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(*D*,*O*)-SPECIES OF BOUNDED REPRESENTATION TYPE

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Abstract. We describe weak (D,O)-species of bounded representation type in terms of Dynkin diagrams and diagrams with weights. We reduce the problem of their description to flat mixed matrix problems over discrete valuation rings and their common skew field of fractions.

Keywords: *species, (D,O)-species, flat mixed matrix problems, discrete valuation ring, Dynkin diagrams*

1. Introduction

The representations of species which are first introduced by P. Gabriel [1] are closely connected with representations of finitely dimensional algebras and Artinian rings and were studied by many authors (see e.g. [1-3]). This paper is devoted to the study of (D,O)-species which can be considered as a type of a generalization of these species. They play an important role in the representation theory of some classes of rings, for example, of right hereditary SPSD-rings. We use the main definitions and results from the previous paper [4].

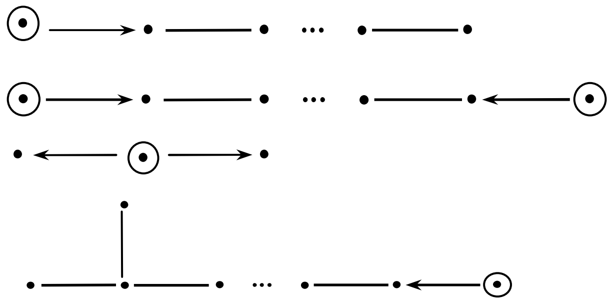
Let $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ be a weak (D,O)-species, where all F_i are equal to O_i or D, and ${}_iM_j$ is a (D, D)-bimodule that is finite dimensional both as a right and left D-vector space (see [1]). Let $I = \{1,2, \ldots, n\}$. The diagram of Ω is a directed graph $Q(\Omega)$ whose vertices are indexed by the numbers 1, 2, ..., n, and the number of arrows from a vertex i to a vertex j is

$$t_{ij} = \dim_D({}_iM_j) \times \dim({}_iM_j)_D + \dim_D({}_jM_i) \times \dim({}_jM_i)_D$$

A vertex i of the graph $Q(\Omega)$ is said to be **marked** if $F_i = O_i$ and we say that F_i is a weight of this vertex. A marked vertex is denoted by $\odot$. By definition of a weak (D,O)-species, each marked vertex is minimal in $Q(\Omega)$. The graph obtained from the graph $Q(\Omega)$ with marked vertices by deleting the orientation of all arrows that connect the unmarked vertices of $Q(\Omega)$ will be called the **diagram with weights** of a (D,O)-species Ω. Recall that a (D,O)-species is of bounded representation type if the dimensions of its indecomposable finite dimensional representations have an upper bound.

In this paper we prove the following theorem, which describes the structure of (D,O)-species of bounded representation type in terms of Dynkin diagrams and diagrams with weights. The proof of this theorem is reduced to solving some flat mixed matrix problems over discrete valuation rings and their common skew field of fractions as it was developed in [4].

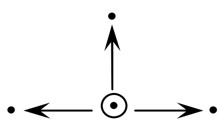
Theorem 1. *Let $\{O_i\}_{i\in I}$ be a family of discrete valuation rings with a common skew field of fractions D. A weak (D,O)-species is of bounded representation type if and only if its diagram $Q(\Omega)$ is a finite disjoint union of Dynkin diagrams of the form A_n, D_n, E_6, E_7, E_8 and the following diagrams with weights:*



Throughout this paper all rings are assumed to be associative with identity.

2. Proof of the necessity in Theorem 1

Lemma 2.1. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$, $I = \{1,2,3,4\}$. Then a (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ whose diagram with weights has the form*



(2.2)

is of unbounded representation type.

Proof. We prove this lemma for the case $F_1 = O$, $F_2 = F_3 = F_4 = D$, ${}_1M_2 = {}_1M_3 = {}_1M_4 = {}_OD_D$ and ${}_iM_j = 0$ for other $i,j \in I$. Write $J = \{1,2,3\}$.

Let X be a right O-module, Y_i a right D-vector space, and let $\varphi_i : X \otimes_O ({}_1M_i) \to Y_i$ be a D-linear mapping for $i \in J$.

Consider the category Rep(Ω) whose objects are representations $M = (X, Y_i, \varphi_i,)_{i\in J}$, and a morphism from an object M to an object $M' = (X', Y'_i, \varphi'_i,)_{i\in J}$ is a set of homomorphisms $(\alpha, \beta_i)_{i\in J}$, in which $\alpha: X \to X'$ is a homomorphism of O-modules, β_i: $Y_i \to Y'_i$ is a homomorphism of D-vector spaces ($i \in J$), and the following equalities hold:

$$\beta_i \varphi_i = \varphi'_i(\alpha \otimes 1) \quad (i = 1,2,3) \tag{2.3}$$

Let us show that the category Rep(Ω) is of unbounded representation type. Let X be a finitely generated free O-module with basis $\omega_1, \ldots, \omega_n$; and let Y_i be a finite dimensional D-space with basis $\tau_1^{(i)}, \ldots, \tau_{k_i}^{(i)}$ ($i = 1,2,3$). Suppose

$$\varphi_i(\omega_s \otimes 1) = \sum_{j=1}^{k_i} \tau_j^{(i)} a_{js}^{(i)} \quad (i \in J) \tag{2.4}$$

where $a_{js}^{(i)} \in D$ for $i \in J$, $s = 1, \ldots, n$. Then the matrices $\mathbf{A}_i = (a_{js}^{(i)})$, $i \in J$, define the representation M uniquely up to equivalence.

Let $\mathbf{U} \in \mathrm{M_n}(O)$ be the matrix corresponding to the isomorphism α, and let $\mathbf{V}_i \in M_{k_i}(D)$ be the matrices corresponding to the isomorphisms β_i, $i \in J$. If $\mathbf{A}'_i$ are the matrices corresponding to a representation M', then equalities (2.3) have the following matrix form:

$$\mathbf{V}_i \mathbf{A}_i \mathbf{U}^{-1} = \mathbf{A}'_i, \quad i \in J. \tag{2.5}$$

We obtain the following matrix problem:

Given a block-rectangular matrix $\mathbf{T} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}$ *with the following admissible transformations:*

1. *Right O-elementary transformations of columns of* $\mathbf{T}$.
2. *Left D-elementary transformations of rows within each block* $\boldsymbol{A}_i$ ($i = 1,2,3$).

Set

$$\mathbf{A}_1 = \left[\begin{array}{cccc||cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right],$$

$$\mathbf{A}_2 = \left[\begin{array}{cccc||cccc} \pi^{-2} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \pi^{-4} & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{-2n} & 0 & 0 & \cdots & 1 \end{array}\right], \quad \mathbf{A}_3 = \left[\begin{array}{c||c} \pi^{n-1} & 0 \\ \pi^{n-2} & 0 \\ \vdots & \vdots \\ 1 & 0 \end{array}\right],$$

where $\pi \in R = \operatorname{rad} O$, $\pi \neq 0$. By [5, Lemma 4], the matrix $\mathbf{T}$ is indecomposable and therefore the species with diagram (2.2) is of unbounded representation type.

Lemma 2.6. *Let O be a discrete valuation ring with a common skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$, $I = \{1,2,3,4\}$. Then a (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j\,\in I}$ whose diagram with weights has the form*

$$\bullet \longleftarrow \odot \longrightarrow \bullet \;\rule[0.5ex]{2em}{0.4pt}\; \bullet \tag{2.7}$$

is of unbounded representation type.

Proof. Analogously as in the proof of Lemma 2.1, we obtain the following matrix problem:

Given a block-triangular matrix $\mathbf{T} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}$. *The following transformations are admitted:*

1. *Right O-elementary transformations of columns within the first vertical strip of the matrix* $\mathbf{T}$.
2. *Right D-elementary transformations of columns within the second vertical strip of the matrix* $\mathbf{T}$.
3. *Left D-elementary transformations of rows within each horizontal strip of the matrix* $\mathbf{T}$.

Reduce $\mathbf{A}_3$ to the form $\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}$ and obtain $\mathbf{A}_2$ of the form $\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$. It is possible to add any row of the matrix $\mathbf{B}_2$ multiplied on the left by elements of D to any column of the matrix $\mathbf{B}_1$. Thus the matrices $\mathbf{A}_1$, $\mathbf{B}_1$ and $\mathbf{B}_2$ form the matrix problem II from [5] and so the (D,O)-species with diagram (2.7) is of unbounded representation type.

Lemma 2.8. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$, $I = \{1,2,3,4,5\}$. Then a (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j\,\in I}$ whose diagram with weights has the form*

$$\begin{tikzcd} & \odot \arrow[d] & & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd} \tag{2.9}$$

is of unbounded representation type.

Proof. Write $J = \{1,2,3,4\}$. Let X be a right O-module, let Y_i be a right D-vector space for $i \in J$, and let $\varphi_1 : X \otimes_O ({}_1M_4) \to Y_3$, $\varphi_i : Y_{i-1} \otimes_O ({}_iM_4) \to Y_3$ (i = 2,3), $\varphi_4 : Y_2 \otimes_O ({}_3M_4) \to Y_4$ be D-linear mappings.

Let $\mathbf{A}_i$ be the matrices corresponding to homomorphisms φ_i ($i \in J$) that define the representation M, and $\mathbf{A}'_i$ ($i \in J$) be the matrices corresponding to the representation M'. If $\mathbf{U} \in \mathrm{M}_n(O)$ is the matrix corresponding to the isomorphism α, and $\mathbf{V}_i \in M_{k_i}(D)$ are the matrices corresponding to the isomorphisms β_i, $i \in J$, then we have the following matrix equalities:

$$\mathbf{V}_3\mathbf{A}_1\mathbf{U}^{-1} = \mathbf{A}'_1, \quad \mathbf{V}_3\mathbf{A}_2\mathbf{V}_1^{-1} = \mathbf{A}'_2, \tag{2.10}$$

$$\mathbf{V}_3\mathbf{A}_3\mathbf{V}_2^{-1} = \mathbf{A}'_3, \quad \mathbf{V}_4\mathbf{A}_4\mathbf{V}_2^{-1} = \mathbf{A}'_4. \tag{2.11}$$

We obtain the following matrix problem:

Given a block-rectangular matrix $\mathbf{T} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ 0 & 0 & \mathbf{A}_4 \end{bmatrix}$ *which partitioned into* 2 *horizontal and* 3 *vertical strips. The following transformations are admitted:*

1. *Right O-elementary transformations of columns within the first vertical strip of* $\mathbf{T}$.
2. *Right D-elementary transformations of columns within the second and third vertical strips of* $\mathbf{T}$.
3. *Left D-elementary transformations of rows within each horizontal strip of* $\mathbf{T}$.

Reduce $\mathbf{A}_2$ and $\mathbf{A}_4$ to the form $\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{A}_3$ to the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

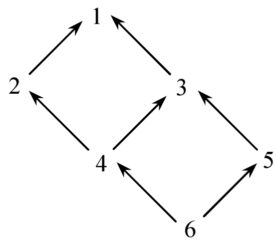
In according with this reducing the matrix $\mathbf{A}_1$ is divided into 6 horizontal strips and we obtain the following matrix problem.

Given a block-rectangular matrix $\mathbf{B}$ *partitioned into* 6 *vertical strips*:

$\mathbf{B}_1$	$\mathbf{B}_2$	$\mathbf{B}_3$	$\mathbf{B}_4$	$\mathbf{B}_5$	$\mathbf{B}_5$

The following transformations are admitted:

1. *Left O-elementary transformations of rows of* $\mathbf{B}$.
2. *Right D-elementary transformations of columns within each vertical strip* $\mathbf{B}_i$ ($i = 1,2,..,6$).
3. *If* $\alpha_i \leq \alpha_j$ *in the poset* S:

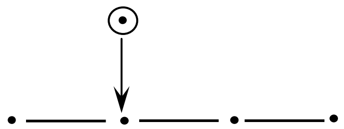


then any column of the vertical strip $\mathbf{B}_i$ *multiplied on the right by the arbitrary element of D can be added to any column of the vertical strip* $\mathbf{B}_j$.

It is easy to see that the blocks $\mathbf{B}_2$, $\mathbf{B}_4$, $\mathbf{B}_5$ form the matrix problem II from [5]. Therefore the species with diagram (2.9) is of unbounded representation type.

Analogously one can prove the following lemma:

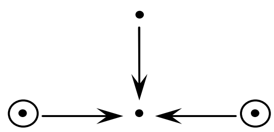
Lemma 2.12. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical* $R = \pi O = O\pi$, $I = \{1,2,3,4,5\}$. *Then a* (D,O)*-species* $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ *whose diagram with weights has the form*



(2.13)

is of unbounded representation type for any directions of arrows.

Lemma 2.14. *Let* O_i *be a discrete valuation ring with a common skew field of fractions D and the Jacobson radical* $R_i = \pi_i O_i = O_i\pi_i$, *for* $i = 1,2$; *and* $I = \{1,2,3,4\}$. *Then a* (D,O)*-species* $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ *whose diagram with weights has the form*



(2.15)

is of unbounded representation type.

Proof. Write $J = \{1,2,3\}$. Let X_i be a right O_i-module for $i = 1,2$; Y_j a right D-vector space for $j = 1,2$; and let $\varphi_i : X \otimes_{O_i} ({}_iM_3) \to Y_1$, $(i = 1,2)$, $\varphi_3 : Y_2 \otimes_D ({}_4M_3) \to Y_1$ be D-linear mappings.

If $M = (X_1, X_2, Y_1, Y_2, \varphi_i,)_{i \in J}$ and $M' = (X_1', X_2', Y_1', Y_2', \varphi'_i,)_{i \in J}$ are two equivalent representations of a (D,O)-species Ω and $\alpha_i: X_i \to X_i'$ are isomorphisms of O-modules, $\beta_i: Y_i \to Y'_i$ are isomorphisms of D-vector spaces $(i = 1,2)$, then the following equalities hold:

$$\beta_1\varphi_i = \varphi_i'(\alpha_i \otimes 1), \quad (i = 1,2) \tag{2.16}$$

$$\beta_1\varphi_3 = \varphi_3'(\beta_2 \otimes 1). \tag{2.17}$$

Let $\mathbf{A}_i$ be the matrices corresponding to the homomorphisms φ_i ($i \in J$) that define the representation M, and $\mathbf{A}_i'$ ($i \in J$) be the matrices corresponding to the representation M'. If $\mathbf{U}_i$ is the matrix with entries in O_i corresponding to the isomorphism α_i, and $\mathbf{V}_i \in M_{k_i}(D)$ are the matrices corresponding to the isomorphisms β_i (i = 1,2), then equalities (2.16) and (2.17) have the following matrix form:

$$\mathbf{V}_1\mathbf{A}_i\mathbf{U}_i^{-1} = \mathbf{A}_i', \quad (i = 1,2) \tag{2.18}$$

$$\mathbf{V}_3\mathbf{A}_3\mathbf{V}_2^{-1} = \mathbf{A}_3'. \tag{2.19}$$

We obtain the following matrix problem:

Given a block-rectangular matrix $\mathbf{T} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3\end{bmatrix}$. *The following transformations are admitted:*

1. *Left D-elementary transformations of rows of* $\mathbf{T}$.
2. *Right* O_1*-elementary transformations of columns of* $\mathbf{A}_1$.
3. *Right* O_2*-elementary transformations of columns of* $\mathbf{A}_2$.
4. *Right D-elementary transformations of columns of* $\mathbf{A}_3$.

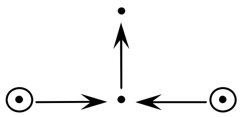
Consider two possible cases.
Case 1: Assume that $O_1 = O_2$. Set

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \pi^2 & 0 & \cdots & 0 \\ 0 & \pi^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{2n} \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 1 \\ \pi \\ \vdots \\ \pi^{n-1} \end{bmatrix}$$

By [5, lemma 3], the matrix $\mathbf{T}$ is indecomposable. Thus the corresponding representation M of the species Ω is indecomposable and the species Ω with diagram (2.15) is of unbounded representation type.

Case 2: Assume that $O_1 \neq O_2$. Set $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{I}$. Then $\mathbf{T} = \begin{bmatrix}\mathbf{I} & \mathbf{I} & \mathbf{A}_3\end{bmatrix}$ and for the matrix $\mathbf{A}_3$ we have the matrix problem II from [6]. By [6, Lemma 4.2], this matrix problem is of unbounded representation type. Therefore, the species with diagram (2.15) is of unbounded representation type.

Lemma 2.20. *Let O_i be a discrete valuation ring with a common skew field of fractions D and the Jacobson radical $R_i = \pi_i O_i = O_i\pi_i$, for $i = 1,2$; and $I = \{1,2,3,4\}$. Then a (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ whose diagram with weights has the form*



(2.21)

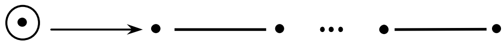
is of unbounded representation type.

This lemma is proved as Lemma 2.14.

Note that all diagrams which are not presented in the diagrams of Theorem 1 have a subdiagram of one of the types considered in this section and hence they are of bounded representation type. Therefore, the necessity of Theorem 1 follows from Lemmas 2.1, 2.6, 2.8, 2.12, 2.14, 2.20 and [4, proposition 4.4].

3. Proof of the sufficiency in Theorem 1

Lemma 3.1. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$, and let $I = \{1,2,\ldots,n, n+1\}$. Then a weak (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ whose diagram with weights has the form*



(3.2)

is of bounded representation type.

Proof. Write $J = \{1, 2, \ldots, n\}$. Let X be a right O-module, let Y_i be a right D-vector space for $i \in J$, and let $\varphi_1 : X \otimes_O ({}_1M_2) \to Y_1$; $\varphi_i : Y_{i-1} \otimes_D ({}_iM_{i+1}) \to Y_i$ be D-linear mapping for $i = 2, \ldots, n$.

If $M = (X, Y_i, \varphi_i)_{i\in J}$ and $M' = (X', Y'_i, \varphi'_i)_{i\in J}$ are two equivalent representations of a (D,O)-species Ω, α: $X \to X'$ is an isomorphism of O-modules and β_i: $Y_i \to Y'_i$ are isomorphisms of D-vector spaces ($i \in J$), then the following equalities hold:

$$\beta_1\varphi_1 = \varphi'_1(\alpha \otimes 1), \tag{3.3}$$

$$\beta_i\varphi_i = \varphi'_i(\beta_{i-1} \otimes 1) \quad (i = 2,\ldots,n). \tag{3.4}$$

Let $\mathbf{A}_i$ be the matrices corresponding to the homomorphisms φ_i ($i \in J$) that define the representation M, and $\mathbf{A}'_i$ ($i\in J$) be the matrices corresponding to the representation M'. If $\mathbf{U}\in \mathrm{M}_s(O)$ is the matrix corresponding to the isomorphism α, and $\mathbf{V}_i \in M_{k_i}(D)$ are the matrices corresponding to the isomorphisms β_i ($i\in J$), then equalities (3.3) and (3.4) have the following matrix form:

$$\mathbf{V}_1\mathbf{A}_1\mathbf{U}^{-1} = \mathbf{A}'_1, \quad \mathbf{V}_i\mathbf{A}_i\mathbf{V}_{i-1}^{-1} = \mathbf{A}'_i, \quad i = 2,\ldots,n \tag{3.5}$$

Note that the problem of reducing a set of matrices $\mathbf{A}_i$ by matrices $\mathbf{V}_i$ (i = 2, ..., n) satisfying equalities (3.5) leads to the problem of classifying representations of the quiver Q with diagram $A_{n\text{-}1}$. By the Gabriel theorem [1], this quiver has $\frac{n(n-1)}{2}$ indecomposable representations. In accordance with these representations, the matrix $\mathbf{A}_1$ is partitioned into 2n-2 vertical strips, and the partial ordering relation between these strips is linear. Therefore, the matrix problem (3.5) leads to the following matrix problem.

Given a rectangular matrix $\mathbf{T}$ *with entries in a skew field D that is partitioned into* 2n-2 *vertical strips*:

$\mathbf{T}_1$	$\mathbf{T}_2$	…	$\mathbf{T}_{2n\text{-}3}$	$\mathbf{T}_{2n\text{-}2}$

The transformations of the following types are admitted:

1. *Right O-elementary transformations of rows of* $\mathbf{T}$.
2. *Left D-elementary transformations of columns within each vertical strip* $\mathbf{T}_i$.
3. *Addition of columns of the i-th vertical strip* $\mathbf{T}_i$ *multiplied on the right by elements of D to columns in the j-th vertical strip* $\mathbf{T}_j$ *if* $i \leq j$.

The matrix $\mathbf{T}$ can be reduced by these transformations to the form in which any block $\mathbf{T}_i$ has the form $\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}$ and all blocks over and under $\mathbf{I}$ are zero. Thus, for any indecomposable representation M the corresponding matrix $\mathbf{T}$ has a finite fixed number of nonzero elements and this number depends only on n. Therefore, the species Ω with diagram (3.2) is of bounded representation type.

Lemma 3.6. *Let* O_i *be a discrete valuation ring with a common skew field of fractions D and the Jacobson radical* $R_i = \pi_i O_i = O_i \pi_i$ *for i* = 1,2; *and let* $I = \{1,2,\ldots, n, n+1\}$. *Then a weak* (*D*,*O*)*-species* $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ *whose diagram with weights has the form*

$$\odot \longrightarrow \bullet \frac{\qquad}{} \bullet \;\cdots\; \bullet \frac{\qquad}{} \bullet \longleftarrow \odot \tag{3.7}$$

is of bounded representation type.

Proof. Renumber the vertices of the diagram (2.16) in such a way that the vertex 1 corresponds to $F_1 = O_1$ and the vertex 2 corresponds to $F_2 = O_2$. We obtain the diagram

$$\begin{tikzcd} \underset{1}{\odot} \arrow[r] & \underset{3}{\bullet} \arrow[r, dash] & \bullet & \cdots & \bullet \arrow[r, dash] & \underset{n+1}{\bullet} & \underset{2}{\odot} \arrow[l] \end{tikzcd}$$

Write $J = \{1, 2, ..., n–1\}$. Let X_i be a right F_i-module (i = 1,2), let Y_j be a right D-vector space for $j \in J$, and let $\varphi_1 : X_1 \otimes_{F_1} ({}_1M_3) \to Y_3$; $\varphi_2 : X_2 \otimes_{F_2} ({}_2M_{n+1}) \to Y_{n+1}$; $\varphi_{i+1} : Y_{i-1} \otimes_D ({}_{i+1}M_{i+2}) \to Y_i$ be D-linear mapping for i = 2, ..., n-1.

If $M = (X_1, X_2, Y_i, \varphi_i, \varphi_n)_{i \in J}$ and $M' = (X_1', X_2', Y'_i, \varphi'_i, \varphi_n')_{i \in J}$ are two equivalent representations of a (D,O)-species Ω and $\alpha_i : X_i \to X_i'$ is an isomorphism of O-modules (i = 1,2), $\beta_j : Y_j \to Y'_j$ is an isomorphism of D-vector spaces ($j \in J$), then the following equalities hold:

$$\beta_1\varphi_1 = \varphi_1'(\alpha_1 \otimes 1), \quad \beta_{n-1}\varphi_2 = \varphi_2'(\alpha_2 \otimes 1), \tag{3.8}$$

$$\beta_j\varphi_{j+1} = \varphi'_{j+1}(\beta_{j-1} \otimes 1) \quad (j = 2,...,n-1) \tag{3.9}$$

Suppose the isomorphism α_i is given by the matrix $\mathbf{U}_i$ with entries in O_i for i = 1,2; and the isomorphism β_j is given by the matrix $\mathbf{V}_j$ with entries in D for $j \in J$. If $\mathbf{A}_i$, $\mathbf{A}_i'$ ($i \in I$) are the matrices corresponding to the representations M, M' then equalities (3.8) and (3.9) are equivalent to the following matrix equalities:

$$\mathbf{V}_1\mathbf{A}_1\mathbf{U}_1^{-1} = \mathbf{A}_1', \quad \mathbf{V}_{n-1}\mathbf{A}_2\mathbf{U}_2^{-1} = \mathbf{A}_2' \tag{3.10}$$

$$\mathbf{V}_j\mathbf{A}_{j+1}\mathbf{V}_{j-1}^{-1} = \mathbf{A}'_{j+1}, \quad (j = 2,...,n-1) \tag{3.11}$$

Note that the reduction of a set of matrices $\mathbf{A}_i$, i = 1,3,4, ..., n, by admissible transformations (3.10) and (3.11) leads to the matrix problem of classifying representations of a (D,O)-species Ω with diagram (3.2) described in Lemma 3.1. After reducing the matrices $\mathbf{A}_i$, i = 1,3,4, ..., n, we get $\mathbf{A}_2$ partitioned into n vertical strips and the following matrix problem for $\mathbf{A}_2$.

Given a block matrix $\mathbf{T}$ *with entries in a skew field D that is partitioned into n vertical strips*:

$\mathbf{T}_1$	$\mathbf{T}_2$	…	$\mathbf{T}_n$

The transformations of the following types are allowed:

1. *Left* O_2*-elementary transformations of rows of* $\mathbf{T}$.
2. *Right* O_1*-elementary transformations of columns within a vertical strip* $\mathbf{T}_k$, *for some fixed* $1 \le k \le n$.
3. *Right D-elementary transformations of columns within each vertical strip* $\mathbf{T}_i$, *if* $i \ne k$.
4. *Addition of columns in the vertical strip* $\mathbf{T}_i$ *multiplied on the right by elements of D to columns in the vertical strip* $\mathbf{T}_j$ *if* $i \le j$.

Using these transformations and taking into account [5, Lemma 4.1], we can reduce $\mathbf{T}$ to the form in which every block $\mathbf{T}_i$ has the form: $\mathbf{U} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}$ or $\mathbf{W} = \begin{bmatrix} \pi^m\mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}$ and all matrices over, under, on the left, and on the right of the matrix $\mathbf{I}$ or, respectively, $\pi^m\mathbf{I}$ are zero. This means that the matrix $\mathbf{A}_2$ is decomposed into a direct sum of matrices of the forms $\mathbf{U}$ and $\mathbf{W}$.

Thus, the species Ω with diagram (3.7) is of bounded representation type.

Analogously we can prove the following lemma:

Lemma 3.12. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$, and let $I = \{1,2,\ldots,n, n+1\}$. Then a weak (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ whose diagram with weights has the form*

(3.13)

is of bounded representation type.

Lemma 3.14. *Let O be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi O = O\pi$, and $I = \{1,2,\ldots,n, n+1\}$. Then a weak (D,O)-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ whose diagram with weights has the form*

(3.15)

is of bounded representation type.

Proof. Write $J = \{1,2\}$. Let X be a right O-module, let Y_i be a right D-vector space, and let $\varphi_i : X \otimes_{F_1} ({}_1M_{i+1}) \to Y_1$ be a D-linear mapping for $i \in J$.

If $M = (X, Y_i, \varphi_i)_{i\in J}$ and $M' = (X', Y'_i, \varphi'_i)_{i\in J}$ are two equivalent representations of a (D,O)-species Ω and $\alpha: X \to X'$ is an isomorphism of O-modules, $\beta_i : Y_i \to Y'_i$ are isomorphisms of D-vector spaces ($i \in J$), then the following equalities hold:

$$\beta_i\varphi_i = \varphi_i'(\alpha \otimes 1) \quad (i = 1,2). \tag{3.16}$$

Let $\mathbf{U} \in \mathrm{M}_s(O)$ be the matrix corresponding to the isomorphism α, and let $\mathbf{V}_i \in M_{k_i}(D)$ be the matrices corresponding to the isomorphisms β_i, $i \in J$. If $\mathbf{A}_i$, $\mathbf{A}_i'$ $(i \in J)$ are the matrices corresponding to the representations M, M' then equalities (3.14) are equivalent to the following matrix equalities

$$\mathbf{V}_i\mathbf{A}_i\mathbf{U}^{-1} = \mathbf{A}_i', \quad (i = 1,2). \tag{3.17}$$

We obtain the following matrix problem.

Given a matrix $\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix}$ *with entries in D. The transformations of the following types are allowed:*

1. *Right D-elementary transformations of columns within any vertical strip.*
2. *Left O-elementary transformations of rows of* $\mathbf{T}$.

The matrix $\mathbf{T}$ can be reduced by these transformations to a direct sum of the following matrices:

$$\left[1\mid 0\right],\quad \left[0\mid 1\right],\quad \left[1\mid 1\right],\quad \left[\begin{array}{c|c} 1 & \pi^k \\ 0 & 1 \end{array}\right].$$

Thus, a (D,O)-species with diagram (3.15) is of bounded representation type. The sufficiency in Theorem 1 follows from Lemmas 3.1, 3.6, 3.12, and 3.16.

4. Conclusion

We have described all weak (D, O)-species of bounded representation type. The proof is given by reduction to mixed matrix problems over discrete valuation rings and their common skew field of fractions. We use some facts about representations of quivers and species from [1-3].

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