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NONLINEAR FREE VIBRATION ANALYSIS OF MICRO-BEAMS RESTING ON VISCOELASTIC FOUNDATION BASED ON THE MODIFIED COUPLE STRESS THEORY

In this paper, nonlinear free vibration analysis of micro-beams resting on the viscoelastic foundation is investigated by the use of the modified couple stress theory, which is able to capture the size effects for structures in micron and sub-micron scales. To this aim, the gov-erning equation of motion and the boundary conditions are derived using the Euler–Bernoulli beam and the Hamilton's principle. The Galerkin method is employed to solve the governing nonlinear differential equation and obtain the frequency-amplitude algebraic equation. Final-ly, the effects of different parameters, such as the mode number, aspect ratio of length to height, the normalized length scale parameter and foundation parameters on the natural fre-quency-amplitude curves of doubly simply supported beams are studied.

1. Introduction

Micro-beams play an important role in micro and nano-electromechanical systems (MEMs and NEMs) e.g., biosensors, micro-resonators, Atomic Force Microscopes (AFMs) and actuators [1–4]. Some experiments [5, 6] were accomplished on the size-dependent mechanical behavior of micro-scale structures. The results of these tests are not in agreement with real mechanical design parameters of micro structures, such as deflection, natural frequencies and buckling load, predicted by formulations obtained using the classical continuum theory. To interpret these size effects, the length scale parameters are required to exist in constitutive relations which are just assumed in non-classical (higher-order) elasticity theories. Due to this weakness of the classical continuum theory to capture the experimentally-detected small-scale effects in the size dependent behavior of structures, various

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non-classical theories, such as the nonlocal [7], strain gradient [8], and couple stress [9], were developed to eliminate the shortcoming in dealing with micro-structures.

In early 1960s, Toupin [9] introduced the couple stress theory in which higher-order stresses, known as the couple stress tensor, were taken into account, besides the classical force stress tensor. Yang et al. [10] suggested a modified couple stress theory in which a new higher-order equilibrium equation, i.e., the moment equilibrium equation of couple stresses, was considered, as well as the classical equilibrium equations. This consideration causes the symmetry of the couple stress tensor. In recent years, vast studies including the static, dynamic, and thermal analyses have been accomplished on micro–structures (for instance, see these studies based on the nonlocal [11, 12], strain gradient [13, 14], and non-Fourier heat conduction theories, [15, 16]).

In what follows, some works are mentioned that investigated the size dependent mechanical behavior of micro-beams using modified couple stress theory. In this regards, Ke et al. [17, 18] studied the size effect on the dynamic stability of functionally-graded micro beams, and nonlinear vibration behaviors of micro-beams, respectively. Park and Gao [19], and Kong et al. [20] studied the static and vibration analysis of size-dependent Bernoulli–Euler beams. Taati et al. [21] developed a formulation for static behavior of the viscoelastic Euler-Bernoulli micro-beams. Ma et al. [22] presented a microstructure-dependent Timoshenko beam model, which can be used to obtain the static and free vibration parameters of the simply supported micro-beams. Ke and Wang [17] studied the dynamic stability of FG micro-beams based on the modified couple stress theory. Asghari and Taati [23] developed a size-dependent formulation for mechanical analyses of FG micro-plates based on the modified couple stress theory. The plate properties can arbitrarily vary through the thickness. Moreover, the boundary conditions were provided at smooth parts of the plate periphery and also at the sharp. Reddy and Kim [24] formulated a general third-order model of FG plates with microstructure-dependent length scale parameter and the von Kármán nonlinearity. This model accounted for temperature dependent properties of the constituents in the functionally graded material. Taati et al. [25] developed a size-dependent, explicit formulation for coupled thermoelasticity addressing a Timoshenko microbeam. This novel model combines modified couple stresses and non-Fourier heat conduction to capture size effects in the micro-scale. Thai and Choi [26] presented size dependent models for bending, buckling, and vibration of functionally graded Kirchhoff and Mindlin plates utilizing a modified couple stress theory. The numerical results showed that the small scale effect leads to a reduction in the magnitude of deflection. Taati [27] obtained analytical solutions for the buckling and post-buckling analysis of FG micro-plates under different kinds of traction on the edges by the modified couple stress theory. The static equilibrium equations of an FG rectangular microplate, as well as the boundary conditions, were derived using the principle of minimum total potential energy. Eltaher et al. [28] studied vibration behavior of a nonlocal Euler–Bernoulli beam by employing finite element method. They

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investigated the effects of nonlocal parameter, slenderness ratios, rotator inertia, and boundary conditions on the natural frequencies of the beam. Akgöz et al. [29] presented bending analysis of FG microbeams embedded in an elastic medium based on modified strain gradient elasticity theory in conjunctions with various beam theories. Togun et al. [30] analyzed nonlinear free and forced vibration of a nanobeam on a Pasternak elastic foundation based on non-local Euler-Bernoulli beam theory. Civalek and Akgöz [31] developed a size-dependent beam model on the basis of hyperbolic shear deformation beam and modified strain gradient theory. They provided the analytical solutions for the static bending and buckling loads of simply supported microbeams embedded in an elastic medium. Civalek and Akgöz [32] examined a microstructure-dependent trigonometric beam model for buckling of microbeams using modified strain gradient theory. This model is able to take into consideration size and shear deformation effects. Shafiei et al. [33] solved the nonlinear size-dependent governing equations on vibration of a non-uniform axially functionally graded (AFG) microbeam. Euler–Bernoulli beam theory, the modified couple stress theory and von-Kármán's geometric nonlinearity were assumed in this study. Ansari et al. [34] investigated nonlinear vibration analysis of Timoshenko nanobeams with different types of end conditions based on surface stress elasticity theory. A numerical method was applied to solve the problem in which the generalized differential quadrature method was used to discretize the governing equations and boundary conditions. To the best of authors' knowledge, no study has been reported to deal with nonlinear free vibration analysis of micro-beams resting on the viscoelastic foundation using nonclassical continuum theory. This paper tries to fulfill the gap in the open literature by deriving the governing equation of motion and the boundary conditions employing the Euler–Bernoulli beam and the Hamilton's principle. Furthermore, the Galerkin method is utilized to solve the governing nonlinear differential equation and obtain the frequency-amplitude algebraic equation. Eventually, the effects of various parameters such as the mode number, aspect ratio of length to height, the normalized length scale parameter and foundation parameters on the natural frequency-amplitude curves of doubly simply supported beams are investigated.

2. Preliminary problem definition

Consider a micro-beam as shown in Fig. 1 with simply supported boundary conditions at its two ends, which is rest on a viscoelastic foundation. The coordinate system and geometric specifications of the micro-beam, including length L, and the rectangular cross-section with width b, and thickness of h, is shown. Here, c indicates the viscosity coefficient of the viscoelastic foundation.

The modified couple stress theory developed by Yang et al. [8] is employed to present formulations. This theory is derived from the classical couple stress theory [7], which has been well established by some researchers.

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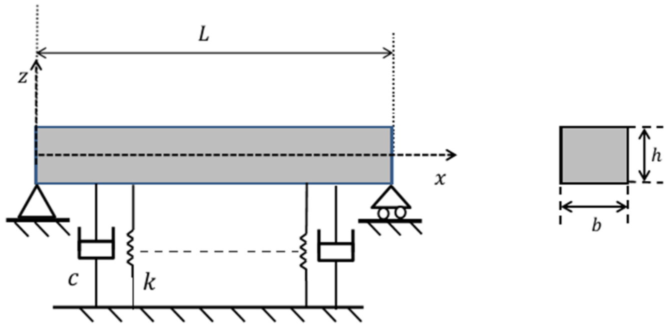

Fig. 1. Coordinate system, and geometric dimensions of the micro-beam

Based on the theory, an additional equilibrium equation is considered for the moments of the couple, which causes the couple stress tensor to be symmetric. Moreover, the strain energy density function is only dependent on the strain and the symmetric part of the curvature tensor, and hence, only one length scale parameter is involved in the constitutive relations. According to the theory, the variation of the strain energy for an anisotropic linear elastic material occupying region can be written as [8]:

$$\delta U = \int_{\Omega} (\sigma_{ij}\delta\varepsilon_{ij} + m_{ij}\delta\chi_{ij})\mathrm{d}\Omega. \tag{1}$$

In Eq. (1), ε_{ij} and χ_{ij} denote the components of the strain tensor $\boldsymbol{\varepsilon}$, and the symmetric part of the curvature tensor $\boldsymbol{\chi}$, which are defined as:

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\ \chi_{ij} &= \frac{1}{2}\left(\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i}\right). \end{aligned} \tag{2}$$

Also, the components of the infinitesimal rotation vector $\boldsymbol{\theta} = \frac{1}{2}curl(\mathbf{u})$ are introduced by θ_i. For linear isotropic elastic materials, constitutive relations of the symmetric part of the force stress and the deviatoric part of the couple stress tensor with the kinematic parameters are given as [8]:

$$\begin{aligned} \sigma_{ij} &= \lambda tr(\varepsilon)\delta_{ij} + 2\mu\varepsilon_{ij}, \\ m_{ij} &= 2\mu l^2 \chi_{ij}, \end{aligned} \tag{3}$$

where σ_{ij} and m_{ij} are called the force and higher-order stresses, respectively. Furthermore, the parameters λ and μ in the constitutive equation of the classical stress σ are Lame constants. The parameter l, which appears in the constitutive Eq. (4), is the material length scale parameter. It should be noticed that the Lame

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constants can be represented in terms of the Young's modulus E, and Poisson's ratio ν as $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \dfrac{E}{2(1+\nu)}$.

3. Governing dynamic equilibrium equations

Based on Euler–Bernoulli beam model, components of the displacement vector field can be stated as:

$$\begin{aligned} u_1 &= u(x,t) - z\frac{\partial w(x,t)}{\partial x}, \\ u_2 &= 0, \\ u_3 &= w(x,t), \end{aligned} \tag{4}$$

where, $u(x,t)$ function indicates the in-plane displacement of the particles on the mid-plane of the beam, which is perpendicular to the e_3 direction. This mid-plane is usually called the bending plate. The side cross-sections of the beam which are under pure bending, just only have rotation around lines on the bending-plane. Parameter Z indicates the distance of each point from the bending-plate (mid-plane). Also, it is necessary to state that parameter t indicates time.

By substitution of Eq. (4) into Eq. (2), the non-zero component of strain is calculated, as follows:

$$\varepsilon_{11} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}. \tag{5}$$

Regarding the fact that the analysis is non-linear, therefore non-linear strain terms must be considered. Considering the boundary conditions of the beam and its applications, in most analysis researches, the non-linear von Kármán term is used. Therefore, we have:

$$\varepsilon_{11} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \tag{6}$$

Substitution of Eq. (4) into equation $\boldsymbol{\theta} = \frac{1}{2}curl(\mathbf{u})$ yields non-zero components of rotation vector, as follows:

$$\theta_2 = -\frac{\partial w}{\partial x}. \tag{7}$$

As a result, substitution of rotation component from equation $\boldsymbol{\theta} = \frac{1}{2}curl(\mathbf{u})$ into relation (2) delivers the non-zero component of curvature, as:

$$\chi_{12} = \chi_{21} = -\frac{1}{2}\frac{\partial^2 w}{\partial x^2}. \tag{8}$$

Now, by substituting the non-zero components of strain and curvature from relations (6) and (5) into the introduced constitutive Eq. of (3), the non-zero stress

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components are achieved as follows:

$$\sigma_{11} = E\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right),$$
$$m_{12} = m_{21} = -\mu l^2 \frac{\partial^2 w}{\partial x^2}. \tag{9}$$

Using the relation for variations of strain energy based on the modified couple stress for linear elastic materials is stated in relation (1), and the relations for the components of the strain and curvature, and variation of strain energy are stated as follows:

$$\delta U = \int_0^L \left\{ N_{11}\delta\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) - \left(M_{11} + M_{12}^m\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right\} \mathrm{d}x. \tag{10}$$

Force and moment resultants of stress in the above relation are defined as:

$$N_{11} = \int_A \sigma_{11}\mathrm{d}z = EA\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right),$$
$$M_{11} = \int_A \sigma_{11} z\mathrm{d}z = -EI\left(\frac{\partial^2 w}{\partial x^2}\right), \tag{11}$$
$$M_{12}^m = \int_A m_{12}\mathrm{d}z = -\mu l^2 A\left(\frac{\partial^2 w}{\partial x^2}\right),$$

where, in relation (11), A and I indicate the area of the cross-section and moment of inertia of the beam, respectively, and are calculated as follows:

$$A = \int_{-h/2}^{h/2} b\mathrm{d}z = bh, \qquad I = \int_A z^2\mathrm{d}A = \frac{bh^3}{12}. \tag{12}$$

By taking part by part integration on Eq. (10), variation of strain energy is achieved as follows:

$$\delta U = \int_0^L \left\{ -\left(\frac{\partial N_{11}}{\partial x}\right)\delta u(x,t) - \left(\frac{\partial}{\partial x}\left(N_{11}\frac{\partial w}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(M_{11} + M_{12}^m\right)\right)\delta w(x,t)\right\}\mathrm{d}x$$
$$+ \left(N_{11}\delta u(x,t)\right)_{x=0}^{x=L} + \left(\left(N_{11}\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(M_{11} + M_{12}^m\right)\right)\delta(w(x,t))\right)_{x=0}^{x=L} \tag{13}$$
$$- \left(\left(M_{11} + M_{12}^m\right)\delta\left(\frac{\partial w(x,t)}{\partial x}\right)\right)_{x=0}^{x=L}.$$

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Kinetic energy of the beam is also achieved as:

$$\begin{aligned} T =& \frac{1}{2}\int_0^L \int_A \rho\left((\dot{u}_1)^2 + (\dot{u}_2)^2 + (\dot{u}_3)^2\right) \mathrm{d}A\mathrm{d}x \\ =& \frac{1}{2}\int_0^L \rho\left(A(\dot{u}(x,t))^2 + I\left(\frac{\partial \dot{w}(x,t)}{\partial x}\right)^2 + A(\dot{w}(x,t))^2\right)\mathrm{d}x. \end{aligned} \tag{14}$$

Here, ρ indicates the density of the material, and as also stated earlier, I and A indicate the moment of inertia and area of the cross-section of the beam, respectively. Regarding relation (14), variation of the kinetic energy is seen as follows:

$$\begin{aligned} \delta T =& \int_0^L \rho\Bigg(A\dot{u}(x,t)\delta\dot{u}(x,t) + I\left(\frac{\partial \dot{w}(x,t)}{\partial x}\right)\delta\left(\frac{\partial \dot{w}(x,t)}{\partial x}\right) \\ &+ A\dot{w}(x,t)\delta\dot{w}(x,t)\Bigg)\mathrm{d}x, \end{aligned} \tag{15}$$

where, by taking part by part integration on the time variable of t in Eq. (15), it can be rewritten as follows:

$$\begin{aligned} \delta T =& \int_0^L \rho\Bigg[A\left(-\ddot{u}(x,t)\delta u(x,t) + \frac{\partial}{\partial x}\left(\dot{u}(x,t)\delta u(x,t)\right)\right) \\ &+ I\left(-\left(\frac{\partial \ddot{w}(x,t)}{\partial x}\right)\delta\left(\frac{\partial w(x,t)}{\partial x}\right) + \frac{\partial}{\partial t}\left(\left(\frac{\partial \dot{w}(x,t)}{\partial x}\right)\delta\left(\frac{\partial w(x,t)}{\partial x}\right)\right)\right) \\ &+ A\left(-\ddot{w}(x,t)\delta w(x,t) + \frac{\partial}{\partial T}\left(\dot{w}(x,t)\delta w(x,t)\right)\right)\Bigg]\mathrm{d}x. \end{aligned} \tag{16}$$

Later, similarly by applying the part by part integration on the variable x of the first term of the above, the final form of the variation of kinetic energy is resulted as:

$$\begin{aligned} \delta T =& \int_0^L \rho\Bigg[-A(\ddot{u}(x,t))\delta u(x,t) + \left(I\frac{\partial^2 \ddot{w}(x,t)}{\partial x^2} - A\ddot{w}(x,t)\right)\delta w(x,t) \\ &+ \frac{\partial}{\partial t}\Bigg(I\left(\frac{\partial \ddot{w}(x,t)}{\partial x}\right)\delta\left(\frac{\partial w(x,t)}{\partial x}\right) + \\ &+ A\left(\dot{u}(x,t)\delta u(x,t) + \dot{w}(x,t)\delta w(x,t)\right)\Bigg)\Bigg]\mathrm{d}x. \end{aligned} \tag{17}$$

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Finally, variations of the work done by the viscoelastic foundation can be calculated by:

$$\delta W = -\int_0^L \left(kw(x,t) + c\dot{w}(x,t)\right)\delta w(x,t)\mathrm{d}x. \tag{18}$$

Governing dynamic equations based on Hamilton's principle stated as follows, can be achieved by:

$$\int_{t_1}^{t_2} \delta(T + W - U)\mathrm{d}t = 0. \tag{19}$$

By replacement of variations of strain energy (13), variations of kinetic energy (17), and variations of the work done by external forces (18) in Hamilton's principle (19), and regarding the basic Lemma of the differential equation governing the dynamic equilibrium is obtained as:

δu:

$$\frac{\partial N_{11}}{\partial x} - \rho A\ddot{u}(x,t) = 0, \tag{20}$$

δw:

$$\begin{aligned}\frac{\partial^2(M_{11} + M_{12}^m)}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{11}\frac{\partial w(x,t)}{\partial x}\right) - kw(x,t)\\ - c\dot{w}(x,t) + \rho\left(I\frac{\partial^2\ddot{w}(x,t)}{\partial x^2} - A\ddot{w}(x,t)\right) = 0.\end{aligned} \tag{21}$$

Similarly, for the boundary conditions at both ends of the beam, we have:

$$\begin{aligned} N_{11} &= 0, &\quad \text{or} \quad & \delta u(x,t) = 0,\\ M_{11} + M_{12}^m &= 0, &\quad \text{or} \quad & \delta\left(\frac{\partial w(x,t)}{\partial x}\right) = 0,\\ \frac{\partial}{\partial x}\left(M_{11} + M_{12}^m\right) + N_{11}\frac{\partial w}{\partial x} &= 0, &\quad \text{or} \quad & \delta w(x,t) = 0.\end{aligned} \tag{22}$$

The dynamic equilibrium equation must be calculated in terms of side displacement of w, and in order to do this, by substituting stress resultants from Eq. (11) into the Eqs. (20) and (21), the equations are express in the following form:

δu:

$$E\frac{\partial}{\partial x}\left[\frac{\partial u(x,t)}{\partial x} + \frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2\right] - \rho\ddot{u}(x,t) = 0, \tag{23}$$

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δw:

$$-(EI+\mu l^2 A)\frac{\partial^4 w(t,x)}{\partial x^4}+EA\frac{\partial}{\partial x}\left[\left(\frac{\partial u(x,t)}{\partial x}+\frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2\right)\frac{\partial w(x,t)}{\partial x}\right]$$
$$-kw(x,t)-c\dot{w}(x,t)+\rho\left(I\frac{\partial\ddot{w}(x,t)}{\partial x^2}-A\ddot{w}(x,t)\right)=0. \tag{24}$$

Similarly for the boundary conditions at the both ends of the beam, one can get:

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2=0 &&\text{or}\quad \delta u(x,t)=0,\\
&\frac{\partial^2 w}{\partial x^2}=0 &&\text{or}\quad \delta\left(\frac{\partial w(x,t)}{\partial x}\right)=0,\\
&-(EI+\mu l^2 A)\frac{\partial^2 w}{\partial x^2}+EA\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\left(\frac{\partial w}{\partial x}\right)=0 &&\text{or}\quad \delta w(x,t)=0.
\end{aligned} \tag{25}$$

4. Solution of governing equations

For the solution of governing equations, first normalization must be carried out, therefore the following non-dimensional parameters are defined:

$$(\zeta,\tilde{u})=\frac{1}{L}(x,u),\qquad \tilde{w}=\frac{w}{h},$$
$$\eta=\frac{h}{L},\qquad \tau=\frac{t}{L}\sqrt{\frac{E}{\rho}}. \tag{26}$$

Using the above non-dimensional parameters, Eqs (23), and (24) can be rewritten as:

$\delta\tilde{u}$:

$$\frac{\partial}{\partial\zeta}\left[\frac{\partial\tilde{u}(\zeta,\tau)}{\partial\zeta}+\frac{1}{2}\eta^2\left(\frac{\partial\tilde{w}(\zeta,\tau)}{\partial\zeta}\right)^2\right]-\frac{\partial^2\tilde{u}(\zeta,\tau)}{\partial\tau^2}=0, \tag{27}$$

$\delta\tilde{w}$:

$$-\tilde{D}\eta^2\frac{\partial^4\tilde{w}(\zeta,\tau)}{\partial\zeta^4}+\frac{\partial}{\partial\zeta}\left[\left(\frac{\partial\tilde{u}(\zeta,\tau)}{\partial\zeta}+\frac{1}{2}\eta^2\left(\frac{\partial\tilde{w}(\zeta,\tau)}{\partial\zeta}\right)^2\right)\frac{\partial\tilde{w}(\zeta,\tau)}{\partial\zeta}\right]$$
$$-\tilde{k}\tilde{w}(\zeta,\tau)-\tilde{c}\frac{\partial w(\zeta,\tau)}{\partial\tau}+\frac{\eta^2}{12}\frac{\partial^4\tilde{w}(\zeta,\tau)}{\partial\zeta^2\partial\tau^2}-\frac{\partial^2\tilde{w}(\zeta,\tau)}{\partial\tau^2}=0, \tag{28}$$

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where, in Eq. (28), we have:

$$\tilde{D} = \left(\frac{1}{12} + \frac{\mu}{E}\left(\frac{l}{h}\right)^2\right),$$
$$\tilde{k} = \frac{kL}{k^*}, \qquad \text{in which} \quad k^* = \frac{EA}{L}, \tag{29}$$
$$\tilde{c} = \frac{cL}{c^*}, \qquad \text{in which} \quad c^* = A\sqrt{\rho e}.$$

Similarly, for the boundary conditions, one can get:

$$\frac{\partial \tilde{u}}{\partial \zeta} + \frac{\eta^2}{2}\left(\frac{\partial \tilde{w}}{\partial \zeta}\right)^2 = 0 \qquad \text{or} \qquad \delta\tilde{u}(\zeta,\tau) = 0,$$
$$\frac{\partial^2 \tilde{w}}{\partial \zeta^2} = 0 \qquad \text{or} \quad \delta\left(\frac{\partial \tilde{w}(\zeta,\tau)}{\partial \zeta}\right) = 0, \tag{30}$$
$$-\tilde{D}\eta^2\frac{\partial^2 \tilde{w}}{\partial \zeta^2} + \left(\frac{\partial \tilde{u}}{\partial \zeta} + \frac{\eta^2}{2}\left(\frac{\partial \tilde{w}}{\partial \zeta}\right)^2\right)\left(\frac{\partial \tilde{w}}{\partial \zeta}\right) = 0, \quad \text{or} \qquad \delta\tilde{w}(\zeta,\tau) = 0.$$

For this study, it is considered that the micro-beams have two ends of simply supported boundary conditions are expressed, as follows:

$$\tilde{u}(\zeta = 0,\tau) = \tilde{w}(\zeta = 0,\tau) = \frac{\partial^2 \tilde{w}(\zeta = 0,\tau)}{\partial \zeta^2} = 0,$$
$$\tilde{u}(\zeta = 1,\tau) = \tilde{w}(\zeta = 1,\tau) = \frac{\partial^2 \tilde{w}(\zeta = 1,\tau)}{\partial \zeta^2} = 0. \tag{31}$$

Since the kinematic variable $\tilde{u}$ is considered to be in the form of a harmonic function such that no sign of the axial displacement at any point of the beam happens during deformation. Regarding the boundary conditions presented in (31), for the displacement components we will have:

$$\tilde{u}(\zeta,\tau) = \Lambda_n^{\tilde{u}} \sin(n\pi\zeta)\cos^2(\Omega_n\tau),$$
$$\tilde{w}(\zeta,\tau) = \Lambda_n^{\tilde{w}} \sin(n\pi\zeta)\cos(\Omega_n\tau). \tag{32}$$

Based on Galerkin method, the integral form of Eqs. (27) and (28) are presented as:

$$\int_0^1\int_0^{2\pi}\left\{\frac{\partial}{\partial \zeta}\left[\frac{\partial \tilde{u}(\zeta,\tau)}{\partial \zeta} + \frac{1}{2}\eta^2\left(\frac{\partial \tilde{w}(\zeta,\tau)}{\partial \zeta}\right)^2\right] - \frac{\partial^2 \tilde{u}(\zeta,\tau)}{\partial \tau^2}\right\}\sin(n\pi\zeta)\cos^2(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = 0, \tag{33}$$

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$$\int_0^1 \int_0^{2\pi} \left\{ -\tilde{D}\eta^2 \frac{\partial^4 \tilde{w}(\zeta,\tau)}{\partial \zeta^4} + \frac{\partial}{\partial \zeta}\left[\left(\frac{\partial \tilde{u}(\zeta,\tau)}{\partial \zeta} + \frac{1}{2}\eta^2 \left(\frac{\partial \tilde{w}(\zeta,\tau)}{\partial \zeta}\right)^2\right)\frac{\partial \tilde{w}(\zeta,\tau)}{\partial \zeta}\right] \right.$$
$$- \tilde{k}\tilde{w}(\zeta,\tau) - \tilde{c}\frac{\partial \tilde{w}(\zeta,\tau)}{\partial \tau} + \frac{\eta^2}{12}\frac{\partial^4 \tilde{w}(\zeta,\tau)}{\partial \zeta^2 \partial \tau^2} \tag{34}$$
$$\left. -\frac{\partial^2 \tilde{w}(\zeta,\tau)}{\partial \tau^2}\right\} \sin(n\pi\zeta)\cos(\Omega_n \tau) \mathrm{d}\tau\, \mathrm{d}\zeta = 0,$$

where, by substitution of displacement component s from Eq. (32) into Eqs. (33) and (34), one gets:

$$\left(g_1 + g_2\Omega_n^2\right)\Lambda_n^{\tilde{u}} + g_3\left(\Lambda_n^{\tilde{w}},\right)^2 = 0, \tag{35}$$

$$g_4\Lambda_n^{\tilde{w}}\Lambda_n^{\tilde{u}} + \left(g_5 + g_6\Omega_n + g_7\Omega_n^2\right)\Lambda_n^{\tilde{w}} + g_8\left(\Lambda_n^{\tilde{w}}\right)^3 = 0, \tag{36}$$

where, in relations (35) and (36), we have:

$$\begin{aligned}
g_1 &= -(n\pi)^2 \int_0^1 \int_0^{2\pi} \sin^2(n\pi\zeta)\cos^4(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = -\frac{3\pi}{8}(n\pi)^2,\\
g_2 &= 2\int_0^1 \int_0^{2\pi} \sin^2(n\pi\zeta)\cos(2\Omega_n\tau)\cos^2(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = \frac{1}{4},\\
g_3 &= -\frac{1}{2}\eta^2(n\pi)^3 \int_0^1 \int_0^{2\pi} \sin^3(n\pi\zeta)\cos^4(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta =\\
&\quad -\frac{\pi}{4}\eta^2(n\pi)^3(1+(-1)^{n+1}),\\
g_4 &= -(n\pi)^2 \int_0^1 \int_0^{2\pi} \sin(2n\pi\zeta)\sin(n\pi\zeta)\cos^4(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = 0,\\
g_5 &= -(\tilde{D}\eta^2(n\pi)^4 + \tilde{k})\int_0^1 \int_0^{2\pi} \sin^2(n\pi\zeta)\cos^2(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta =\\
&\quad -\frac{\pi}{2}(\tilde{D}\eta^2(n\pi)^4 + \tilde{k}),\\
g_6 &= -4\tilde{c}\int_0^1 \int_0^{\pi/2} \sin^2(n\pi\zeta)\sin(\Omega_n\tau)\cos(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = -4\tilde{c}\pi,
\end{aligned} \tag{37}$$

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$$g_7 = \left(\frac{\eta^2}{12}(n\pi)^2 + 1\right)\int_0^1\int_0^{2\pi} \sin^2(n\pi\zeta)\cos^2(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = \frac{\pi}{2}\left(\frac{\eta^2}{12}(n\pi)^2 + 1\right),$$

$$g_8 = -\frac{3}{2}\eta^2(n\pi)^3\int_0^1\int_0^{2\pi} \sin^2(n\pi\zeta)\cos^2(n\pi\zeta)\cos^2(\Omega_n\tau)\mathrm{d}\tau\,\mathrm{d}\zeta = -\frac{3\pi}{16}\eta^2(n\pi)^3.$$

Considering Eqs. (35) and (36), we will have:

$$[A]\begin{Bmatrix}\Lambda_n^{\tilde{u}}\\ \Lambda_n^{\tilde{w}}\end{Bmatrix} = 0,$$
$$[A] = \begin{bmatrix} g_1 + g_2\Omega_n^2 & g_3\Lambda_n^{\tilde{w}} \\ g_4\Lambda_n^{\tilde{w}} & g_5 + g_6\Omega_n + g_7\Omega_n^2 + g_8(\Lambda_n^{\tilde{w}})^2 \end{bmatrix}. \tag{38}$$

By equating the determinant of $[A]$ matrix to zero, the frequency equation in terms of different values of maximum displacement domain is produced:

$$a_1\Omega_n^4 + a_2\Omega_n^3 + a_3\Omega_n^2 + a_4\Omega_n^2(\Lambda_n^{\tilde{w}})^2 + a_5\Omega_n + a_6(\Lambda_n^{\tilde{w}})^2 + a_7 = 0, \tag{39}$$

$$\begin{aligned} &a_1 = g_2g_7, \quad a_2 = g_2g_6, \quad a_3 = g_2g_5 + g_1g_7,\\ &a_4 = g_2g_8, \quad a_5 = g_1g_6, \quad a_6 = g_1g_8 - g_3g_4,\\ &a_7 = g_1g_5. \end{aligned} \tag{40}$$

5. Numerical results

In this section, the numerical results are presented for the micro-beam is made of Silicon with properties specified in Table 1.

Table 1.

Mechanical properties of Silicon material [13, 25]

ρ (kg/m^3)	E (N/m^2)	ν
2330	169e9	0.22

Since the natural frequencies of micro beam resting on the viscoelastic have not been reported, the linear and nonlinear frequencies of micro-beam without foundation are employed to valid results of this study. In Table 2, the linear frequencies of micro-beam for various values of the normalized length scale parameter are compared to Kong's study [20].

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Table 2.
The linear frequencies of micro-beam for various values of the normalized length scale parameter

Study	$l/h = 0$	$l/h = 0.1$	$l/h = 0.2$	$l/h = 0.4$
Present	0.5607	0.5743	0.6133	0.7495
Kong et a. [20]	0.598	0.5837	0.6234	0.7617

Table 3.
The nonlinear frequencies of a size dependent micro-beam for various values of the maximum amplitude

$\Lambda_n^{\tilde{w}}/h$	Present study	Wang et al. [35]
0.2	1.04	1.02
0.6	1.181	1.177
0.8	1.31	1.30

A comparison of nonlinear frequencies of a size dependent micro-beam with $l/h = 0.5$, $b/h = 2$, $L/h = 20$ is given in Table 3.

In the results delivered as a graph, if the problem parameters are constant, their values are shown in Table 4.

Table 4.
Values of design parameters

$\tilde{k}$	$\tilde{c}$	b/h	L/h	l/h	h	Mode shape (1)
1	1	3	5	0.3	1e-6	1

Variations of the maximum amplitude versus the normalized frequency of the micro-beams for values of mode shapes are shown in Figs. 2 and 3. As can be seen, the values of normalized frequencies become larger when the number of mode shape and the normalized length scale parameter l/h are increased. Moreover, the length scale parameter has more significant influence for larger mode shapes.

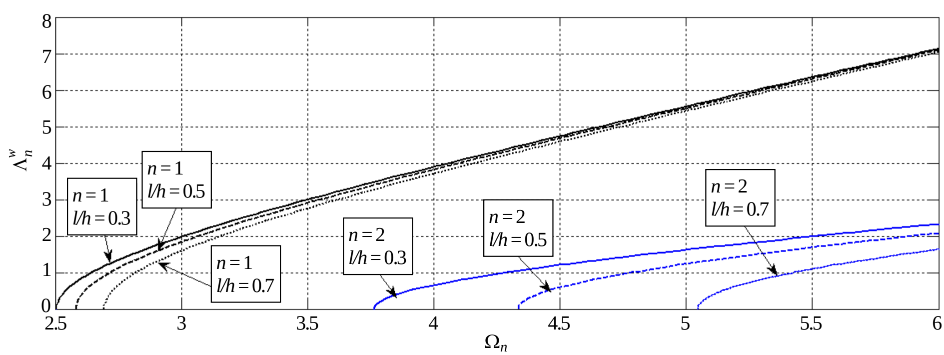

Fig. 2. Variations of the maximum amplitude versus normalized frequency in mode shapes of $n = 1$ and $n = 2$

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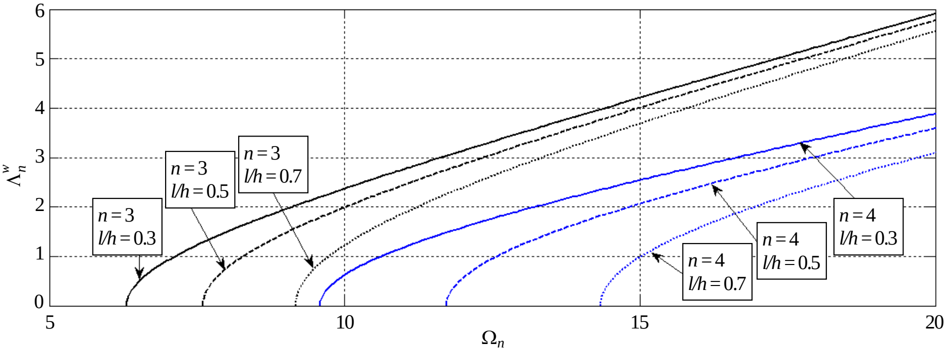

Fig. 3. Variations of the maximum amplitude versus normalized frequency in mode shapes of $n = 3$ and $n = 4$

In Fig. 4, the effect of the length scale parameter on curves of the maximum amplitude versus normalized frequency is studied. From this figure, it can be readily concluded that values of the maximum amplitude are reduced by increase of l/h at any natural frequency.

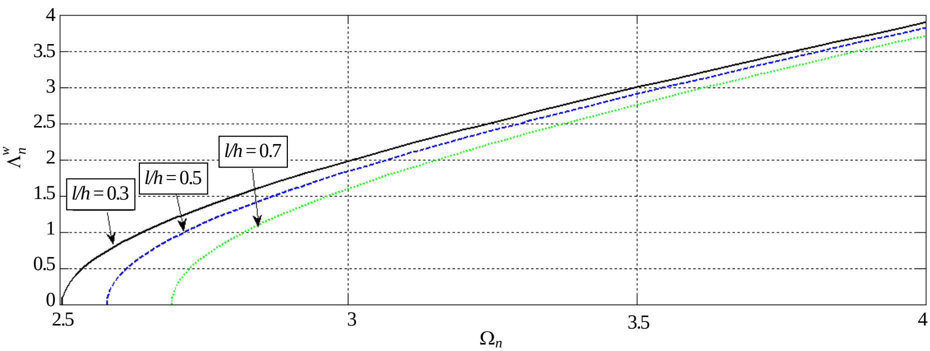

Fig. 4. Variations of the maximum amplitude versus normalized frequency for various values of the normalized length scale parameters

In Fig. 5, curves of the maximum amplitude versus normalized frequency are demonstrated for different values of aspect ratio L/h. To this figure, it can be readily found that values of the normalized nonlinear frequencies are decreased as aspect ratio L/h increase at any maximum amplitude.

In Figs. 6 and 7, the effects of the normalized elastic $\tilde{k}$ and viscoelastic $\tilde{c}$ foundation coefficients, respectively, on curves of the maximum amplitude versus normalized frequency are investigated for different values of the normalized length scale parameter l/h. As can be observed, the values of the normalized nonlinear frequencies are strongly varied by increase in values of $\tilde{k}$ and $\tilde{c}$ for any value of the

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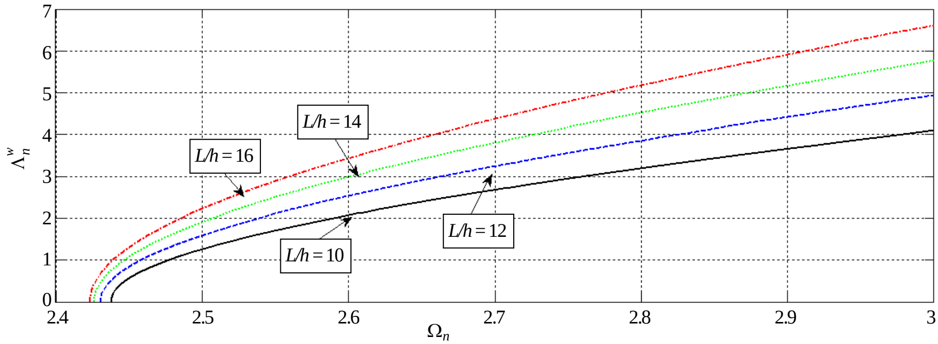

Fig. 5. Variations of the maximum amplitude versus normalized frequency for various values of aspect ratio L/h

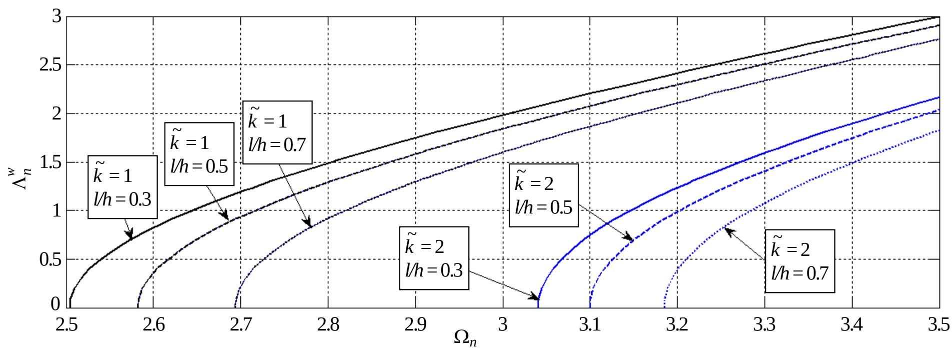

Fig. 6. Variations of the maximum amplitude versus normalized frequency for various values of $\tilde{k}$

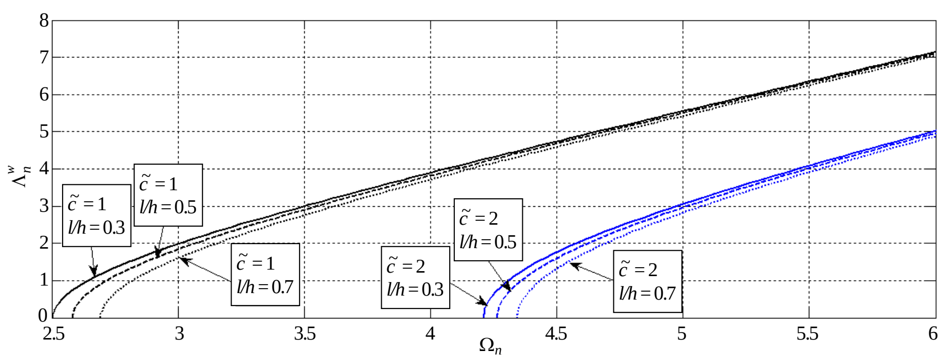

Fig. 7. Variations of the maximum amplitude versus normalized frequency for various values of $\tilde{c}$

normalized length scale parameter. Furthermore, the influences of the normalized elastic foundation coefficients on the normalized nonlinear frequencies are many times more impressive than that of l/h.

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6. Conclusion

In this work, the nonlinear free vibration behavior of micro-beams resting on the viscoelastic foundation has been investigated using the modified couple stress theory. The dynamic equilibrium equation as well as the boundary conditions was derived by means of the Euler–Bernoulli beam and the Hamilton's principle. In addition, the Galerkin method was applied to solve the governing nonlinear differential equation and obtain the frequency-amplitude algebraic equation. Also, the numerical results were presented to illustrate the effects of important parameters such as the mode number, aspect ratio of length to height, the normalized length scale parameter and foundation parameters on the natural frequency-amplitude curves. The findings indicated that

- The values of normalized frequencies become larger when the number of mode shape and the normalized length scale parameter l/h are increased.
- The length scale parameter has more significant influence for larger mode shapes.
- Values of the normalized nonlinear frequencies are decreased as aspect ratio L/h increase at any maximum amplitude.
- The values of the normalized nonlinear frequencies are strongly varied by the increase in values of $\tilde{k}$ and $\tilde{c}$ for any value of the normalized length scale parameter.
- The influences of the normalized elastic foundation coefficients on the normalized nonlinear frequencies are many times more impressive than other parameters.

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