\bar{A} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (13)$$

Using (10) and (13) we obtain

$$\begin{aligned} M &= (\bar{A}A^T + \bar{B}B^T) (AA^T + BB^T)^{-1} \\ &= \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ &\quad \times \left[\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right]^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}. \end{aligned} \quad (14)$$

According to Procedure 2, we choose the matrix M for example in the form

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}. \quad (15)$$

In this case

$$\begin{aligned} \bar{A} &= MA = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \\ \bar{B} &= MB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \quad (16)$$

The pair (16) is controllable since

$$\text{rank} \begin{bmatrix} B & AB \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2 = n. \quad (17)$$

Note that if we choose

$$M = A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad (18)$$

then

$$\bar{A} = MA = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{B} = MB = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \quad (19)$$

and the pair (\bar{A}, \bar{B}) is uncontrollable.

The above considerations for the pair (A, B) can be extended to the pair (A, C) of the system (1).

Definition 2. [4, 5, 7, 9, 10, 13] The linear system (1) is called observable if knowing its input $u(t) \in \mathfrak{R}^m$ and its output $y(t) \in \mathfrak{R}^p$ for $t \in [0 \ t_f]$ it is possible find its unique initial condition $x(0) \in \mathfrak{R}^n$.

Theorem 4. If the pair (A, C) is unobservable but satisfies the condition

$$\text{rank} \begin{bmatrix} A \\ C \end{bmatrix} = n, \quad A \in \mathfrak{R}^{n \times n}, \quad C \in \mathfrak{R}^{p \times n}, \quad (20)$$

then there exists a nonsingular matrix $\bar{M} \in \mathfrak{R}^{n \times n}$ such that the pair

$$(\bar{A}, \bar{C}), \quad \bar{A} = A\bar{M}, \quad \bar{C} = C\bar{M} \quad (21)$$

is observable.

The proof is similar (dual) to the proof of Theorem 3.

3. STABILIZATION OF THE UNCONTROLLABLE LINEAR SYSTEMS BY STATE FEEDBACKS

Consider the linear system (1) with an uncontrollable pair (A, B) . We are looking for the state feedback matrix K such that the closed-loop matrix

$$\hat{A}_c = \bar{A} - \bar{B}K \quad (22)$$

has the desired eigenvalues (Fig. 1).

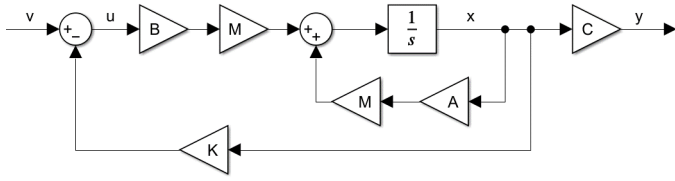


Fig. 1. Linear system with state feedback

First, we choose the matrix M such that the pair

$$\bar{A} = MA, \quad \bar{B} = MB \quad (23)$$

is controllable and next using one of the well-known approaches [3, 5, 6, 9, 10, 12] of the eigenvalues assignment we choose the matrix K such that the matrix \hat{A}_c has the desired eigenvalues.

To solve the problem the following procedure can be applied.

Procedure 3.

Step 1. Using the approach of Section 2 find the matrix M such that the pair (23) is controllable.

Step 2. Using one of the well-known approaches of the eigenvalues assignment find the matrix K such that the closed-loop matrix (22) has the desired eigenvalues.

Example 2. (Continuance of Example 1) For the uncontrollable system with the matrices (11) find the state feedback matrix $K = [k_1 \ k_2]$ such that the close-loop matrix (22) has the eigenvalues $s_1 = -2, s_2 = -3$.

Using the procedure we obtain the following:

Step 1. For the matrix M of the form (15), we have obtained the controllable pair

$$\bar{A} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (24)$$

Step 2. In this case, the close-loop matrix has the form

$$\begin{aligned} \hat{A}_c &= \bar{A} - \bar{B}K = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 - k_1 & -1 - k_2 \\ 3 - k_1 & -1 - k_2 \end{bmatrix} \end{aligned} \quad (25)$$

and

$$\begin{aligned} \det [I_2s - \hat{A}_c] &= \begin{vmatrix} s - 2 + k_1 & 1 + k_2 \\ -3 + k_1 & s + 1 + k_2 \end{vmatrix} \\ &= s^2 + (k_1 + k_2 - 1)s + k_2 + 1 \\ &= (s - s_1)(s - s_2) = s^2 - (s_1 + s_2)s + s_1s_2. \end{aligned} \quad (26)$$

From (26) for the desired eigenvalues $s_1 = -2, s_2 = -3$ we have $k_1 = 1, k_2 = 5$ and the desired state feedback matrix has the form $K = [k_1 \ k_2] = [1 \ 5]$.

4. CONCLUDING REMARKS

It has been shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedback if condition (5) is satisfied. It is also shown that if condition (5) is satisfied, then there exists a nonsingular matrix $M \in \mathcal{R}^{n \times n}$ such that the pair (MA, MB) is controllable (Theorem 3). Two procedures for the computation of the matrix $M \in \mathcal{R}^{n \times n}$ have been proposed and illustrated by a simple numerical example. This approach can be extended to linear discrete-time linear systems and fractional orders linear systems.

ACKNOWLEDGEMENTS

This work was carried out in the framework of work No. WZ/WE – IA/6/2020.

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