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Nonoverlapping domain decomposition for optimal control problems governed by semilinear models for gas flow in networks*

by

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Abstract: We consider optimal control problems for gas flow in pipeline networks. The equations of motion are taken to be represented by a first-order system of hyperbolic semilinear equations derived from the fully nonlinear isothermal Euler gas equations. We formulate an optimal control problem on a network and introduce a tailored time discretization thereof. In order to further reduce the complexity, we consider an instantaneous control strategy. The main part of the paper is concerned with a nonoverlapping domain decomposition of the optimal control problem on the graph into local problems on smaller sub-graphs—ultimately on single edges. We prove convergence of the domain decomposition method on networks and study the wellposedness of the corresponding time-discrete optimal control problems. The point of the paper is that we establish virtual control problems on the decomposed subgraphs such that the corresponding optimality systems are in fact equal to the systems obtained via the domain decomposition of the entire optimality system.

Keywords: optimal control, gas networks, Euler's equation, semilinear PDE, nonoverlapping domain decomposition

1. Introduction

We consider a semilinear hyperbolic system for gas flow in a network of pipes that is derived from the Euler equations for compressible fluids in cylindrical pipes. The overall goal is to control the flow of gas in an optimal way such that at the so-called entry nodes the gas is provided at a certain pressure and at a set of so-called exit nodes the pressure and flow conditions are realized.

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The control instruments in the system are valves and compressors which, in turn, are modeled as switching boundary conditions, followed by continuous profile controls. Indeed, the decision to open a valve is followed by a continuous opening of the valve, and, correspondingly, once a decision is made to close the valve, the valve actually closes continuously. A similar explanation holds for the action of compressors; see the mathematical description below. The control costs are taken to be tracking cost for the flow and the pressure plus a penalization of the control costs. The entire optimal control problem can be put into the framework of mixed-integer nonlinear optimal control for partial differential equations (MINOC-PDE)—an extension of finite-dimensional mixed-integer nonlinear programming (MINLP). Clearly, there is no general theory for this kind of problem available. See, e.g., the recent survey paper by Hante et al. (2017) for further information.

The aim of this article to create an avenue, along which one can proceed in order to reduce the size and the complexity of the problem until current methods from the literature become feasible in order to handle the problem. This is organized as follows: The first step is to introduce a proper time discretization of the problem, namely a semi-implicit-explicit Euler discretization, which turns the problem into a sequence of static semilinear problems. The second step is to apply the concept of "instantaneous" or "rolling horizon control" that turns the problem into a sequence of one-step optimal control problems for a given time level. The third step, and this is the essence of this paper, is to apply a tailored nonoverlapping domain decomposition in a way similar to Lagnese and Leugering (2004) and, more recently, Leugering (2017), in order to reduce the size and the complexity of the problem to reasonably small networks—even to a single pipe. This is done via an iterative scheme: first for the mere simulation problem and then for the corresponding optimality systems. We will show that in both cases the iterations converge, so that in the limit the solutions satisfy the original problem or the original optimality system on the entire network. It is important to note that thereby the optimal control problem on the entire network is iteratively decoupled to optimal control problems on the smaller sub-networks by using the so-called "virtual controls". See, e.g., Hundhammer and Leugering (2001) for an application of the instantaneous control paradigm in the context of domain-decomposition to wave equations on networks. The paper, therefore, aims at both the parallelization of the optimal original control problem and a size reduction in order to finally apply tailored MINLP methods (as developed in, e.g., Gugat et al., 2016; Schmidt et al., 2017) to the smaller sub-networks. These actual MINLP techniques are, however, not in the scope of the present paper and, thus, we refer to a forthcoming publication for the fully discrete-continuous problem.

In Leugering (2017) one of the present authors followed the described concept for a semilinear elliptic model derived from the one under study in this article. The results in Leugering (2017) paved the way to the analysis of the current article. However, the proofs are different and so are the regularity results. Moreover, in the current article we provide the modeling and the corresponding mathematical handling of "discrete elements", like valves and compressors. In this respect, the results obtained in the current article are novel and better tuned to the actual gas network problem arising in the considered application. For the sake of brevity, in this paper, we refrain from elaborating on the actual convergence behavior of the instantaneous control scheme. Suffice it to say that as long one is interested in controlling between equilibrium configurations, which actually is the main focus in gas network optimization, one typically enjoys fast convergence; see Hundhammer and Leugering (2001) and Lagnese and Leugering (2004). The application of the instantaneous control method to time-varying real-world gas network problems is subject to a forthcoming publication. See, however, Gugat et al. (2017), where the instantaneous control method has been applied for small networks.

2. Modeling of single pipes and entire networks

We now provide the modeling necessary in order to formulate the optimal control problems.

2.1. Modeling of gas flow in a single pipe

The Euler equations are given by a system of nonlinear hyperbolic partial differential equations (PDEs), which represent the motion of a compressible non-viscous fluid or gas. They consist of the continuity equation, the balance of moments, and the energy equation. The full set of equations is given by

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0, \\ \partial_t(\rho v) + \partial_x(p + \rho v^2) &= -\frac{\lambda}{2D}\rho v|v| - g\rho h', \\ \partial_t \left(\rho(\frac{1}{2}v^2 + e)\right) + \partial_x \left(\rho v(\frac{1}{2}v^2 + e) + pv\right) &= -\frac{k_w}{D}\left(T - T_w\right); \end{aligned}$$

see Smoller (1983), Brouwer et al. (2011), LeVeque (1992, 2002). Let ρ denote the density, v the velocity, and p the pressure of the gas. We further denote by λ the friction coefficient and by D the diameter of the pipe. The gas temperature is denoted by T, the temperature of the pipe's wall by T_w, and e denotes the internal energy of the gas. Finally, g is the gravitational acceleration, $h' = h'(x)$ is the constant slope of the pipe, and k_w is the pipe's heat transfer coefficient. The variables of the system are ρ, T, and the mass flow $q = a\rho v$, where a is the cross-sectional area of the pipe. We also denote by c the speed of sound, i.e., $c^2 = \partial_\rho p$ (for constant entropy). In particular, in the subsonic case ($|v| < c$) that we consider in the sequel, two boundary conditions have to be imposed on the left end and one at the right end of the pipe. We consider here the isothermal case only. Thus, for horizontal pipes, i.e., $h' = 0$, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0, \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) &= -\frac{\lambda}{2D}\rho v|v|. \end{aligned}$$

In the particular case, where we have a constant speed of sound $c = \sqrt{p/\rho}$ and only consider small velocities $|v| \ll c$, we arrive at the semilinear model; see Osiadacz (1996):

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \\ &\frac{\partial}{\partial t}(\rho v) + \frac{\partial p}{\partial x} = -\frac{\lambda}{2D}\rho v|v|. \end{aligned} \tag{1}$$

2.2. Network modeling

Let $G=(V,E)$ denote the here considered graph of the gas network with nodes $V = \{n_1, n_2, ..., n_{|V|}\}$ and edges $E = \{e_1, e_2, \ldots, e_{|E|}\}$. Node indices are denoted $j \in \mathcal{J} = \{1, \ldots, |V|\}$, while edges are labeled with $i \in \mathcal{I} = \{1, \ldots, |E|\}$. For the sake of uniqueness, we associate to each edge a direction. Accordingly, we introduce the edge-node incidence matrix with entries

$$d_{ij} = \begin{cases} -1, & \text{if node } n_j \text{ is the left node of the edge } e_i, \\ 1, & \text{if node } n_j \text{ is the right node of the edge } e_i, \\ 0, & \text{otherwise.} \end{cases}$$

In contrast to the classical notion of graphs in discrete mathematics, the graphs considered here are known as metric graphs, which means that the edges are continuous curves. In fact, we consider straight edges along which differential equations hold. The pressure variables $p_i(n_j)$ coincide for all edges incident at node n_j, i.e., for all edge indices $i \in \mathcal{I}_j := \{i = 1, \ldots, |E| \colon d_{ij} \neq 0\}$. We express the transmission conditions at the nodes in the following way. We introduce the edge degree $\delta_j := |\mathcal{I}_j|$ and distinguish between multiple nodes n_j with $\delta_j > 1$, whereas for simple nodes n_j we have $\delta_j = 1$. The corresponding index sets are denoted by $\mathcal{J}^{\mathrm{M}}$ and $\mathcal{J}^{\mathrm{S}}$. The set of multiple nodes contains serial nodes, i.e., nodes with edge degree $\delta_j = 2$. The set of simple nodes further decomposes into those simple nodes $\mathcal{J}^{\mathrm{S}}_{\mathrm{D}}$ at which Dirichlet (i.e., pressure) conditions hold, and Neumann nodes $\mathcal{J}^{\mathrm{S}}_{\mathrm{N}}$ that are flow-controlled. With this, the continuity conditions across an uncontrolled node reads

$$p_i(n_j, t) = p_k(n_j, t), \quad j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{c}} \cup \mathcal{J}_{\mathrm{v}}),\ i, k \in \mathcal{I}_j,$$

where $\mathcal{J}_{\mathrm{c}}$ and $\mathcal{J}_{\mathrm{v}}$ denote the serial nodes of compressors and valves that we interpret as controlled transmission conditions; see below for the details. The nodal balance equation for the flows can be written as a classical Kirchhoff-type condition

$$\sum_{i\in\mathcal{I}_j} d_{ij} q_i(n_j,t) = 0, \quad j \in \mathcal{J}^{\mathrm{M}}.$$

As already mentioned, we assume that valves and compressors are serial nodes n_j, i.e., $j \in \mathcal{J}^{\mathrm{M}}$ with $\delta_j = 2$. At such a node we have an incoming edge with unique index $i \in \mathcal{I}_j^+$, where $\mathcal{I}_j^+ := \{i \in \mathcal{I}_j : d_{ij} = 1\}$, and an outgoing edge with unique index $k \in \mathcal{I}_j^- := \{k \in \mathcal{I}_j : d_{kj} = -1\}$.

We now provide the network model of (1), see its complete description under System 1. It is obvious from System 1 that for $s_j^{\mathrm{v}}(t) = 1$, i.e., the case, in which the valve at node n_j is open, the classical transmission conditions hold, while for $s_j^{\mathrm{v}}(t) = 0$, the outgoing flow and—according to the Kirchhoff condition, which still holds—the incoming flow is zero. Similarly, for $s_j^{\mathrm{c}}(t) = 1$, the compressor is active, resulting in pressure control such that the pressure in the outgoing pipe is increased with respect to (w.r.t.) the pressures of the incoming pipes. To the best knowledge of the authors, System 1 with the switching functions $s_j^{\mathrm{v}}(t), s_j^{\mathrm{c}}(t) \in \{0,1\}$, even for the simplest possible network, namely - a two-link system with a compressor or valve at the connection point, has not been considered for the semilinear problem so far. Even for smooth relaxations of $s_j^{\mathrm{v}}(\cdot)$ and $s_j^{\mathrm{c}}(\cdot)$, no published result seems to be available.

System 1. Gas network model; $x \in (0,\ell_i)$ and $t \in (0,T)$

$$\begin{aligned}
\partial_t p_i(x,t) + \frac{c_i^2}{a_i}\partial_x q_i(x,t) &= 0, && i \in \mathcal{I}\\
\partial_t q_i(x,t) + \partial_x p_i(x,t) &= -\frac{\lambda c_i^2}{2D_i a_i^2}\frac{q_i(x,t)|q_i(x,t)|}{p_i(x,t)}, && i \in \mathcal{I}\\
p_i(n_j,t) &= p_k(n_j,t), && j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{c}} \cup \mathcal{J}_{\mathrm{v}}), i,k \in \mathcal{I}_j\\
g_j(p_i(n_j,t), q_i(n_j,t)) &= u_j(t), && j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j\\
\sum_{i\in\mathcal{I}_j} d_{ij} q_i(n_j,t) &= 0, && j \in \mathcal{J}^{\mathrm{M}}\\
s_j^{\mathrm{v}}(t)\,(p_i(n_j,t) - p_k(n_j,t)) + (1 - s_j^{\mathrm{v}}(t)) q_i(n_j,t) &= 0, && j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
s_j^{\mathrm{c}}(t)\left(u_j(t) - C\left(\left(\frac{p_k(n_j,t)}{p_i(n_j,t)}\right)^{\mathrm{sign}(q_k(n_j,t))\kappa} - 1\right)\right) & &&\\
+(1 - s_j^{\mathrm{c}}(t))\,(p_i(n_j,t) - p_k(n_j,t)) &= 0 && j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
p_i(x,0) = p_{i,0}(x),\ q_i(x,0) &= q_{i,0}(x), && i \in \mathcal{I}
\end{aligned}$$

For more details on the compressor model see, e.g., Schmidt et al. (2015), Rose et al. (2016), or the chapter by Fügenschuh et al. (2015) of the recent book by Koch et al. (2015). Note that we can replace the transmission conditions at the compressor node by the bilinear transmission conditions as follows:

$$\begin{aligned} & u_j(t) - C\left(\left(\frac{p_k(n_j,t)}{p_i(n_j,t)}\right)^{\operatorname{sign}(q_k(n_j,t))\kappa} - 1\right) = 0 \\ \iff & \left(\frac{u_j(t)+C}{C}\right)^{\operatorname{sign}(q_k(n_j,t))/\kappa} = \frac{p_k(n_j,t)}{p_i(n_j,t)}. \end{aligned}$$

If we replace $u_j(t)$ by

$$u_j(t) = \left(\frac{u_j + C}{C}\right)^{\operatorname{sign}(q_k(n_j,t))/\kappa}$$

and ensure $u_j \geq 1$, the original transmission condition at the compressor node can be replaced with

$$p_i(n_j,t)u_j(t) - p_k(n_j,t) = 0$$

if the compressor is active. Otherwise, the classical continuity condition for the pressure holds. This results in a bilinear boundary control.

3. The optimal control problem, time discretizations, and an instantaneous control approach

We are now in the position to formulate optimal control problems on the level of entire gas networks. There are many different approaches towards optimizing and/or control the flow of gas through pipeline networks. One of these approaches aims at optimizing discrete decision variables, such as on-off-states for valves and compressors. We refer to Hante et al. (2017); Gugat et al. (2016, 2017); Schmidt et al. (2017), refrain in the sequel from discussing issues of valves and compressors in detail, and focus on the continuous aspects of the problem. The combined discrete and continuous optimization will be the subject of future research. We now describe the general format of an optimal control problem associated with the semilinear model equations of the previous section:

$$\min_{(p,q,u,s)\in\Xi} \quad I(p,q,u,s) \quad \text{s.t.} \quad (p,q,u,s) \text{ satisfies System 1}, \tag{2}$$

where

$$\begin{aligned} I(p,q,u,s) := & \sum_{i\in\mathcal{I}} \int_0^T \int_0^{\ell_i} I_i(p_i,q_i)\,\mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2} \sum_{j\in\mathcal{J}^{\mathrm{S}}\cup\mathcal{J}_{\mathrm{c}}} \int_0^T |u_j(t)|^2\,\mathrm{d}t \\ & + \frac{1}{2}\int_0^T \sum_{j\in\mathcal{J}_{\mathrm{v}}} |s_j^{\mathrm{v}}(t)|^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \sum_{j\in\mathcal{J}_{\mathrm{c}}} |s_j^{\mathrm{c}}(t)|^2\,\mathrm{d}t \end{aligned} \tag{3}$$

and

$$\begin{aligned}\Xi := \{(p,q,u,s) \ : \ & p_i \in [\underline{p}_i, \bar{p}_i], \ q_i \in [\underline{q}_i, \bar{q}_i], \ i \in \mathcal{I}, \\ & u_j \in [\underline{u}_j, \bar{u}_j], \ j \in \mathcal{J}^{\mathrm{S}} \cup \mathcal{J}_{\mathrm{c}}, \\ & s_j^{\mathrm{v}} \in \{0,1\}, \ j \in \mathcal{J}_{\mathrm{v}}, \ s_j^{\mathrm{c}} \in \{0,1\}, \ j \in \mathcal{J}_{\mathrm{c}}\}\end{aligned} \tag{4}$$

holds. In (3), $\nu > 0$ is a penalty parameter and $I_i(\cdot,\cdot)$ is a continuous function on the pair (p_i, q_i). In (4), the quantities $\underline{p}_i, \underline{q}_i, \bar{p}_i, \bar{q}_i$ are given constants that determine the feasible pressures and flows in the pipes, while $\underline{u}_j, \bar{u}_j$ describe control constraints. In the continuous-time case the inequalities are considered as being satisfied for all times and everywhere along the pipes. In the sequel, we will not consider control and state constraints and even reduce to a time semi-discretization.

To this end, we consider a time discretization of System 1 such that $[0,T]$ is decomposed into break points $0 = t_0 < t_1 < \cdots < t_N = T$ with $\Delta t_n := t_{n+1} - t_n$ for $n = 0, \ldots, N-1$. Accordingly, we abbreviate $p_{i,n}(x) := p_i(x, t_n)$, $q_{i,n}(x) := q_i(x, t_n)$. Next, we apply a semi-implicit Euler scheme, which takes p_i in the friction term in an explicit manner. The resulting semi-discretized system provided in the form of System 2. With this, we obtain the optimal

System 2. Semi-discretized model; $x \in (0, \ell_i)$, $n = 0, \ldots, N-1$

$$\begin{aligned}
\frac{1}{\Delta t_n} p_{i,n+1}(x) + \frac{c_i^2}{a_i}\partial_x q_{i,n+1}(x) &= \frac{1}{\Delta t_n} p_{i,n}(x), \quad i \in \mathcal{I} \\
\frac{1}{\Delta t_n} q_{i,n+1}(x) &+ \partial_x p_{i,n+1}(x) \\
= -\frac{\lambda c_i^2}{2 D_i a_i^2} \frac{q_{i,n+1}(x)|q_{i,n+1}(x)|}{p_{i,n}(x)} &+ \frac{1}{\Delta t_n} q_{i,n}(x), \quad i \in \mathcal{I} \\
p_{i,n+1}(n_j) &= p_{k,n+1}(n_j), \quad j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{c}} \cup \mathcal{J}_{\mathrm{v}}), \ i,k \in \mathcal{I}_j \\
g_j(p_{i,n+1}(n_j), q_{i,n+1}(n_j)) &= u_{j,n+1}, \quad j \in \mathcal{J}^{\mathrm{S}}, \ i \in \mathcal{I}_j \\
\sum_{i \in \mathcal{I}_j} d_{ij} q_{i,n+1}(n_j) &= 0, \quad j \in \mathcal{J}^{\mathrm{M}} \\
s_{j,n+1}^{\mathrm{v}} \left(p_{i,n+1}(n_j) - p_{k,n+1}(n_j)\right) & \\
+(1 - s_{j,n+1}^{\mathrm{v}}) q_{i,n+1}(n_j) &= 0, \quad j \in \mathcal{J}_{\mathrm{v}}, \ i \in \mathcal{I}_j^+, \ k \in \mathcal{I}_j^- \\
s_{j,n+1}^{\mathrm{c}} \left(p_{i,n+1}(n_j) u_j - p_{k,n+1}(n_j)\right) & \\
+(1 - s_{j,n+1}^{\mathrm{c}}) \left(p_{i,n+1}(n_j) - p_{k,n+1}(n_j)\right) &= 0, \quad j \in \mathcal{J}_{\mathrm{c}}, \ i \in \mathcal{I}_j^+, \ k \in \mathcal{I}_j^- \\
p_i(x,0) = p_{i,0}(x), \ q_i(x,0) &= q_{i,0}(x), \quad i \in \mathcal{I}
\end{aligned}$$

control problem on the time-discrete level:

$$\begin{aligned}\min_{(p,q,u,s)\in\hat{\Xi}} \quad & \hat{I}(p,q,u,s) := \sum_{i\in\mathcal{I}}\sum_{n=1}^{N}\int_0^{\ell_i}\hat{I}_i(p_{i,n},q_{i,n})\,\mathrm{d}x + \frac{\nu}{2}\sum_{n=1}^{N}\sum_{j\in\mathcal{J}^{\mathrm{S}}\cup\mathcal{J}_{\mathrm{c}}}|u_{j,n}|^2 \\ & \quad + \frac{1}{2}\sum_{n=1}^{N}\sum_{j\in\mathcal{J}_{\mathrm{v}}}|s^{\mathrm{v}}_{j,n}|^2 + \frac{1}{2}\sum_{n=1}^{N}\sum_{j\in\mathcal{J}_{\mathrm{c}}}|s^{\mathrm{c}}_{j,n}|^2 \\ \text{s.t.} \quad & (p,q,u,s) \text{ satisfies System 2.}\end{aligned} \tag{5}$$

In (5), we consider discretized and edge-wise given cost functions, e.g.,

$$\hat{I}_i(p_{i,n},q_{i,n})(x) := \frac{1}{2}\left(|p_{i,n}(x)-p^d_{i,n}(x)|^2 + |q_{i,n}(x)-q^d_{i,n}(x)|^2\right)$$

for $x\in(0,\ell_i)$, $i\in\mathcal{I}$, and tracking targets $p^d_{i,n}$ and $q^d_{i,n}$. Moreover, $\hat{\Xi}$ is the discretized version of Ξ. It is clear that (5) involves all time steps in the cost functional. We would like to reduce the complexity of the problem even further. To this aim, we consider what has come to be known as *instantaneous control*; see Choi et al. (1993, 1999). This approach has also been used for the control of vibrating string networks in Hundhammer and Leugering (2001), for the control of wave equations in networks in Hinze (2002), for traffic flows in Herty et al. (2007), or for the control of linear wave equations in Altmueller et al. (2010). Very recently, a similar approach has been applied for MPEC-type optimal control problems in Antil et al. (2017) and for mixed-integer optimal control problems with PDEs in Gugat et al. (2017). The approach amounts to reducing the sums in the cost function of (5) to the time-level t_{n+1}. This strategy is known as *rolling horizon* approach, the simplest case of the *moving horizon* paradigm; consult, e.g., Hinze and Volkwein (2002), Hundhammer and Leugering (2001). Thus, for each $n=0,\dots,N-1$ and given $p_{i,n},q_{i,n}$, we consider the problems

$$\begin{aligned}\min_{(p,q,u,s)\in\hat{\Xi}} \quad & \tilde{I}(p,q,u,s) := \sum_{i\in\mathcal{I}}\int_0^{\ell_i}\hat{I}_i(p_{i,n+1},q_{i,n+1})\,\mathrm{d}x \\ & \quad + \frac{\nu}{2}\sum_{j\in\mathcal{J}^{\mathrm{S}}\cup\mathcal{J}_{\mathrm{c}}}|u_{j,n+1}|^2 + \frac{1}{2}\sum_{j\in\mathcal{J}_{\mathrm{v}}}|s^{\mathrm{v}}_{j,n+1}|^2 + \frac{1}{2}\sum_{j\in\mathcal{J}_{\mathrm{c}}}|s^{c}_{j,n+1}|^2 \\ \text{s.t.} \quad & (p,q,u,s) \text{ satisfies System 2 at time level } n+1.\end{aligned} \tag{6}$$

It is now convenient to discard the actual time level index $n+1$ and redefine the states at the former time as input data. To this end, we introduce

$$\alpha_i := \frac{1}{\Delta t_n}, \quad \beta_i = \frac{\alpha_i a_i}{c_i^2}, \quad f_i^1 := \beta_i p_{i,n}(x),$$
$$f_i^2 := \alpha_i q_{i,n}(x), \quad g_i(x;q_i(x)) := \frac{\lambda c_i^2}{2D_i a_i^2}\frac{q_i(x)|q_i(x)|}{p_{i,n}(x)},$$

and rewrite System 2 as System 3.

System 3. Constraint system of Problem (7); $x \in (0, \ell_i)$

$$\begin{aligned}
\beta_i p_i(x) + \partial_x q_i(x) &= f_i^1, && i \in \mathcal{I}\\
\alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i(x)) &= f_i^2, && i \in \mathcal{I}\\
p_i(n_j) &= p_k(n_j), && j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{c}} \cup \mathcal{J}_{\mathrm{v}}),\ i,k \in \mathcal{I}_j\\
g_j(p_i(n_j), q_i(n_j)) &= u_j, && j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j\\
\sum_{i \in \mathcal{I}_j} d_{ij} q_i(n_j) &= 0, && j \in \mathcal{J}^{\mathrm{M}}\\
s_j^{\mathrm{v}} \left(p_i(n_j) - p_k(n_j)\right) + (1 - s_j^{\mathrm{v}}) q_i(n_j) &= 0, && j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
s_j^{\mathrm{c}} \left(p_i(n_j) u_j - p_k(n_j)\right) + (1 - s_j^{\mathrm{c}}) \left(p_i(n_j) - p_k(n_j)\right) &= 0, && j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-
\end{aligned}$$

This results in the final optimal control problem to be discussed below:

$$\min_{(p,q,u,s) \in \hat{\Xi}} \quad \hat{I}(p,q,u,s) \quad \text{s.t.} \quad (p,q,u,s) \text{ satisfies System 3.} \tag{7}$$

4. Domain decomposition

4.1. Initial definitions and formulations

In this section, we provide an iterative nonoverlapping domain decomposition that can be interpreted as an Uzawa method; compare Algorithm 3 in Glowinski and Le Tallec (1989) and see the monograph of Lagnese and Leugering (2004) for details. The idea for this algorithm originates from a decoupling of the transmission conditions at all multiple nodes. In order to present the main ideas, we concentrate on that case first in Section 4.2. After that, we decompose the full graph into sub-graphs in Section 4.3, where we cut the connecting edges at possibly artificial serial nodes. To this end, we define the flow vector $q^k := (d_{ik} q_i(n_k))_{i \in \mathcal{I}_k}^\top$ and the pressure vectors $p^k := (p_i(n_k))_{i \in \mathcal{I}_k}^\top$ at a given node n_k, $k \in \mathcal{J}^{\mathrm{M}}$. Moreover, given a vector $z := (z_i)_{i \in \mathcal{I}_k}$, we define

$$\mathcal{S}^k(z)_i := \frac{2}{d_k} \sum_{j \in \mathcal{I}_k} z_j - z_i.$$

Then, $(S^k)^2 = I$, i.e., the mapping is idempotent, and $S^k(e) = 1$ for $e := (1, \dots, 1)^\top \in \mathbb{R}^{d_k}$. Using this notation, we now establish the general concept. For any $\sigma > 0$ we set

$$-q^k + \sigma p^k = \sigma \mathcal{S}^k(p^k) + \mathcal{S}^k(q^k). \tag{8}$$

Applying $\mathcal{S}^k$ to both sides of (8), we obtain

$$\sum_{i \in \mathcal{I}_k} d_{ik} q_i(n_k) = 0. \tag{9}$$

With this, (8) reduces to

$$p_i(n_k) = \frac{1}{d_k} \sum_{j \in \mathcal{I}_k} p_j(n_k), \quad i \in \mathcal{I}_k,$$

which, in turn, implies

$$p_i(n_k) = p_j(n_k), \quad k \in \mathcal{J}^{\mathrm{M}}, \ i, j \in \mathcal{I}_k. \tag{10}$$

Clearly, if the transmission conditions (9) and (10) hold at the multiple node n_k, then (8) is also fulfilled. Thus, (8) is equivalent to the transmission conditions (9), (10). This new condition (8) is now relaxed in an iterative scheme (using l as iteration number) as follows:

$$-(q^k)^{l+1} + \sigma(p^k)^{l+1} = \sigma \mathcal{S}^k((p^k)^l) + \mathcal{S}^k((q^k)^l) =: (g^k)^{l+1}, \quad g^k = (g_{ik})^\top_{i \in \mathcal{I}_k}. \tag{11}$$

We obtain the relation

$$(g^k)^{l+1} = \mathcal{S}^k(2\sigma(p^k)^l - (g^k)^l). \tag{12}$$

This gives rise to the definition of a fixed point mapping. To this end, we need to look into the behavior of the interface, i.e., the transmission nodes, in terms of g^k, $k \in \mathcal{J}^{\mathrm{M}}$, i.e.,

$$g \in \mathcal{X} := \Pi_{k \in \mathcal{J}^{\mathrm{M}}} \Pi_{i \in \mathcal{I}_k} \mathbb{R}, \quad \|g\|^2_{\mathcal{X}} := \sum_{k \in \mathcal{J}^{\mathrm{M}}} \sum_{i \in \mathcal{I}_k} \frac{1}{\sigma} |g_{ik}|^2 \tag{13}$$

and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ with

$$\begin{aligned} (\mathcal{T}g)_{i,k} &= \mathcal{S}^k(2\sigma(p^k) - g^k)_i, \quad k \in \mathcal{J}^{\mathrm{M}}, \ i \in \mathcal{I}_k, \\ (\mathcal{T}g)_k &= \{(\mathcal{T})_{i,k}, \ i \in \mathcal{I}_k\}, \\ \mathcal{T}g &= \{(\mathcal{T}g)_k, \ k \in \mathcal{J}^{\mathrm{M}}\}. \end{aligned} \tag{14}$$

Now,

$$\|\mathcal{T}g\|^2_{\mathcal{X}} = \sum_{k \in \mathcal{J}^{\mathrm{M}}} \sum_{i \in \mathcal{I}_k} \frac{1}{\sigma_k} |\mathcal{S}^k(2\sigma(p^k) - g^k)_i|^2$$

holds. We use the facts

$$\sum_{i \in \mathcal{I}_k} (\mathcal{S}^k g^k)_i^2 = \sum_{i \in \mathcal{I}_k} (g^k)_i^2$$

and

$$\sum_{i \in \mathcal{I}_k} (\mathcal{S}^k q^k)_i (\mathcal{S}^k g^k)_i = \sum_{i \in \mathcal{I}_k} q_i^k g_i^k$$

to obtain

$$\|\mathcal{T}g\|_{\mathcal{X}}^2 = \|g\|_{\mathcal{X}}^2 - 4\sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k}(g_i^k - \sigma_k p_i(n_k))p_i(n_k). \tag{15}$$

We now formulate a relaxed version of a fixed point iteration: For $\varepsilon \in [0,1)$, we set

$$g^{l+1} = (1-\varepsilon)\mathcal{T}(g^l) + \varepsilon g^l. \tag{16}$$

So far, the relations concerning the iteration at the interfaces do not involve the state equation explicitly. For the analysis of the convergence of the iterates, we need to specify the equations.

4.2. The nonoverlapping domain decomposition

For the ease of presentation, we first look at a graph that does not contain valves or compressors and we only consider the situation of flow-controlled boundary nodes. Thus, at this point we consider an edge that connects two multiple nodes or one multiple node and a controlled simple node. We are interested in the errors between the solutions of System 3 and the solutions of

$$\begin{aligned}
\beta_i p_i^{l+1}(x) + \partial_x q_i^{l+1}(x) &= f_i^1, \quad x\in(0,\ell_i),\ i\in\mathcal{I},\\
\alpha_i q_i^{l+1}(x) + \partial_x p_i^{l+1}(x) + g_i(x; q_i^{l+1}(x)) &= f_i^2, \quad x\in(0,\ell_i),\ i\in\mathcal{I},\\
-d_{ij}q_i^{l+1}(n_j) + \sigma p_i^{l+1}(n_j) &= g_{kj}^{l+1}, \quad j\in\mathcal{J}^{\mathrm{M}},\ i,k\in\mathcal{I}_j,\\
q_i(n_j) &= u_j, \quad i\in\mathcal{I}_j,\ j\in\mathcal{J}_{\mathrm{N}}^{\mathrm{S}},
\end{aligned} \tag{17}$$

where g_{kj}^{l+1} satisfies (12). Notice that the third position in (17) describes a set of equations, one for each edge incident at node n_j. Thus, we introduce $\hat{q}^{l+1} := q^{l+1} - q$ and $\hat{p}^{l+1} := p^{l+1} - p$. Then, $\hat{q}^{l+1}$ and $\hat{p}^{l+1}$ solve a nonlinear differential equation with nonlinearity $g_i(\hat{q}_i^{l+1} + q_i) - g_i(q_i)$, zero right-hand sides and homogeneous boundary conditions at the simple nodes. As we noted above, the full transmission conditions are equivalent to (8). Hence, the error satisfies the same iterative Robin-type boundary conditions as q^{l+1} and p^{l+1}. We consider the following integration by parts formula after multiplying by a test function ϕ:

$$\begin{aligned}
0 &= \sum_{i\in\mathcal{I}}\int_0^{\ell_i}\left(\beta_i\hat{p}_i^{l+1} + \partial_x\hat{q}_i^{l+1}\right)\phi_i\,\mathrm{d}x\\
&= \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k} d_{ik}\hat{p}_i^{l+1}(n_k)\phi_i(n_k) + \sum_{i\in\mathcal{I}}\int_0^{\ell_i}\left(\beta_i\hat{p}_i^{l+1}\phi_i - \hat{q}_i^{l+1}\partial_x p_i\right)\mathrm{d}x,\\
0 &= \sum_{i\in\mathcal{I}}\int_0^{\ell_i}\left(\alpha_i\hat{q}_i^{l+1} + \partial_x\hat{p}_i^{l+1} + g_i(\hat{q}_i^{l+1} + q_i) - g_i(q_i)\right)q_i\,\mathrm{d}x.
\end{aligned}$$

We obtain

$$\begin{aligned} &-\sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k} d_{ik}\hat{q}_i^{l+1}(n_k)\hat{p}_i^{l+1}(n_k) \\ &=\sum_{i\in\mathcal{I}}\int_0^{\ell_i}\left(\beta_i(\hat{p}_i^{l+1})^2+\alpha_i(\hat{q}_i^{l+1})^2+(g_i(\hat{q}_i^{l+1}+q_i)-g_i(q_i))\hat{q}_i^{l+1}\right)\,\mathrm{d}x. \end{aligned}$$

Moreover, we have

$$\sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k} d_{ik}\hat{q}_i^{l+1}(n_k)\hat{p}_i^{l+1}(n_k)=-\sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k}(g_{ik}-\sigma p_i(n_k))p_i(n_k).$$

This identity is used in (15), evaluated for the error

$$\|\mathcal{T}g\|_{\mathcal{X}}^2=\|g\|_{\mathcal{X}}^2-4\sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k}((g_i^k)^l-\sigma_k(\hat{p}_i^k)^l)(\hat{p}e_i^k)^l.$$

We obtain

$$\begin{aligned} &\|g^{l+1}\|_{\mathcal{X}}^2=\|\mathcal{T}g^l\|_{\mathcal{X}}^2 \\ &=\|g^l\|_{\mathcal{X}}^2-4\sum_{i\in\mathcal{I}}\int_0^{\ell_i}\left(\beta_i(\hat{p}_i^l)^2+\alpha_i(\hat{q}_i)^2+(g_i(\hat{q}_i^l+q_i)-g_i(q_i))\hat{q}^l\right)\mathrm{d}x. \end{aligned}$$

We assume monotonicity of the nonlinear term

$$(g_i(x;s)-g_i(x;t))(s-t)\geq 0,\quad x\in(0,\ell_i),\ i\in\mathcal{I}. \tag{18}$$

Then, the error iteration is

$$\|g^{l+1}\|_{\mathcal{X}}^2\leq\|\mathcal{T}g^l\|_{\mathcal{X}}^2=\|g^l\|_{\mathcal{X}}^2-4\sum_{i\in\mathcal{I}}\int_0^{\ell_i}\left(\beta_i(\hat{p}_i^l)^2+\alpha_i(\hat{q}_i)^2\right)\,\mathrm{d}x \tag{19}$$

and, thus, the error does not increase. That it actually decreases to zero is shown below. Before, we look at the relaxed version of the iteration (16). Taking norms, we obtain

$$\|g^{l+1}\|_{\mathcal{X}}^2\leq\|g^l\|_{\mathcal{X}}^2-4(1-\varepsilon)\sum_{i\in\mathcal{I}}\|\hat{q}_i\|^2+\|\hat{p}_i\|^2. \tag{20}$$

We iterate in (19) or (20) down from l to zero and obtain

$$\{g^l\}\text{ is bounded},\quad\|\hat{p}_i^l\|^2,\|\hat{q}_i^l\|^2\to 0,\quad l\to\infty.$$

But according to the error equations, if $\hat{p}_i\to 0$ holds strongly, then also $\partial_x\hat{q}_i$, and in a similar way also $\partial_x\hat{p}_i$, strongly tends to zero. Thus, the full sequence of traces converges.

Theorem 1 *Under the monotonicity assumption* (18)*, for each* $\varepsilon\in[0,1)$ *the iteration* (16) *with* (11)*,* (13)*, and* (14) *converges as* $l\to\infty$*. The convergence of the solutions is in the* H^1*-sense (see* (29)*) on the entire network. Moreover, the traces at the decomposition nodes converge.*

Before we embark on the domain decomposition of the optimal control problems, we discuss the extension to sub-graph decomposition.

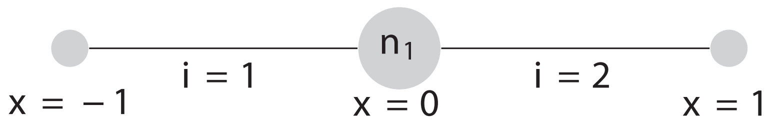

Figure 1. Two-link network of Example 1.

4.3. Sub-graph decomposition

We consider the graph $G = (V, E)$ being decomposed into sub-graphs $G_m = (V_m, E_m)$ for $m = 1, \dots, K$. For the ease of presentation, we split the original graph only at serial nodes $j \in \mathcal{J}^{\mathrm{M}}$. We assume that the sub-graphs are connected according to an adjacency structure $A_{m,n} = 1$ if the two sub-graphs G_m and G_n with $m, n \in \{1, \dots, K\}$ are connected. Otherwise, $A_{m,n} = 0$ holds. We denote the edge sets of sub-graph G_m, $m \in \{1, \dots, K\}$, by $\mathcal{I}^m$. The serial transmission nodes between sub-graph G_m and G_n are denoted by the set $\mathcal{J}^{\mathrm{M}}_{m,n}$. Moreover, we assume that all valves and compressors are contained in the interior of the sub-graphs. To express this, we introduce the set $\mathcal{J}^{\mathrm{M},o}_m$ of multiple nodes of G_m that are not in $\mathcal{J}^{\mathrm{M}}_{m,n}$. Accordingly, $\mathcal{J}_{m,\mathrm{c}}$ and $\mathcal{J}_{m,\mathrm{v}}$ are the compressor and valve nodes contained in G_m. Thus, after domain decomposition, System 3 then yields System 4.

System 4. Domain-decomposed system; $x \in (0, \ell_i)$, $m = 1, \dots, K$

$$
\begin{aligned}
\beta_i p_i^{l+1}(x) + \partial_x q_i^{l+1}(x) &= f_i^1, && i \in \mathcal{I}^m \\
\alpha_i q_i^{l+1}(x) + \partial_x p_i^{l+1}(x) + g_i(x; q_i^{l+1}(x)) &= f_i^2, && i \in \mathcal{I}^m \\
p_i^{l+1}(n_j) &= p_k^{l+1}(n_j), && j \in \mathcal{J}^{\mathrm{M},o}_m \setminus (\mathcal{J}_{m,\mathrm{c}} \cup \mathcal{J}_{m,\mathrm{v}}), i, k \in \mathcal{I}_j \\
g_j(p_i^{l+1}(n_j), q_i^{l+1}(n_j)) &= u_j, && j \in \mathcal{J}^{\mathrm{S}}_m,\ i \in \mathcal{I}_j \\
\sum_{i \in \mathcal{I}_j} d_{ij} q_i^{l+1}(n_j) &= 0, && j \in \mathcal{J}^{\mathrm{M},o}_m \\
s_j^{\mathrm{v}} \left(p_i^{l+1}(n_j) - p_k^{l+1}(n_j) \right) + (1 - s_j^{\mathrm{v}}) q_i^{l+1}(n_j) &= 0, && j \in \mathcal{J}_{m,\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \\
s_j^{\mathrm{c}} \left(p_i^{l+1}(n_j) u_j - p_k^{l+1}(n_j) \right) & \\
+ (1 - s_j^{\mathrm{c}}) \left(p_i^{l+1}(n_j) - p_k^{l+1}(n_j) \right) &= 0, && j \in \mathcal{J}_{m,\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \\
-d_{ij} q_i^{l+1}(n_j) + \sigma p_i(n_j)^{l+1} & \\
= \sigma p_k(n_j)^l + d_{kj} q_k(n_j)^l &=: g_{kj}^{l+1}, && j \in \mathcal{J}^{\mathrm{M}}_{m,n},\ n : A_{m,n} = 1, i, k \in \mathcal{I}_j
\end{aligned}
$$

EXAMPLE 1 *We consider a serial situation, consisting of two links, labeled with $i = 1, 2$, that are coupled at $x = 0$. The first link stretches from $x = -1$ to $x = 0$, while the second stretches from $x = 0$ to $x = 1$; see Fig. 1. We choose $\alpha_i = \beta_i = 1$, $\gamma_i = \lambda c_i^2 / (2 D_i a_i^2) = 0$, and the distributed loads are given by*

Table 1. Iteration history of g_{11}, g_{21} in Example 1.

Iteration:	0	1	2	3	4	5
g_{11}	0.000000	0.890987	1.009400	1.072349	1.080715	1.085163
g_{21}	0.000000	0.445494	0.682320	0.713794	0.730526	0.732750

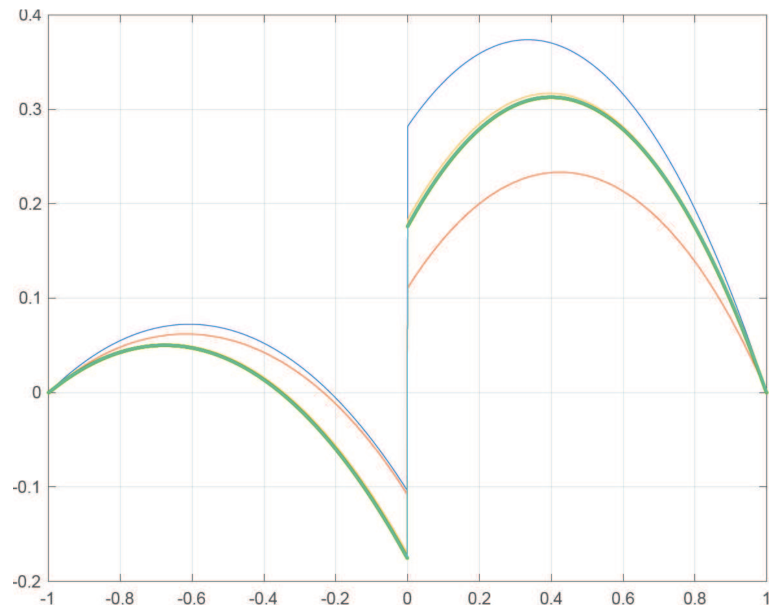

Figure 2. The two-link serial network of Example 1; see Fig. 1. x-axis: spatial coordinate $x \in [-1, 1]$. y-axis: mass flow. The reference solution is printed in bold.

$f_1^1(x) = 1, f_1^2(x) = 1,\ f_2^1 = 1, f_2^2 = 2$. *We plot the first five iterations of the domain decomposition and provide the nodal errors. The reference solution is obtained using the* MATLAB *routine* bvp4c *with a tolerance of* 10^{-4}*; see the bold lines in Fig. 2. For the fixed point behavior of the* g_{kj} *at the interface, consult Table 1.*

The error after five iterations in the continuity conditions for the pressures is 8.87×10^{-4} *and the final error in the flow is* 2.25×10^{-4}. *After* 20 *iterations, the corresponding errors are* 7.92×10^{-12} *and* 1.71×10^{-12}*, respectively. If we now choose* $\gamma = 5$ *and take* 20 *iterations we obtain the errors* 2.84×10^{-9} *and* 4.08×10^{-10}*, respectively. The corresponding plots in Fig. 2 do not show any difference w.r.t. the reference solution.*

Example 2 *We now consider the situation in which a compressor is located in the middle of the two links. Otherwise, the model is as in Example 4.1. The*

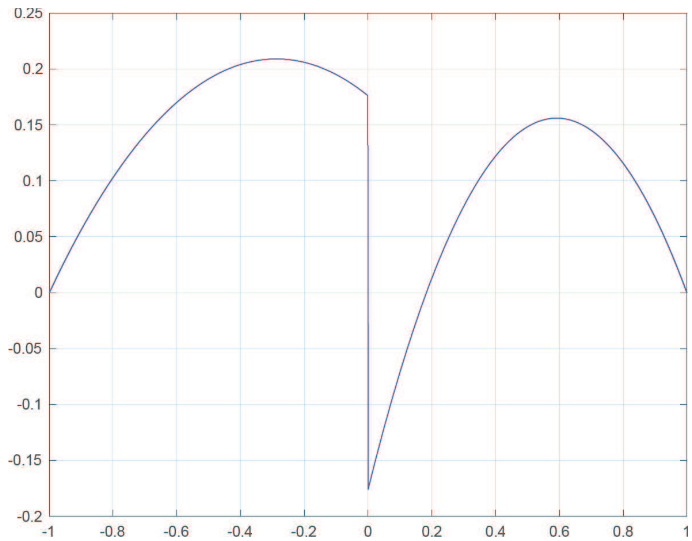

Figure 3. The two-link serial network with activated compressor ($u_0 = 5$). x-axis: spatial coordinate $x \in [-1, 1]$. y-axis: mass flow.

domain decomposition is as follows:

$$\begin{aligned} \beta_i p_i^{l+1}(x) + \partial_x q_i^{l+1}(x) &= f_i^1, \quad i = 1, 2, \\ \alpha_i q_i^{l+1}(x) + \partial_x p_i^{l+1}(x) + g_i(x; q_i^{l+1}(x)) &= f_i^2, \quad i = 1, 2, \\ q_1^{l+1}(-1) = 0, &\quad q_2^{l+1}(1) = 0, \\ -q_1^{l+1}(0) + \sigma p_1(0)^{l+1} u_0 &= \sigma p_2(0)^l + q_2(0)^l, \\ -q_2^{l+1}(0) + \sigma p_2(0)^{l+1} &= \sigma p_1(0)^l u_0 + q_1(0)^l. \end{aligned}$$

We take 20 iterations and put the control $u_0 = 5$. *The flow of the last iteration is plotted on top of the MATLAB reference solution, obtained as above; for illustration see Fig. 3. The errors are* $3.550273 \cdot 10^{-7}$ *and* $1.665880 \cdot 10^{-7}$, *respectively.*

REMARK 1 *We consider the situation of the last example, analyze a particular iteration* $l+1$ *and omit this index, while keeping the previous index, in order to identify the data of the problem. In particular, for the edge* 2 *we have*

$$\begin{aligned} \beta_2 p_2(x) + \partial_x q_2(x) &= f_2^1, \quad x \in (0, \ell_2), \\ \alpha_2 q_2(x) + \partial_x p_2(x) + g_2(x; q_2(x)) &= f_2^2, \quad x \in (0, \ell_2), \\ -q_2(0) + \sigma p_2(0) = \sigma p_1(0)^l (s(u-1)+1) + q_1(0)^l, &\quad q_2(1) = \bar{q}_2. \end{aligned} \tag{21}$$

If $s = 1$, *the control* $u \geq 1$ *is applied, as the pressure is then higher than in the previous pipe. Otherwise, the control* 1 *is applied, as then the pressures are the* *same. We may introduce $v = u - 1$ and have $v \geq 0$. The control v then appearing in the Robin-type boundary condition is multiplied by the binary variable s and by $\sigma p_1(0)^l$ from the previous iteration. Thus, the constellation above is a Robin-type boundary control problem for a single link. In Gugat et al. (2016), the authors have established particular situations in which a master-sub-problem-strategy, where the master problem consists in optimizing the discrete variables, i.e., deciding whether the compressor is active or not, and the sub-problem concerns the continuous optimization, i.e., the pump control, converges. In that study it was required that the control-to-state map of the sub-problem is smooth, strictly monotone, and either convex or concave. Further developments that alleviate the assumptions were presented in Schmidt et al. (2017). A similar situation has been studied in Buchheim, Kuhlmann and Meyer (2016) for an integer control problem for a semilinear Laplace boundary value problem, where also the concavity of the control-to-state-map turned out to be the crucial argument. We therefore ask the question whether the flow q is concave as a function of u. For its answer, we would like to resort to a maximum principle and transform problem (21) into a second-order problem. This is done by differentiating the first equation of (21) with respect to x and inserting the resulting expression for $\partial_x p_2$ into the second equation. The pressure terms in the boundary and transmission conditions are then $p_i(n_j) = -\partial_x q_i(n_j)/\beta_i$. We ignore the edge index and formulate an optimal control problem for the single edge 2:*

$$\begin{aligned} \min_{s\in\{0,1\},u\in[1,\bar{u}]} \quad & \|q - q^d\|_{L^2(0,1)} + \frac{\nu}{2}(s^2 + u^2) \\ s.t. \quad & \alpha\beta q - \partial_{xx} q + \beta g(x;q) = \beta f^2 - \partial_x f^1, \\ & q(0) + \frac{\sigma}{\beta}\partial_x q(0) = \phi s(u-1) + \mu, \\ & q(1) = \bar{q}. \end{aligned} \tag{22}$$

Here, $\phi = \sigma/\beta_1\, \partial_x q_1(0)^l$, $\mu = \phi - q_1(0)^l$. In order to demonstrate the concavity of q as a function of u, using differential calculus, we need to show that $\partial_{uu} q(u) < 0$. This, however, requires that $g(x;\cdot)$ be twice differentiable. Obviously, the function $g(x;q) = \gamma(x)q(x)|q(x)|$ is first-order continuously differentiable, while the second derivative is not well defined at $x = 0$, being otherwise identical to the Heavyside function. Its Bouligand second derivative is the set $\{-1, 1\}$. We now use the smoothed function $g_\varepsilon(x;q) = \gamma(x)(\varepsilon + |q(x)|^2)^{\frac{1}{2}} q(x)$. We can now differentiate the constraints of (22) w.r.t. u and obtain for $w := D_u q(u)$ and $z := D_{uu} q(u)$:

$$\alpha\beta z - \partial_{xx} z + \beta D_2 g_\varepsilon(x; q(u)) z = -\beta D_2^2 g_\varepsilon(x; q(u)) w^2, \tag{23a}$$

$$q(0) + \frac{\sigma}{\beta}\partial_x z(0) = 0, \tag{23b}$$

$$z(1) = 0. \tag{23c}$$

As the flow is in the positive direction by construction, $q(u)$ is positive for positive controls. This can also be proven using the maximum principle for (22).

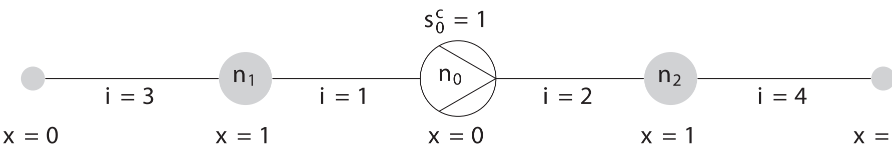

Figure 4. Network of Example 3.

The term $\beta D_2^2 g_\varepsilon(x; q(u))w^2$ is positive and, hence, the right-hand side of (23a) *is negative. According to the maximum principle, z is negative and, therefore, $q(u)$ is concave as a function of $u > 0$. Thus, for $\varepsilon > 0$, we have achieved the situation alluded to above. This amounts to saying that up to a relaxation parameter, we can achieve a global solution at the iteration level $l+1$ and $\varepsilon > 0$ using the techniques of Gugat et al (2016). This property is reminiscent to the results in Ahmad Ali et al. (2016), where additional control and state constraints are considered. However, the nonlinearity does not formally fit into the framework of Ahmad Ali et al. (2016). The extension of these results for constrained problems with the nonlinearity discussed here is the subject of a forthcoming publication. Having achieved the optimal control in* (22) *for edge* 2*, we can use it in the iteration for the edge* 1*, according to* (21)*. It ought to be noted that the question if the global optimum is stable as the domain decomposition iteration converges is open. See, e.g., Pertsinidis et al. (1998) for a sensitivity analysis for MINLPs.*

Example 3 *We consider the serial situation displayed in Fig. 4, where the edges* $1, 2$ *are connected by the node n_0 (at $x = 0$), the edges* $3, 4$ *are connected to edge* 1 *via node n_1 (at $x = 1$) and to edge* 2 *via node n_2 (at $x = 1$). At $x = 0$, i.e., the node between edges* $1, 2$*, we have an active compressor, i.e., $s_0^c = 1$. We decompose the network at the two serial nodes between edges* $1, 3$ *and* $2, 4$ *at $x = 1$, respectively. With this configuration, we have $\mathcal{I}_0 = \{1, 2\}$, $\mathcal{I}_1 = \{1, 3\}$, $\mathcal{I}_2 = \{2, 4\}$. At the simple nodes of edges* 3 *and* 4*, we consider controlled boundary flows. We write down the system in a more explicit way:*

$$
\begin{aligned}
\beta_i p_i^{l+1}(x) + \partial_x q_i^{l+1}(x) &= f_i^1, \quad x \in (0, \ell_i),\ i = 1, \dots, 4,\\
\alpha_i q_i^{l+1}(x) + \partial_x p_i^{l+1}(x) + g_i(x; q_i^{l+1}(x)) &= f_i^2, \quad x \in (0, \ell_i),\ i = 1, \dots, 4,\\
p_1^{l+1}(0) u_0 &= p_2^{l+1}(0),\\
q_1^{l+1}(0) + q_2^{l+1}(0) &= 0,\\
q_3^{l+1}(0) = u_3,\ q_4^{l+1}(0) &= u_4,\\
-q_1^{l+1}(1) + \sigma p_1(1)^{l+1} &= \sigma p_3(1)^l + q_3(1)^l =: g_{31}^{l+1},\\
-q_3^{l+1}(1) + \sigma p_3(1)^{l+1} &= \sigma p_1(1)^l + q_1(1)^l =: g_{11}^{l+1},\\
-q_2^{l+1}(1) + \sigma p_2(1)^{l+1} &= \sigma p_4(1)^l + q_4(1)^l =: g_{42}^{l+1},\\
-q_4^{l+1}(1) + \sigma p_4(1)^{l+1} &= \sigma p_2(1)^l + q_2(1)^l =: g_{22}^{l+1}.
\end{aligned}
$$

It is then obvious that the domain decomposition method converges.

Examples 1, 2 and 3 show that a network with compressors and valves can be decomposed into sub-graphs down to individual edges using the nonoverlapping domain decomposition procedure.

Theorem 2 *Let the assumption of Theorem 1 be valid. Then, the sub-graph iteration* (4) *converges as* $l \to \infty$ *in the* H^1*-sense.*

5. Domain decomposition for optimal control problems

We now consider the optimal control problem (7) with two modifications: First, we fix a given switching structure s. Second, we only consider flow boundary controls. The latter means that we replace $g_j(p_i(n_j), q_i(n_j)) = u_j$ by $q_i(n_j) = u_j$ for $j \in \mathcal{J}^{\mathrm{S}}$, $i \in \mathcal{I}_j$. The corresponding optimality system provided in the form of System 5.

System 5. Optimality system of Problem (7) with fixed switching structure and flow boundary control; $x \in (0, \ell_i)$

$$
\begin{aligned}
\beta_i p_i(x) + \partial_x q_i(x) &= f_i^1, && i \in \mathcal{I}\\
\alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i) &= f_i^2, && i \in \mathcal{I}\\
\beta_i \phi_i(x) - \partial_x \psi_i(x) &= -\kappa_i (p_i - p_i^0), && i \in \mathcal{I}\\
\alpha_i \psi_i(x) - \partial_x \phi_i(x) + \partial_q g_i(x; q_i)\phi_i &= -\kappa_i (q_i - q_i^0), && i \in \mathcal{I}\\
q_i(n_j) = u_j,\ \psi_i(n_j) &= 0, && j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j\\
p_i(n_j) = p_k(n_j),\ \phi_i(n_j) &= \phi_k(n_j), && j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{c}} \cup \mathcal{J}_{\mathrm{v}}), i, k \in \mathcal{I}_j\\
\sum_{i \in \mathcal{I}_j} d_{ij} q_i(n_j) = 0,\ \sum_{i \in \mathcal{I}_j} d_{ij} \psi_i(n_j) &= 0, && j \in \mathcal{J}^{\mathrm{M}}\\
s_j^{\mathrm{v}} \left(p_i(n_j) - p_k(n_j)\right) + (1 - s_j^{\mathrm{v}}) q_i(n_j) &= 0, && j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
s_j^{\mathrm{v}} \left(\phi_i(n_j) - \phi_k(n_j)\right) + (1 - s_j^{\mathrm{v}}) \psi_i(n_j) &= 0, && j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
s_j^{\mathrm{c}} \left(p_i(n_j) u_j - p_k(n_j)\right) + (1 - s_j^{\mathrm{c}}) \left(p_i(n_j) - p_k(n_j)\right) &= 0, && j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
s_j^{\mathrm{c}} \left(\psi_i(n_j) u_j - \psi_k(n_j)\right) + (1 - s_j^{\mathrm{c}}) \left(\phi_i(n_j) - \phi_k(n_j)\right) &= 0, && j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-\\
u_j &= -\frac{1}{\nu} \phi_i(n_j), && j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j\\
u_j &= -s_j^{\mathrm{c}} p_i(n_j) \psi_i(n_j), && j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j
\end{aligned}
$$

The idea is to use a domain decomposition similar to the one discussed so far. We design a method that allows to interpret the decomposed optimality System 5 as an optimality system of an optimal control problem formulated on a sub-graph or, ultimately, on an individual edge. To fix the ideas, we first concentrate on systems without valves and compressors as before. The reason is that we do not intend to decompose the systems at such nodes. Instead, we focus on the decomposition at serial nodes again. To this end, we introduce the following local system, involving two edges, labeled with $i,k \in \mathcal{I}_j$:

$$\begin{aligned}
\beta_i p_i^{l+1}(x) + \partial_x q_i^{l+1}(x) &= f_i^1, \quad i \in \mathcal{I},\\
\alpha_i q_i^{l+1}(x) + \partial_x p_i^{l+1}(x) + g_i(x; q_i^{l+1}) &= f_i^2, \quad i \in \mathcal{I},\\
\beta_i \phi_i^{l+1}(x) - \partial_x \psi_i^{l+1}(x) &= -\kappa_i(p_i^{l+1} - p_i^0), \quad i \in \mathcal{I},\\
\alpha_i \psi_i^{l+1}(x) - \partial_x \phi_i^{l+1}(x) + \partial_q g_i(x; q_i^{l+1})\phi_i^{l+1} &= -\kappa_i(q_i^{l+1} - q_i^0), \quad i \in \mathcal{I},\\
-d_{ij} q_i^{l+1}(n_j) + \sigma p_i^{l+1}(n_j) - \mu \phi_i^{l+1}(n_j) &= g_{kj}^{l+1}, \quad i,k \in \mathcal{I}_j,\\
d_{ij} \psi_i^{l+1}(n_j) + \sigma \phi_i^{l+1}(n_j) + \mu p_i^{l+1}(n_j) &= h_{kj}^{l+1}, \quad i,k \in \mathcal{I}_j,\\
g_{kj}^{l+1} &= d_{kj} q_k^l(n_j)) + \sigma p_k^l(n_j) - \mu \phi_k^l(n_j), \quad i,k \in \mathcal{I}_j,\\
h_{kj}^{l+1} &= -d_{kj} \psi_k^l(n_j) + \sigma \phi_k^l(n_j) + \mu p_k^l(n_j), \quad i,k \in \mathcal{I}_j,
\end{aligned} \tag{24}$$

where $x \in (0, \ell_i)$. System (24) reflects a situation, where the domain decomposition is applied at a serial node that connects two edges.

EXAMPLE 4 *We consider a serial situation, where two links are coupled at $x = 0$ and the pressure is controlled at the two ends with $x = 1$. The transmission node at $x = 0$ is the one where we apply the domain decomposition. We have the following academic scenario for demonstrating the domain decomposition for optimality systems. On both edges we apply a distributed load $f_i^1(x) = 0$, $f_i^2(x) = 1000$ for all $x \in (0,1)$ and $i = 1,2$. We would like to track the constant targets $f_i^{2,d}(x) = 1, x \in (0,1), i = 1,2$, and choose $\beta_i = 1$, $\alpha_i = 1000$, and $\kappa_i = 100$ for $i = 1,2$. As iteration parameters, we use $\mu = 0$ and $\sigma = 1$. As above, we solve the optimality system using the MATLAB routine bvp4c for obtaining the reference solution and compare it with the result of our domain decomposition method. We print the solution of the domain decomposition iterations on top of the reference solutions, for the optimal states and the adjoints, respectively. For the results see Figs. 5 and 6 for the states, the adjoints, and the nodal errors, respectively. Since the situation is fully symmetric, we only plot the solution in $x \in [0,1]$.*

EXAMPLE 5 *Here, we consider the same network as in the previous example but change the physical data. We recall that $f_i^{1,2}$ represent previous pressure and flow functions along the edges $i = 1,2$. Assume those are constant and equal, say, $f_1^1 = f_2^1 = 1$ for all $x \in (0,1)$, while $f_1^2 = -f_2^2 = \alpha$. We may take $\alpha\beta =: c = 1000$, which is fine for the time discretizations discussed above, in particular if we choose the spatial discretization $\Delta x = 1/1000$. We first ignore the nonlinearity. Then, the flow is 1 and -1 on edges 1 and 2, respectively, while the pressure is equal to 1 in both pipes. If we take these as tracking goals, the domain decomposition iteration should finally reveal these solutions with controls $u_i = 1$. This is what we observe in Fig. 7. The error behavior is as above.*

We now take the same configuration and tune the nonlinearity. This gives new equilibria. By setting $\gamma = 0.1$, we obtain the results shown in Fig. 8, where also the change in the adjoints can be seen.

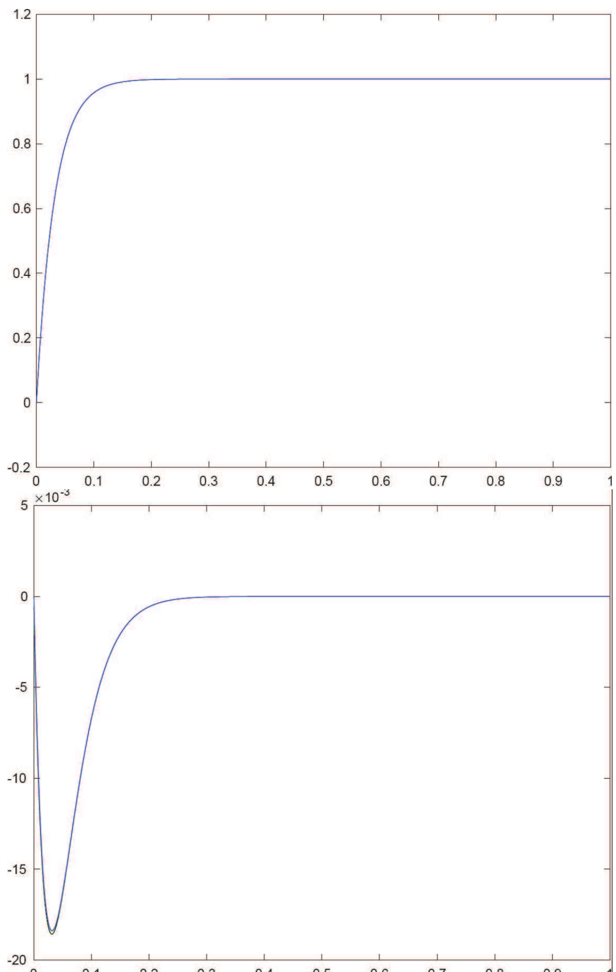

Figure 5. Optimal control of the two-link serial network of Example 4. Upper diagram: optimal states (with mass flow on the y-axis). Lower diagram: adjoints. x-axis: spatial coordinate $x \in [0, 1]$.

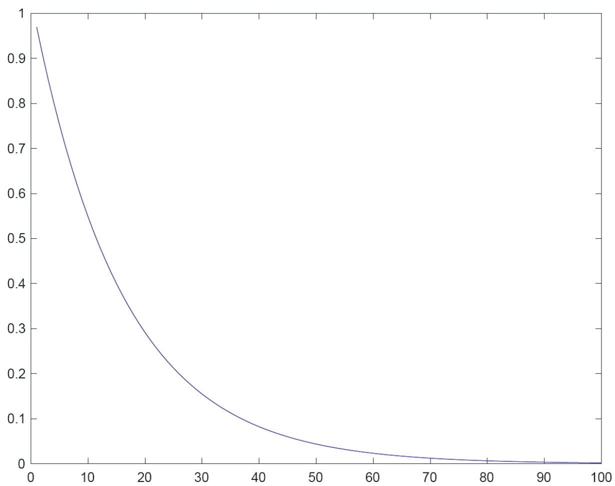

Figure 6. Optimal control of the two-link serial network of Example 4. Nodal errors over the course of the iterations (x-axis).

Let us now consider the following optimization problems on a single edge. The idea is to introduce a *virtual control* that aims at controlling the classical inhomogeneous Neumann conditions, including the iteration history at the interface as the inhomogeneity to the Robin-type condition that appears in the decomposition. To this end, it is sufficient to consider three cases:

a) The edge i connects a controlled flow-node $j \in \mathcal{J}_{\rm N}^{\rm S}$ node with a multiple (serial) node $k \in \mathcal{J}^{\rm M}$ at which the domain decomposition is active.
b) The edge i connects a controlled pressure-node $j \in \mathcal{J}_{\rm D}^{\rm S}$ with multiple (serial) node $k \in \mathcal{J}^{\rm M}$ at which the domain decomposition is active.
c) The edge i connects two multiple (serial) nodes $j, k \in \mathcal{J}^{\rm M}$.

We concentrate on the last case as it is the most complex one. The two other cases are completely analogous. Thus, in the case of a single edge i with no connection to a controlled node, we consider the problem

$$\begin{aligned}\min_{q_i,p_i,v_{ij},v_{ik}} I(q_i,p_i,v_{ij},v_{ik}) &:= \frac{\kappa}{2}\left(\|q_i-q_i^0\|^2+\|p_i-p_i^0\|^2\right)+\frac{1}{2\mu}v_{ij}^2+\frac{1}{2\mu}v_{ik}^2\\ &+\frac{1}{2\mu}(\mu p_i(n_k)-h_{ik})^2+\frac{1}{2\mu}(\mu p_i(n_j)-h_{ij})^2\\ \text{s.t.}\quad & \beta_i p_i+\partial_x q_i=f_i^1,\quad x\in(0,\ell_i),\\ & \alpha_i q_i+\partial_x p_i+g_i(x;q_i)=f_i^2,\quad x\in(0,\ell_i),\\ & -d_{ij}q_i(n_j)+\sigma p_i(n_j)=g_{kj}+v_{ij},\quad i,k\in\mathcal{I}_j,\\ & -d_{ik}q_i(n_k)+\sigma p_i(n_k)=g_{jk}+v_{ik},\quad i,j\in\mathcal{I}_k,\end{aligned}$$

where the h_{ij}, h_{ik} appear in the domain decomposition of the optimality system in (24) and are taken at the iteration level l. We now also involve valves and

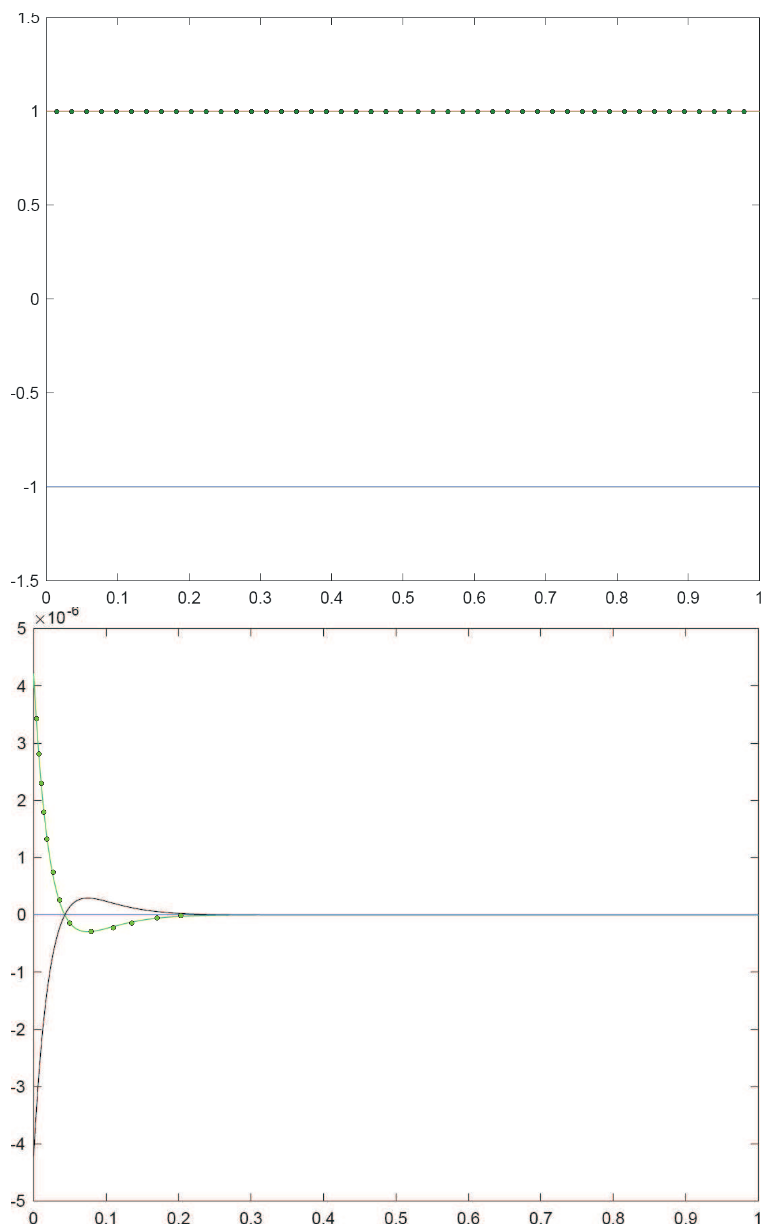

Figure 7. Optimal control of the two-link serial network of Example 5 without nonlinearity. Upper diagram: optimal states (with mass flow on the y-axis). Lower diagram: adjoints. x-axis: spatial coordinate $x \in [0, 1]$.

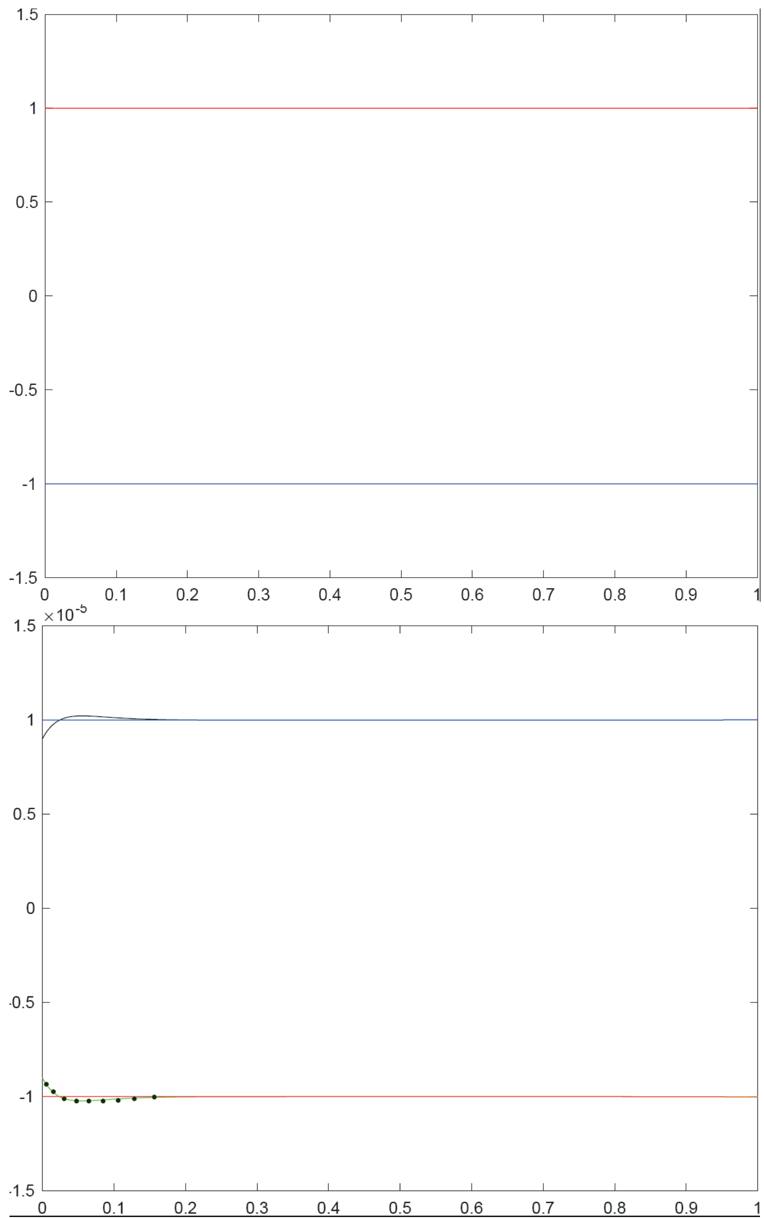

Figure 8. Optimal control of the two-link serial network of Example 5 with nonlinearity. Upper diagram: optimal states (with mass flow on the y-axis). Lower diagram: adjoints. x-axis: spatial coordinate $x \in [0, 1]$.

compressors that are present in the sub-graphs G_m and formulate the analogous optimal control problem on the sub-graph G_m:

$$
\begin{aligned}
\min_{q_i, p_i, v_{ij}, v_{ik}} \quad & I(q_i, p_i, v_i, v_j) := \frac{\kappa}{2} \sum_{i \in \mathcal{I}^m} \left(\|q_i - q_i^0\|^2 + \|p_i - p_i^0\|^2 \right) \\
& + \frac{1}{2\mu} \sum_{j \in \mathcal{J}^{\mathrm{M}}_{m,n}, A_{m,n}=1} \; \sum_{i \in \mathcal{I}_j} \left(v_{ij}^2 + (\mu p_i(n_j) - h_{ij})^2 \right) \\
\text{s.t.} \quad & \beta_i p_i(x) + \partial_x q_i(x) = f_i^1, \quad x \in (0, \ell_i), \ i \in \mathcal{I}^m, \\
& \alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i(x)) = f_i^2, \quad x \in (0, \ell_i), \ i \in \mathcal{I}^m, \\
& q_i(n_j) = u_j, \quad j \in \mathcal{J}^{\mathrm{S}}_m, \ i \in \mathcal{I}_j, \\
& p_i(n_j) = p_k(n_j), \quad j \in \mathcal{J}^{\mathrm{M},o}_m \setminus (\mathcal{J}_{m,\mathrm{c}} \cup \mathcal{J}_{m,\mathrm{v}}), \ i, k \in \mathcal{I}_j, \\
& \sum_{i \in \mathcal{I}_j} d_{ij} q_i(n_j) = 0, \quad j \in \mathcal{J}^{\mathrm{M},o}_m, \\
& s_j^{\mathrm{v}} \left(p_i(n_j) - p_k(n_j) \right) + (1 - s_j^{\mathrm{v}}) q_i(n_j) = 0, \\
& \qquad j \in \mathcal{J}_{\mathrm{v},m}, \ i \in \mathcal{I}_j^+, \ k \in \mathcal{I}_j^-, \\
& s_j^{\mathrm{c}} \left(p_i(n_j) u_j - p_k(n_j) \right) + (1 - s_j^{\mathrm{c}}) \left(p_i(n_j) - p_k(n_j) \right) = 0, \\
& \qquad j \in \mathcal{J}_{\mathrm{c},m}, \ i \in \mathcal{I}_j^+, \ k \in \mathcal{I}_j^-, \\
& - d_{ij} q_i(n_j) + \sigma p_i(n_j) = g_{kj} + v_{ij}, \\
& \qquad j \in \mathcal{J}^{\mathrm{M}}_{m,n}, \ n : A_{m,n} = 1, \ i, k \in \mathcal{I}_j.
\end{aligned}
\tag{25}
$$

Here, we omitted the iteration indices l for the sake of convenience. Note that the constraints of (25) are the same as in System 4, except for the case that we only consider flow boundary control here and that we add the virtual controls. The corresponding optimality conditions are given in System 6.

Let us remark the following. Problem (25) and the corresponding optimality System 6 on the sub-graph G_m are now completely decoupled from the analogous problems on all other sub-graphs G_n, $n \neq m$. This means that we can actually decompose the optimization problem given on the graph into a set of local optimization problems given on the sub-graphs.

EXAMPLE 6 *We continue with Example 3. The corresponding virtual control problem regarding the decomposition at the nodes n_1 and n_2, where the edges* 1 *and* 3 *as well as* 2 *and* 4 *meet at $x = 1$, respectively, is then given by*

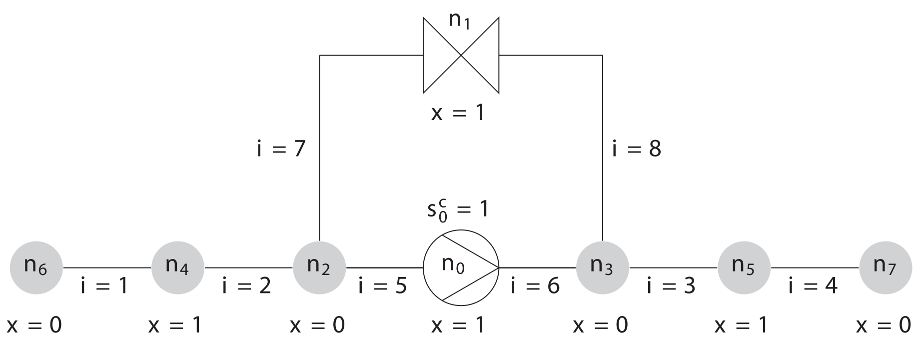

Figure 9. Network of Example 7

System 6. Optimality system of Problem (25); $x \in (0, \ell_i)$

$$
\begin{array}{rc}
\beta_i p_i(x) + \partial_x q_i(x) = f_i^1, & i \in \mathcal{I}^m \\
\alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i(x)) = f_i^2, & i \in \mathcal{I}^m \\
\beta_i \phi_i(x) - \partial_x \psi_i(x) = -\kappa_i (p_i - p_i^0), & i \in \mathcal{I}^m \\
\alpha_i \psi_i(x) - \partial_x \phi_i(x) + \partial_q g_i(x; q_i(x)) \phi_i = -\kappa_i (q_i - q_i^0), & i \in \mathcal{I}^m \\
q_i(n_j) = u_j, \psi_i(n_j) = 0, & j \in \mathcal{J}_m^{\mathrm{S}},\ i \in \mathcal{I}_j \\
p_i(n_j) = p_k(n_j), \phi_i(n_j) = \phi_k(n_j), & j \in \mathcal{J}_m^{\mathrm{M},o} \setminus (\mathcal{J}_{m,\mathrm{c}} \cup \mathcal{J}_{m,\mathrm{v}}), i, k \in \mathcal{I}_j \\
\sum_{i \in \mathcal{I}_j} d_{ij} q_i(n_j) = 0,\ \sum_{i \in \mathcal{I}_j} d_{ij} \psi_i(n_j) = 0, & j \in \mathcal{J}_m^{\mathrm{M},o} \\
s_j^{\mathrm{v}} \left(p_i(n_j) - p_k(n_j) \right) + (1 - s_j^{\mathrm{v}}) q_i(n_j) = 0, & j \in \mathcal{J}_{\mathrm{v},m},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \\
s_j^{\mathrm{v}} \left(\phi_i(n_j) - \phi_k(n_j) \right) + (1 - s_j^{\mathrm{v}}) \psi_i(n_j) = 0, & j \in \mathcal{J}_{\mathrm{v},m},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \\
s_j^{\mathrm{c}} \left(p_i(n_j) u_j - p_k(n_j) \right) + (1 - s_j^{\mathrm{c}}) \left(p_i(n_j) - p_k(n_j) \right) = 0, & j \in \mathcal{J}_{\mathrm{c},m},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \\
s_j^{\mathrm{c}} \left(\psi_i(n_j) u_j - \psi_k(n_j) \right) + (1 - s_j^{\mathrm{c}}) \left(\phi_i(n_j) - \phi_k(n_j) \right) = 0, & j \in \mathcal{J}_{\mathrm{c},m},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \\
-d_{ij} q_i(n_j) + \sigma \phi_i(n_j) + \mu p_i(n_j) = g_{kj}, & j \in \mathcal{J}_{m,n}^{\mathrm{M}},\ n : A_{m,n} = 1,\ i, k \in \mathcal{I}_j \\
d_{ij} \psi_i(n_j) + \sigma \phi_i(n_j) + \mu p_i(n_j) = h_{kj}, & j \in \mathcal{J}_{m,n}^{\mathrm{M}},\ n : A_{m,n} = 1,\ i, k \in \mathcal{I}_j \\
u_j = -\frac{1}{\nu} \phi_i(n_j), & j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j \\
u_j = -s_j^{\mathrm{c}} p_i(n_j) \psi_i(n_j), & j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j
\end{array}
$$

$$
\begin{aligned}
\min_{u,v} \quad & I((q_i,p_i)_{i=1}^4, v_{11}, v_{31}, v_{22}, v_{42}, u_0, u_3, u_4) := \\
& \sum_{i=1}^{4} \frac{\kappa}{2} \left(\|q_i - q_i^0\|^2 + \|p_i - p_i^0\|^2 \right) \\
& + \frac{1}{2\mu} \left(v_{11}^2 + (\mu p_1(1) - h_{11})^2 \right) + \frac{1}{2\mu} \left(v_{31}^2 + (\mu p_3(1) - h_{31})^2 \right) \\
& + \frac{1}{2\mu} \left(v_{22}^2 + (\mu p_2(1) - h_{22})^2 \right) + \frac{1}{2\mu} \left(v_{42}^2 + (\mu p_4(1) - h_{42})^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & \beta_i p_i(x) + \partial_x q_i(x) = f_i^1, \quad x \in (0, \ell_i),\ i = 1, \dots, 4, \\
& \alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i(x)) = f_i^2, \quad x \in (0, \ell_i),\ i = 1, \dots, 4, \\
& - q_1(1) + \sigma p_1(1) = g_{31}^{l+1} + v_{11}, \quad -q_3(1) + \sigma p_3(1) = g_{11}^{l+1} + v_{31} \\
& - q_2(1) + \sigma p_2(1) = g_{42}^{l+1} + v_{12}, \quad -q_4(1) + \sigma p_4(1) = g_{22}^{l+1} + v_{42}, \\
& p_1(0) u_0 = p_2(0), \quad q_1(0) + q_2(0) = 0, \quad q_3(0) = u_3, \quad q_4(0) = u_4.
\end{aligned}
$$

Example 7 *We expand the model of the last example by a valve parallel to the compressor; see Fig. 9.*

We have 8 edges and 8 nodes. Edge 1 has a simple flow-controlled node n_6 at $x = 0$. The edges $1, 2$ are coupled at node n_4, where $x = 1$. Similarly, edge 4 has a simple node at n_7, where $x = 0$, and is coupled to edge 3 at $x = 1$ via node n_5. These two serial links $(1, 2)$ and $(3, 4)$ are connected through nodes n_2, n_3 to edges $5, 7$ and $6, 8$ at $x = 0$, respectively. These are triple junctions. Finally, the compressor is located at $n_0 = n_{\mathrm{c}}$ between links $5, 6$ at $x = 1$, while the valve connects edges $7, 8$ at $n_1 = n_{\mathrm{v}}$, where $x = 1$. The model is given by

$$\beta_i p_i(x) + \partial_x q_i(x) = f_i^1, \tag{26a}$$

$$\alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i(x)) = f_i^2, \tag{26b}$$

$$p_1(1) = p_2(1),\ p_3(1) = p_4(1), \tag{26c}$$

$$q_1(1) + q_2(1) = 0,\ q_3(1) + q_4(1) = 0, \tag{26d}$$

$$p_2(0) = p_5(0) = p_7(0),\ p_6(0) = p_3(0) = p_8(0), \tag{26e}$$

$$q_2(0) + q_5(0) + q_7(0) = 0,\ q_6(0) + q_3(0) + q_8(0) = 0, \tag{26f}$$

$$p_5(1) u_0 = p_6(1),\ q_5(1) + q_6(1) = 0, \quad \text{if } s_0^{\mathrm{c}} = 1, \tag{26g}$$

$$p_5(1) = p_6(1),\ q_5(1) + q_6(1) = 0, \quad \text{if } s_0^{\mathrm{c}} = 0, \tag{26h}$$

$$p_7(1) = p_8(1),\ q_7(1) + q_8(1) = 0, \quad \text{if } s_1^{\mathrm{v}} = 1, \tag{26i}$$

$$q_7(1) = 0,\ q_8(1) = 0, \quad \text{if } s_1^{\mathrm{v}} = 0, \tag{26j}$$

$$q_1(0) = u_6,\ q_4(0) = u_7, \tag{26k}$$

where we again have $x \in (0, 1)$ and $i = 1, \dots, 8$. Constraints (26c) *and* (26d) *describe the serial nodes, where we will apply the domain decomposition. The* *transmission nodes n_2 and n_3 are described in* (26e) *and* (26f)*, the compressor's nodal conditions are given by* (26g) *and* (26h)*. Similarly,* (26i) *and* (26j) *are the valve conditions. Finally, the control and the demand are provided in* (26k)*. The corresponding virtual control problem is given by*

$$\begin{aligned}\min_{u,v}\quad & I((q_i,p_i,u,s)_{i=1}^8, v_{14}, v_{24}, v_{35}, v_{45}, u_0, u_6, u_7)\\ \text{s.t.}\quad & (26\text{a}), (26\text{b}), (26\text{e}), (26\text{f}), (26\text{g}), (26\text{h}), (26\text{i}), (26\text{j}), (26\text{k}),\\ & -q_1(1)+\sigma p_1(1) = g_{14}^{l+1} + v_{14}, \quad -q_2(1)+\sigma p_2(1) = g_{24}^{l+1} + v_{24},\\ & -q_3(1)+\sigma p_3(1) = g_{35}^{l+1} + v_{35}, \quad -q_4(1)+\sigma p_4(1) = g_{45}^{l+1} + v_{45},\end{aligned} \tag{27}$$

where the costs are similar to those of the previous example. In addition, these costs may involve the switching parameters s. Problem (27) *can be seen as a mixed-integer nonlinear program (MINLP) on the sub-graph consisting of the edges $2,3,5,6,7,8$ involving the compressor at node n_0 and the valve at node n_1 with Robin-data*

$$-q_2(1)+\sigma p_2(1) = g_{24}^{l+1} + v_{24} =: r_2^l, \quad -q_4(1)+\sigma p_4(1) = g_{45}^{l+1} + v_{45} =: r_4^l.$$

For each given l, the sub-graph problem admits a minimal solution w.r.t. both u and s. While the optimization w.r.t. the continuous variables results in the decomposed optimality system, a similar conclusion cannot be drawn for the discrete optimization part, as there is no such optimality system w.r.t. the switching variables. Moreover, there is no sensitivity analysis available for such problems. Therefore, even given the fact that the right-hand sides r_2^l, r_4^l converge, as $l \to \infty$, it may happen that the globally optimal switching changes infinitely often in the course of the convergence.

An analysis of the situation addressed at the end of the example and scenarios that avoid this Zeno-phenomenon are the subject of future research.

6. Wellposedness and convergence

6.1. Uniqueness of the primal problem's solution

For a given switching structure $s \in \mathcal{S}$, the flow boundary controlled problem

$$\begin{aligned} \beta_i p_i(x) + \partial_x q_i(x) &= f_i^1, \quad x \in (0,\ell_i),\ i \in \mathcal{I},\\ \alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i(x)) &= f_i^2, \quad x \in (0,\ell_i),\ i \in \mathcal{I},\\ q_i(n_j) &= u_j, \quad j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j,\\ p_i(n_j) &= p_k(n_j), \quad j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{c}} \cup \mathcal{J}_{\mathrm{v}}),\ i,k \in \mathcal{I}_j,\\ \sum_{i\in\mathcal{I}_j} d_{ij} q_i(n_j) &= 0, \quad j \in \mathcal{J}^{\mathrm{M}},\\ s_j^{\mathrm{v}}(p_i(n_j) - p_k(n_j)) + (1-s_j^{\mathrm{v}}) q_i(n_j) &= 0, \quad j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-,\\ s_j^{\mathrm{c}}(p_i(n_j) u_j - p_k(n_j))&\\ +(1-s_j^{\mathrm{c}})(p_i(n_j) - p_k(n_j)) &= 0, \quad j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^- \end{aligned} \tag{28}$$

on G admits a unique solution. In order to prove this, we introduce the first-order differential expression

$$\mathcal{A}(p,q) := \begin{pmatrix} \partial_x q_i \\ \partial_x p_i \end{pmatrix}_{i\in\mathcal{I}}.$$

For defining a proper differential operator, we introduce the spaces

$$\begin{aligned} H &:= \{(p,q) : (p,q) = (p_i,q_i)_{i\in\mathcal{I}} \in \Pi_{i\in\mathcal{I}} L^2(0,\ell_i)^2\}, \\ H^1 &:= H \cap \Pi_{i\in\mathcal{I}} H^1(0,\ell_i)^2, \\ D(\mathcal{A}) &:= \{(p,q) = (p_i,q_i)_{i\in\mathcal{I}} \in H^1 : q_i(n_j) = 0,\ j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j, \\ &\qquad p_i(n_j) = p_k(n_j),\ j \in \mathcal{J}^{\mathrm{M}},\ i,k \in \mathcal{I}_j, \\ &\qquad \sum_{i\in\mathcal{I}_j} d_{ij} q_i(n_j) = 0,\ j \in \mathcal{J}^{\mathrm{M}}\}. \end{aligned} \tag{29}$$

Here we have taken the situation without valves and compressors. For an open valve and a shut-down compressor, we have the canonical pressure and flow transmission conditions as in the definition above. If the valve is closed, we have two extra no-flow conditions at the valve node. If the compressor is switched on, we have a pressure transmission condition involving the control u_j. For a given pressure ratio u_j, the corresponding transmission can be integrated into the domain $D(\mathcal{A})$, otherwise the bilinear term has to be taken into account via shifting it into the state equation. The norm in H is given by

$$\|(p,q)\|_H^2 := \langle (p,q),(p,q)\rangle := \sum_{i\in\mathcal{I}} \int_0^{\ell_i} \left(p_i^2 + q_i^2\right) \mathrm{d}x.$$

Obviously, H is a Hilbert space and we have the dense inclusion $D(\mathcal{A}) \subset H$. A simple calculation shows

$$\langle \mathcal{A}(p,q),(p,q)\rangle = 0,$$

and that, in fact, $\mathcal{A}$ is skew-adjoint. Then, clearly, with $D_i := \operatorname{diag}(\beta_i,\alpha_i)$, $D+\mathcal{A}$ has a bounded inverse on H. Now, the Nemytskii operator $N : H^1 \to H$ with $N(p,q)_i(x) := (0, \gamma_i(x)|q_i(x)|q_i(x))^\top$, $i \in \mathcal{I}$, is compact, as the embedding (in 1d) of $H^1(0,\ell_i) \to L^4(0,\ell_i)$ is compact (and monotone on H). This implies that the equation

$$(D+\mathcal{A}+N)(p,q) = F$$

admits a unique solution for $F \in H$. The same arguments apply for the problems on a sub-graph G_m:

$$
\begin{aligned}
\beta_i p_i(x) + \partial_x q_i(x) &= f_i^1, \quad x \in (0,\ell_i),\ i \in \mathcal{I}^m,\\
\alpha_i q_i(x) + \partial_x p_i(x) + g_i(x; q_i) &= f_i^2, \quad x \in (0,\ell_i),\ i \in \mathcal{I}^m,\\
q_i(n_j) &= u_j, \quad j \in \mathcal{J}_m^{\mathrm{S}},\ i \in \mathcal{I}_j,\\
p_i(n_j) &= p_k(n_j), \quad j \in \mathcal{J}_m^{\mathrm{M},o} \setminus (\mathcal{J}_{\mathrm{c},m} \cup \mathcal{J}_{\mathrm{v},m}),\ i,k \in \mathcal{I}_j,\\
\sum_{i\in\mathcal{I}_j} d_{ij} q_i(n_j) &= 0, \quad j \in \mathcal{J}_m^{\mathrm{M},o},\\
s_j^{\mathrm{v}}(p_i(n_j) - p_k^{l+1}(n_j)) + (1-s_j^{\mathrm{v}}) q_i(n_j) &= 0, \quad j \in \mathcal{J}_{\mathrm{v},m},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-,\\
s_j^{\mathrm{c}}(p_i(n_j) u_j - p_k(n_j))&\\
+(1-s_j^{\mathrm{c}})(p_i(n_j) - p_k(n_j)) &= 0, \quad j \in \mathcal{J}_{\mathrm{c},m},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-,\\
-d_{ij} q_i(n_j) + \sigma p_i(n_j) &= g_{kj} + v_{ij}, \quad j \in \mathcal{J}_{m,n}^{\mathrm{M}},\ n : A_{m,n} = 1,\ i,k \in \mathcal{I}_j.
\end{aligned}
$$

Moreover, we may also apply the same methods in order to show that the corresponding optimality systems admit a unique solution. We skip the details here.

6.2. Smoothness of the control-to-state-map

Let $\hat{q}_t(\hat{u}), \hat{p}_t(\hat{u})$ be the solution of Problem (28) with u replaced by $u + t\hat{u}$ and let q, p be the solution of (28) at $t = 0$. We denote by $\tilde{q} = \hat{q}_t - q, \tilde{p} = \hat{p}_t - p$ the differences of these solutions and obtain

$$
\begin{aligned}
\beta_i \tilde{p}_i(x) + \partial_x \tilde{q}_i(x) &= 0, \quad x \in (0,\ell_i),\ i \in \mathcal{I},\\
\alpha_i \tilde{q}_i(x) + \partial_x \tilde{p}_i(x) + g_i(x; \tilde{q}_i + q_i) - g_i(q_i) &= 0, \quad x \in (0,\ell_i),\ i \in \mathcal{I},\\
\tilde{q}_i(n_j) &= t\hat{u}, \quad j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j,\\
\tilde{p}_i(n_j) &= \tilde{p}_k(n_j), \quad j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{v}} \cup \mathcal{J}_{\mathrm{c}}),\ i,k \in \mathcal{I}_j,\\
\sum_{i\in\mathcal{I}_j} d_{ij} \tilde{q}_i(n_j) &= 0, \quad j \in \mathcal{J}^{\mathrm{M}},\\
s_j^{\mathrm{v}}(\tilde{p}_i(n_j) - \tilde{p}_k(n_j)) + (1-s_j^{\mathrm{v}}) \tilde{q}_i(n_j) &= 0, \quad j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-,\\
s_j^{\mathrm{c}}(\tilde{p}_i(n_j)(u_j + t\hat{u}_j) + \tilde{p}_i(n_j) t\hat{u}_j - \tilde{p}_k(n_j))&\\
+(1-s_j^{\mathrm{c}})(\tilde{p}_i(n_j) - \tilde{p}_k(n_j)) &= 0, \quad j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-.
\end{aligned}
\tag{30}
$$

Dividing by t and letting $t \to 0$, we arrive at the sensitivity problem

$$\begin{aligned}
\beta_i p_i'(x) + \partial_x q_i'(x) = 0, &\quad x \in (0, \ell_i),\ i \in \mathcal{I}, \\
\alpha_i q_i'(x) + \partial_x p_i'(x) + g_i'(x; q_i) q_i' = 0, &\quad x \in (0, \ell_i),\ i \in \mathcal{I}, \\
q_i'(n_j) = \hat{u}, &\quad j \in \mathcal{J}^{\mathrm{S}},\ i \in \mathcal{I}_j, \\
p_i'(n_j) = p_k'(n_j), &\quad j \in \mathcal{J}^{\mathrm{M}} \setminus (\mathcal{J}_{\mathrm{v}} \cup \mathcal{J}_{\mathrm{c}}),\ i, k \in \mathcal{I}_j, \\
\sum_{i \in \mathcal{I}_j} d_{ij} q_i'(n_j) = 0, &\quad j \in \mathcal{J}^{\mathrm{M}}, \\
s_j^{\mathrm{v}}(p_i'(n_j) - p_k'(n_j)) + (1 - s_j^{\mathrm{v}}) q_i'(n_j) = 0, &\quad j \in \mathcal{J}_{\mathrm{v}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-, \\
s_j^{\mathrm{c}}(p_i'(n_j) u_j + p_i(n_j) \hat{u}_j - p_k'(n_j)) & \\
+ (1 - s_j^{\mathrm{c}})(p_i'(n_j) - p_k'(n_j)) = 0, &\quad j \in \mathcal{J}_{\mathrm{c}},\ i \in \mathcal{I}_j^+,\ k \in \mathcal{I}_j^-.
\end{aligned} \tag{31}$$

For the solution q', p' of (31), we may apply standard techniques. As the cost function in (7) is convex, Problem (7) admits a unique solution according to the classical Weierstraass theorem. One can then verify the conditions for the Ioffe–Tichomirov theorem, see Kogut and Leugering (2011), in order to establish the first-order optimality conditions (5). The following theorem summarizes the previous assertions.

Theorem 3 *Under the assumption* (18)*, for* $(f_i^1, f_i^2)_{i \in \mathcal{I}} \in \Pi_{i \in \mathcal{I}} L^2(0, \ell_i)^2$*, there exists a unique solution* $(q, p) \in D(\mathcal{A})$ *of* (30)*. In addition, the mapping from* u *into* q, p *is Gateaux differentiable. Moreover, the optimal control problem* (7) *admits a unique solution. The optimal solution is characterized by the optimality system of first-order* (5)*.*

6.3. Convergence

For the proof of convergence, we concentrate on the decomposition at a serial node. The decomposition of the problem on a graph into separate problems on sub-graphs then follows as described above. To this end, we introduce the errors $\bar{p}^{l+1} := p^{l+1} - p$, $\bar{q}^{l+1} := q^{l+1} - q$, and, accordingly, $\hat{g}^{l+1} := g^{l+1} - g$, which is to be understood in the vectorial sense. We consider serial nodes n_j with adjacent edges $i,k \in \mathcal{I}_j$ and obtain

$$
\begin{aligned}
\beta_i \bar{p}_i^{l+1}(x) + \partial_x \bar{q}_i^{l+1}(x) &= 0, & x \in (0,\ell_i),\ i \in \mathcal{I},\\
\alpha_i \bar{q}_i^{l+1}(x) + \partial_x \bar{p}_i^{l+1}(x) + (g_i(x;\bar{q}_i^{l+1} + q_i) - g_i(x;q_i) &= 0, & x \in (0,\ell_i),\ i \in \mathcal{I},\\
\beta_i \bar{\phi}_i^{l+1}(x) - \partial_x \bar{\psi}_i^{l+1}(x) &= -\kappa_i(\bar{p}_i^{l+1}), & x \in (0,\ell_i),\ i \in \mathcal{I},\\
\alpha_i \bar{\psi}_i^{l+1}(x) - \partial_x \bar{\phi}_i^{l+1}(x) + \partial_q g_i(x;\bar{q}_i^{l+1} + q_i)\psi_i^{l+1} & & \\
+(\partial_q g_i(x;\bar{q}_i^{l+1} - \partial_q g_i(x;q_i))p_i &= -\kappa_i(q_i^{l+1}), & x \in (0,\ell_i),\ i \in \mathcal{I},\\
-d_{ij}\bar{q}_i^{l+1}(n_j) + \lambda \bar{p}_i^{l+1}(n_j) - \mu \bar{\phi}_i^{l+1}(n_j) &= \bar{g}_{kj}^{l+1}, & i,k \in \mathcal{I}_j,\\
d_{ij}\bar{\psi}_i^{l+1}(n_j) + \lambda \bar{\phi}_i^{l+1}(n_j) + \mu \bar{p}_i^{l+1}(n_j) &= \bar{h}_{kj}^{l+1}, & i,k \in \mathcal{I}_j,\\
\bar{g}_{kj}^{l+1} &= d_{kj}\bar{q}_k^{l}(n_j)) + \lambda \bar{p}_k^{l}(n_j) - \mu \bar{\phi}_k^{l}(n_j), & i,k \in \mathcal{I}_j,\\
\bar{h}_{kj}^{l+1} &= -d_{kj}\bar{\psi}_k^{l}(n_j) + \lambda \bar{\phi}_k^{l}(n_j) + \mu \bar{p}_k^{l}(n_j), & i,k \in \mathcal{I}_j.
\end{aligned}
\tag{32}
$$

Moreover, we define

$$
(\bar{g},\bar{h}) \in \mathcal{X} := \Pi_{k\in\mathcal{J}^{\mathrm{M}}}\Pi_{i\in\mathcal{I}_k}\mathbb{R}^2, \quad \|(\bar{g},\bar{h})\|_{\mathcal{X}}^2 := \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}_k}\left(|\bar{g}_{ik}|^2 + |\bar{h}_{ik}|^2\right)
$$

and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ with

$$
(\mathcal{T}(\bar{g},\bar{h})_{i,j} := \left(2(\lambda \bar{p}_k^l(n_j) - \mu\bar{\phi}_k(n_j)) - \bar{g}_{kj}^l, 2(\lambda\bar{\phi}_k(n_j)^l + \mu\bar{p}_k(n_j)^l) - \bar{h}_{kj}^l\right)
$$

for $i,j \in \mathcal{I}_k$. Now,

$$
\begin{aligned}
\|\mathcal{T}(\bar{g},\bar{h})\|_{\mathcal{X}}^2 &= \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left(|\bar{g}_{ik}^{l+1}|^2 + |\bar{h}_{ik}^{l+1}|^2\right)\\
&= \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left((\bar{g}_{ik}^l)^2 - 4d_{ik}\bar{q}_i(n_k)^l\left(\lambda\bar{p}_i(n_k) - \mu\bar{\phi}_i^l(n_k)\right)\right)\\
&\quad + \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left((\bar{h}_{ik}^l)^2 + 4d_{ik}\bar{\psi}_i^l(n_k)\left(\lambda\bar{\phi}_i^l(n_k) + \mu\bar{p}_i^l(n_k)\right)\right)
\end{aligned}
$$

holds. We multiply the state equation for the errors p_i, q_i by ϕ_i and ψ_i, respectively, and perform summations and integration by parts in order to obtain

$$
\begin{aligned}
\|\mathcal{T}(\bar{g},\bar{h})\|_{\mathcal{X}}^2 \quad &= \textstyle\sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left(|\bar{g}_{ik}^{l+1}|^2 + |\bar{h}_{ik}^{l+1}|^2\right) = \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left(|\bar{g}_{ik}^{l}|^2 + |\bar{h}_{ik}^{l}|^2\right)\\
&= -4\lambda\textstyle\sum_{i\in\mathcal{I}}\int_0^{\ell_i}\Big(\beta_i((\bar{p}_i^l)^2 + (\phi_i^l)^2) + \alpha_i(\bar{q}_i^l)^2 + (\bar{\psi}_i^l)^2) + \partial_q g_i(x;\bar{q}_i + q_i)\bar{\psi}_i^2\\
&\qquad +(\partial_q g_i(x;\bar{q}_i + q_i) - \partial_q g_i(x;q_i))\bar{\psi}_i\psi + (g_i(x;\bar{q}_i + q_i) - g_i(x;q_i))\bar{q}_i\\
&\qquad +\kappa(\bar{q}_i\bar{\psi}_i + \bar{\phi}_i\bar{p}_i\Big)\,\mathrm{d}x\\
&\quad -4\mu\textstyle\sum_{i\in\mathcal{I}}\int_0^{\ell_i}\Big(\kappa((\bar{p}_i)^2 + (\bar{q}_i)^2) + \partial_q g_i(x;\bar{q}_i + q_i)\bar{q}_i\bar{\psi}_i\\
&\qquad +(\partial_q g_i(x;\bar{q}_i + q_i) - \partial_q g_i(x;q_i))\psi\bar{q}_i\\
&\qquad -(g_i(x;\bar{q}_i + q_i) - g_i(x;q_i))\bar{\psi}_i\Big)\,\mathrm{d}x.
\end{aligned}
$$

Now,

1. $\partial_q g_i(x;s) \geq |s|$,
2. $g_i(x;\bar{q}_i+q_i) - g_i(x;q_i) = g_i(x;\theta\bar{q}_i+q_i)\bar{q}_i$,
3. $(\partial_q g_i(x;\bar{q}_i+q_i) - (\partial_q g_i(x;\bar{q}_i+q_i) - g_i(x;q_i))\bar{\psi} \leq |(1-\theta)\bar{q}_i|\bar{q}_i\bar{\psi}_i$,
4. $\partial_q g_i(x;\bar{q}_i+q_i) - \partial_q g_i(x;q_i)) \leq \|\gamma_i\||\bar{q}_i| =: L_i|\bar{q}_i|$

hold and with these statements we can estimate

$$
\begin{aligned}
&\|\mathcal{T}(\bar{g},\bar{h})\|_{\mathcal{X}}^2 \\
&= \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left(|\bar{g}_{ik}^{l+1}|^2 + |\bar{h}_{ik}^{l+1}|^2\right) = \sum_{k\in\mathcal{J}^{\mathrm{M}}}\sum_{i\in\mathcal{I}}\left(|\bar{g}_{ik}^{l}|^2 + |\bar{h}_{ik}^{l}|^2\right) \\
&= -4\lambda\sum_{i\in\mathcal{I}}\int_0^{\ell_i}\Bigg((\lambda_i\alpha_i + (\mu - \frac{1}{2}\lambda)\kappa + \lambda\partial_q g_i(x;\theta\bar{q}_i+q_i) \\
&\qquad -L_i((\mu+\frac{1}{2})|\psi_i| + \mu|\bar{\psi}_i|)|\bar{q}_i|^2 + (\lambda\beta_i + (\mu - \frac{1}{2}\lambda)\kappa)|\bar{p}_i|^2 \\
&\qquad +\lambda\beta_i|\bar{\phi}_i|^2 + (\lambda\alpha_i - \frac{1}{2}\lambda\kappa + \lambda\partial_q g_i(x;\bar{q}_i+q_i) - L_i\frac{1}{2}|\psi|)|\psi|_i^2\Bigg)\,\mathrm{d}x.
\end{aligned}
\tag{33}
$$

This estimate has to be adjusted in a straightforward way for the case of a decomposition at a boundary control node. It is obvious from (33) that for sufficiently large α_i, β_i, the coefficients of the quantities with $\bar{p}_i^2, \bar{q}_i^2, \bar{\phi}_i^2, \bar{\psi}_i^2$ in (33) can be made uniformly positive. It is also evident from (33) that the choice of the parameters $\alpha_i, \beta_i, \kappa$ will depend on the flows q_i, ψ_i (in case $\mu = 0$) and additionally on the errors $\bar{q}_i, \bar{\psi}_i$ if $\lambda, \mu > 0$. This means that the convergence is local.

Theorem 4 *Under the positivity assumptions for the coefficients and the monotonicity assumption* (18)*, the iterations converge and the solutions* $q^l = (q_i^l, p_i^l)_{i\in\mathcal{I}}$ *of the iterative process* (32)*, describing the local optimality systems on the individual edges, converge to the solution of the optimality system* (24)*. The convergence takes place in the* H^1*-sense. Moreover, the traces at the decomposition nodes also converge.*

We finally remark that the same theorem applies to the nonoverlapping domain decomposition using sub-graphs.

7. Conclusion

In this paper, we first reduced the original time-dependent optimal control problem (2) for the gas flow in a given network via a semi-implicit-explicit time discretization scheme first to (5) and then to an optimal control problem for a single time step (7). The latter problem has to be solved in an instantaneous control paradigm. We designed an iterative nonoverlapping domain decomposition at multiple nodes in the spirit of Lagnese and Leugering (2004) and Leugering (2017) in order to decompose both the system of equations on the entire network and the optimality System 6 to suitable sub-networks, containing valves and compressors. As a result, the iterations converge in natural norms and, moreover, for the optimality system, the decomposed systems are, in fact, optimality systems for virtual optimal control problems (25) on the corresponding sub-networks. We provide numerical evidence for both iterations, i.e., for the solutions to the system on the network and for the optimal solutions together with their adjoints, respectively. The results pave the way for MINLP solution techniques for problems involving the on-off-control of valves and compressors in combination with continuous controls at simple nodes and, e.g., continuous compressor controls. By using the proposed domain decomposition method, the size and the complexity of the MINLP problems to solve can now be controlled. Besides this, the method developed here provides a completely parallel treatment of the considered optimal control problems.

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