Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XIX (2014)
195–206

A NOTE ON WEAKLY ϱ-UPPER CONTINUOUS FUNCTIONS

KATARZYNA NOWAKOWSKA, MAŁGORZATA TUROWSKA

Abstract

In the article we present definition and some properties of weakly ϱ-upper continuous functions. We find maximal additive and maximal multiplicative families for the class of weakly ϱ-upper continuous functions.

1. Preliminaries

In the article we apply standard symbols and notations. By $\mathbb{R}$ we denote the set of all real numbers, by $\mathbb{N}$ we denote the set of all positive integers. By $\mathcal{L}$ we denote the family of Lebesgue measurable subsets of the real line. The symbol $\lambda(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}$. In the whole article, I will denote an open interval (not necessarily bounded) with ends a, b and f – a real function defined in I. By $\mathcal{A}$ we denote the class of all approximately continuous functions defined in I.

Let E be a measurable subset of $\mathbb{R}$ and x be a real number. According to [1], the numbers

$$\overline{d}^+(E,x) = \limsup_{t\to 0^+} \frac{\lambda(E\cap[x,x+t])}{t}$$

and

$$\overline{d}^-(E,x) = \liminf_{t\to 0^+} \frac{\lambda(E\cap[x-t,x])}{t}$$

are called the right upper density of E at x and left upper density of E at x, respectively. The number

$$\overline{d}(E,x) = \max\left\{\overline{d}^+(E,x), \overline{d}^-(E,x)\right\}$$

is called the upper density of E at x.

• *Katarzyna Nowakowska* — e-mail: nowakowska_k@go2.pl
Pomeranian University in Słupsk.
• *Małgorzata Turowska* — e-mail: malgorzata.turowska@apsl.edu.pl
Pomeranian University in Słupsk.

Recall the definition of ϱ-upper continuous function.

Definition 1. [2] *Let E be a measurable subset of $\mathbb{R}$. If $x \in \mathbb{R}$ and $0 < \varrho < 1$, then we shall say that x is a point of ϱ-type upper density of E if $\overline{d}(E, x) > \varrho$.*

Definition 2. [2] *Let $x \in I$. A real-valued function f defined on I is called ϱ-upper continuous at x provided that there is a measurable set $E \subset I$ such that x is a point of ϱ-type upper density of E, $x \in E$ and $f|_E$ is continuous at x. If f is ϱ-upper continuous at each point of I, we say that f is ϱ-upper continuous.*

By $\mathcal{UC}_\varrho$ we denote the class of all ϱ-upper continuous functions defined in an open interval I.

2. Weakly ϱ-continuous functions

Now, we shall give the basic definitions of this paper.

Definition 3. *Let E be a measurable subset of $\mathbb{R}$ and $x \in \mathbb{R}$. If $\varrho \in (0, 1)$, then we say that x is a point of weak ϱ-type upper density of E if $\overline{d}(E, x) \geq \varrho$.*

Definition 4. *A real-valued function f defined in I is called weakly ϱ-upper continuous at $x \in I$ provided that there is a measurable set $E \subset I$ such that x is a point of weak ϱ-type upper density of E, $x \in E$ and $f|_E$ is continuous at x. If f is weakly ϱ-upper continuous at each point of I, we say that f is weakly ϱ-upper continuous.*

By $w\mathcal{UC}_\varrho$ we denote the class of all weakly ϱ-upper continuous functions defined on an open interval I.

In an obvious way we define one-sided weak ϱ-upper continuity at a point x and f is weakly ϱ-upper continuous at x if and only if it is weakly ϱ-upper continuous at x on the right or on the left.

Corollary 1. *If $0 < \varrho_1 < \varrho_2 < 1$, $x_0 \in I$ and $f \colon I \to \mathbb{R}$ is weakly ϱ_2-upper continuous at x_0, then f is weakly ϱ_1-upper continuous at x_0.*

Corollary 2. *If $0 < \varrho < 1$ and $f \colon I \to \mathbb{R}$ is ϱ-upper continuous at some point x_0 from I, then f is weakly ϱ-upper continuous at x_0.*

Example 1. *Let $\varrho \in (0, 1)$. We shall show that there exists $f \colon \mathbb{R} \to \mathbb{R}$ such that $f \in w\mathcal{UC}_\varrho \setminus \mathcal{UC}_\varrho$.*

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} x_n = 0$ and $x_{n+1} < x_n$ for every $n \geq 1$. For each $n \geq 1$ take any $y_n \in (x_{n+1}, x_n)$ such that $x_n - y_n = \varrho(x_n - x_{n+1})$. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ letting

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \{x_n\} \cup (x_1, \infty), \\ 1 & \text{if } x \in \{0\} \cup \bigcup_{n=1}^{\infty} [y_n, x_n), \\ \text{linear on each interval } [x_{n+1}, y_n], n \geq 1. \end{cases}$$

Clearly, f is ϱ-upper continuous at every point except at 0. Take any $\varepsilon > 0$. Then

$$\lambda\left(\{x \in [x_{n+1}, y_n]\colon |f(x) - 1| < \varepsilon\}\right) = \varepsilon\lambda\left([x_{n+1}, y_n]\right) = \\ = \varepsilon(1-\varrho)\lambda\left([x_{n+1}, x_n]\right).$$

Therefore,

$$\lambda\left(\{x \in [0, x_n]\colon |f(x) - 1| < \varepsilon\}\right) = \\ = \sum_{k=n}^{\infty} \left(\varrho\lambda\left([x_{k+1}, x_k]\right) + \varepsilon(1-\varrho)\lambda\left([x_{k+1}, x_k]\right)\right) = \\ = (\varrho + \varepsilon(1-\varrho)) \sum_{k=n}^{\infty} \lambda\left([x_{k+1}, x_k]\right) = (\varrho + \varepsilon(1-\varrho))\lambda\left([0, x_n]\right).$$

Then

$$\overline{d}\left(\{x\colon |f(x) - 1| < \varepsilon\}, 0\right) = \lim_{n\to\infty} \frac{\lambda\left(\{x \in [0, x_n]\colon |f(x) - 1| < \varepsilon\}\right)}{\lambda\left([0, x_n]\right)} = \\ = \varrho + \varepsilon(1-\varrho).$$

Since $\lim_{\varepsilon\to 0^+} \overline{d}\left(\{x\colon |f(x) - 1| < \varepsilon\}, 0\right) = \lim_{\varepsilon\to 0^+} (\varrho + \varepsilon(1-\varrho)) = \varrho$, we conclude that f is not ϱ-upper continuous at 0 and f is weakly ϱ-upper continuous at 0. Hence $f \in w\mathcal{UC}_\varrho \setminus \mathcal{UC}_\varrho$.

Corollary 3. *If $0 < \varrho_1 < \varrho_2 < 1$ and $f\colon I \to \mathbb{R}$ is weakly ϱ_2-upper continuous at some point x_0 from I, then f is ϱ_1-upper continuous at x_0.*

Example 2. *We shall show that if $0 < \varrho_1 < \varrho_2 < 1$, then there is a function $f\colon (a, b) \to \mathbb{R}$ such that $f \in \mathcal{UC}_{\varrho_1} \setminus w\mathcal{UC}_{\varrho_2}$.*

Let $a < 0 < b$. We can find a sequence $([a_n, b_n])_{n\geq 1}$ of pairwise disjoint closed intervals such that $0 < b_{n+1} < a_n < b_n$ for each n and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty} [a_n, b_n], 0\right) = \frac{\varrho_1 + \varrho_2}{2}.$$

Let $([c_n, d_n])_{n\geq 1}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for every $n \geq 1$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty}([c_n, d_n] \setminus [a_n, b_n]), 0\right) = 0$. Put $I_n = [a_n, b_n]$, $J_n = [c_n, d_n]$ for every $n \geq 1$. Define a function $f\colon (a, b) \to \mathbb{R}$ letting

$$f(x) = \begin{cases} 0 \quad \text{if } x \in \{0\} \cup \bigcup_{n=1}^{\infty} I_n, \\ 1 \quad \text{if } x \in (a, 0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] \cup [d_1, b), \\ \text{linear on each interval } [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

The function f is continuous at every point except at 0. If $E = \bigcup_{n=1}^{\infty} I_n \cup \{0\}$, then the function f restricted to E is constant, so in particular, it is continuous at zero. Moreover,

$$\overline{d}(E, 0) \geq \overline{d}^+(E, 0) = \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) = \frac{\varrho_1 + \varrho_2}{2} > \varrho_1.$$

Hence $f \in \mathcal{UC}_{\varrho_1}$. But

$$\overline{d}^+(\{x\colon f(x) < 1\}, 0) \leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} J_n, 0\right) \leq$$
$$\leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) + \overline{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), 0\right) = \frac{\varrho_1 + \varrho_2}{2} < \varrho_2.$$

Moreover $\overline{d}^-(\{x\colon f(x) < 1\}, 0) = 0$. Thus $\overline{d}(\{x\colon f(x) < 1\}, 0) < \varrho_2$ and f is not weakly ϱ_2-upper continuous at 0. Hence $f \notin w\mathcal{UC}_{\varrho_2}$.

Corollary 4. $\bigcup_{\varrho\in(0,1)} \mathcal{UC}_\varrho = \bigcup_{\varrho\in(0,1)} w\mathcal{UC}_\varrho$.

Corollary 5. $\bigcap_{\varrho\in(0,1)} \mathcal{UC}_\varrho = \bigcap_{\varrho\in(0,1)} w\mathcal{UC}_\varrho$.

Definition 5. *We say that a real-valued function f defined on an open interval I has Denjoy property at $x_0 \in I$ if for each $\varepsilon > 0$ and $\delta > 0$ the set*

$$\{x \in (x_0 - \delta, x_0 + \delta)\colon |f(x) - f(x_0)| < \varepsilon\}$$

contains a measurable subset of positive measure. We say that f has Denjoy property if it has Denjoy property at each point $x \in I$.

Immediately from Theorem 2.1 in [2], Remark 2.1 in [2] and Corollary 2 we obtain the following results.

Corollary 6. *If* $0 < \varrho < 1$ *and* $f \in u\mathcal{UC}_\varrho$, *then* f *is measurable.*

Corollary 7. *If* $0 < \varrho < 1$ *and* $f \in u\mathcal{UC}_\varrho$, *then* f *has Denjoy property.*

The proof of the next corollary follows directly from Theorem 2.4 in [4].

Corollary 8. *There exists function* f *such that* $f \in \bigcap_{\varrho \in (0,1)} u\mathcal{UC}_\varrho$ *and* f *does not belong to the Baire class* 1.

We shall need the following lemma.

Lemma 1. [3] *If* $0 < \varrho \leq 1$ *and* $\{E_n : n \in \mathbb{N}\}$ *is a descending family of measurable sets such that* $x \in \bigcap_{n=1}^{\infty} E_n$ *and* $\overline{d}(E_n, x) \geq \varrho$ *for* $n \geq 1$, *then there exists a measurable set* E *such that* $\overline{d}(E, x) \geq \varrho$, $x \in E$, *and for each* $n \in \mathbb{N}$ *there exists* $\delta_n > 0$ *for which* $E \cap [x - \delta_n, x + \delta_n] \subset E_n$.

We shall give an equivalent condition of weak ϱ-upper continuity at a point.

Theorem 1. *If* $0 < \varrho < 1$ *and* $f \colon I \to \mathbb{R}$ *is a measurable function, then* f *is weakly* ϱ*-upper continuous at* $x \in I$ *if and only if*

$$\overline{d}\,(\{y \in I \colon |f(x) - f(y)| < \varepsilon\}, x) \geq \varrho \quad \textit{for every} \quad \varepsilon > 0.$$

Proof. Assume that f is weakly ϱ-upper continuous at x. Let $E \subset I$ be a measurable set such that $x \in E$, $f|_E$ is continuous at x and $\overline{d}(E, x) \geq \varrho$. Since $f|_E$ is continuous at x, for each $\varepsilon > 0$ we can find $\delta > 0$ such that $[x - \delta, x + \delta] \cap E \subset \{y \in E \colon |f(x) - f(y)| < \varepsilon\}$. Hence for each $\varepsilon > 0$

$$\overline{d}\,(\{y \in I \colon |f(x) - f(y)| < \varepsilon\}, x) \geq \overline{d}\,(\{y \in E \colon |f(x) - f(y)| < \varepsilon\}, x) = \\ = \overline{d}(E, x) \geq \varrho.$$

Finally, assume that for each $\varepsilon > 0$,

$$\overline{d}\big(\{y \in I \colon |f(x) - f(y)| < \varepsilon\}, x\big) \geq \varrho.$$

By Lemma 1 for sets $E_n = \left\{y \in I \colon |f(x) - f(y)| < \frac{1}{n}\right\}$, where $n \in \mathbb{N}$, we can construct a measurable set E such that $x \in E$, $\overline{d}(E, x) \geq \varrho$ and for each n there exists $\delta_n > 0$ for which $E \cap [x - \delta_n, x + \delta_n] \subset E_n$. The last condition implies that $f|_E$ is continuous at x. It follows that f is weakly ϱ-upper continuous at x, what was to be shown. □

Now we will show that the family of weakly ϱ-upper continuous functions is closed under uniform limits, i.e. every limit of uniformly convergent sequence of functions from $u\mathcal{UC}_\varrho$ belongs to this family.

Theorem 2. *If* $0 < \varrho < 1$ *and a sequence* $(f_n)_{n\geq 1}$ *of weakly* ϱ*-upper continuous functions is uniformly convergent to a function* f*, then* f *is weakly* ϱ*-upper continuous.*

Proof. Let $(f_n)_{n\geq 1}$ be a sequence of weakly ϱ-upper continuous functions uniformly converges to f. Let $x_0 \in I$ and $\varepsilon > 0$. There exists $n_0 \geq 1$ such that for every $k > n_0$ and every $x \in I$ the inequality

$$|f_k(x) - f(x)| < \frac{\varepsilon}{3}$$

holds. Fix $n > n_0$. Since f_n is weakly ϱ-upper continuous at x_0, there exists a measurable set $E \subset I$ such that $x_0 \in E$, $f_n|_E$ is continuous at x_0 and $\overline{d}(E, x_0) \geq \varrho$. Then there exists a positive δ such that

$$[x_0 - \delta, x_0 + \delta] \cap E \subset \left\{x \in E\colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\right\}.$$

Notice that

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$

if $x \in [x_0 - \delta, x_0 + \delta] \cap E$. Therefore

$$\left\{x \in E\colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\right\} \subset \{x\colon |f(x) - f(x_0)| < \varepsilon\}.$$

Hence

$$\overline{d}\left(\{x\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\right) \geq \\ \geq \overline{d}\left(\left\{x \in E\colon |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}\right\}, x_0\right) = \overline{d}(E, x_0).$$

Therefore

$$\overline{d}\left(\{x\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\right) \geq \overline{d}(E, x_0) \geq \varrho.$$

It means that the function f is weakly ϱ-upper continuous at x_0. $\square$

Example 3. *We shall show that the family of* ϱ*-upper continuous functions is not closed under the operation of uniform convergence.*

Define the function f in the same way as in Example 1. Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $f_n = \min\{1 - \frac{1}{n}, f\}$ for each $n \geq 1$. Then, clearly, the sequence $(f_n)_{n\geq 1}$ uniformly converges to f and $f \notin \mathcal{UC}_\varrho$. Since

$$\left\{x\colon f_n(x) = 1 - \tfrac{1}{n} = f(0)\right\} = \left\{x\colon |f(x) - 1| < \tfrac{1}{n}\right\}$$

and $\overline{d}\left(\{x\colon |f(x) - 1| < \frac{1}{n}\}, 0\right) = \varrho + \frac{1}{n}(1 - \varrho) > \varrho$, we infer that $f_n \in \mathcal{UC}_\varrho$ for each $n \geq 1$.

3. Maximal additive family

Definition 6. *Let $\mathcal{F}$ be any family of real valued functions defined on I. The set $\mathcal{M}_a(\mathcal{F}) = \{g\colon \forall_{f\in\mathcal{F}}\ f+g \in \mathcal{F}\}$ is called a maximal additive family for $\mathcal{F}$.*

Remark 1. *If a zero (constant) function is a member of a family of functions F, then $\mathcal{M}_a(\mathcal{F}) \subset\ \mathcal{F}$.*

Lemma 2. [3] *Let numbers c and γ fulfil the inequality $0 < c < \gamma < 1$. Moreover, let E be a measurable subset of $\mathbb{R}$ with the property $\overline{d}^+(E, x) = c$ for some point $x \in \mathbb{R}$. Then there exists a measurable set H such that $E \subset H$, $\overline{d}^+(H, x) \geq \gamma$ and $\overline{d}^+(H \setminus E, x) \leq \gamma - c(1-\gamma)$.*

The proof of next theorem is based on the proof of Theorem 2.1 in [3], where the maximal additive class for ϱ-upper continuous functions is discussed.

Theorem 3. *If $0 < \varrho < 1$, then for each $f \in u\mathcal{UC}_\varrho \setminus \mathcal{A}$ there exists $g\colon I \to \mathbb{R}$ such that $g \in u\mathcal{UC}_\varrho$ and $f + g \notin u\mathcal{UC}_\varrho$.*

Proof. Since $f \notin \mathcal{A}$, there exist $x_0 \in I$ and $\varepsilon > 0$ such that

$$\overline{d}^+ (\{x \in I\colon |f(x) - f(x_0)| \geq \varepsilon\}, x_0) > 0$$

or

$$\overline{d}^- (\{x \in I\colon |f(x_0) - f(x)| \geq \varepsilon\}, x_0) > 0.$$

Without loss of generality we may assume that the first inequality holds.

Put $E = \{x \in I\colon |f(x) - f(x_0)| \geq \varepsilon\}$ and $c = \overline{d}^+(E, x_0)$. Therefore $c > 0$. Let γ be a real number satisfying conditions $\gamma \geq \varrho$, $c < \gamma < 1$ and $\gamma - c(1-\gamma) < \varrho$. By Lemma 2, there exists a measurable set H such that $E \subset H$, $\overline{d}^+(H, x_0) \geq \gamma$ and $\overline{d}^+(H \setminus E, x_0) \leq \gamma - c(1-\gamma)$. Next one can find a sequence $([a_n, b_n])_{n\geq 1}$ of closed intervals such that $x_0 < b_{n+1} < a_n < b_n$ for each $n \geq 1$ and

$$\overline{d}^+ \left(\bigcup_{n=1}^{\infty} [a_n, b_n] \setminus H, x_0 \right) = \overline{d}^+ \left(H \setminus \bigcup_{n=1}^{\infty} [a_n, b_n], x_0 \right) = 0.$$

Thus $\overline{d}^+ \left(\bigcup_{n=1}^{\infty} [a_n, b_n], x_0 \right) = \overline{d}^+ (H, x_0) \geq \gamma \geq \varrho$.

Let $([c_n, d_n])_{n\geq 1}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for all n and $\overline{d}^+ \left(\bigcup_{n=1}^{\infty} ([c_n, d_n] \setminus [a_n, b_n]), x_0 \right) = 0$. Put $I_n = [a_n, b_n]$ and $K_n = [c_n, d_n]$ for each $n \geq 1$. Define a function $g\colon (a, b) \to \mathbb{R}$ letting

$$g(x)=\begin{cases} 0 & \text{if } x\in\{x_0\}\cup\bigcup\limits_{n=1}^{\infty} I_n,\\ -f(x)+f(x_0)+\varepsilon & \text{if } x\in(a,x_0)\cup\bigcup\limits_{n=1}^{\infty}[d_{n+1},c_n]\cup[d_1,b),\\ \text{linear on each interval } [c_n,a_n],[b_n,d_n], n\geq 1. \end{cases}$$

Since $f\in u\mathcal{UC}_\varrho$, g is weakly ϱ-upper continuous at each point except at x_0. Applying inequality

$$\overline{d}\left(\{x\colon g(x)=g(x_0)=0\},x_0\right)\geq \overline{d}^{+}\left(\bigcup_{n=1}^{\infty} I_n,x_0\right)\geq\varrho,$$

we conclude that g is weakly ϱ-upper continuous at x_0, too. It means that $g\in u\mathcal{UC}_\varrho$.

Now, we shall show that $f+g$ is not weakly ϱ-upper continuous at x_0. Put

$$F=\{x\in I\colon\ |(f+g)(x)-(f+g)(x_0)|<\varepsilon\}.$$

For $x\notin\bigcup\limits_{n=1}^{\infty}K_n\cup\{x_0\}$, we have $(f+g)(x)-(f+g)(x_0)=\varepsilon$. Therefore $F\subset\bigcup\limits_{n=1}^{\infty}K_n\cup\{x_0\}$. If $x\in\bigcup\limits_{n=1}^{\infty}I_n$, then $(f+g)(x)-(f+g)(x_0)=f(x)-f(x_0)$ and consequently $\bigcup\limits_{n=1}^{\infty}I_n\cap F\subset\bigcup\limits_{n=1}^{\infty}I_n\setminus E$. Thus

$$\begin{aligned}\overline{d}(F,x_0)=\overline{d}^{+}(F,x_0)&\leq\overline{d}^{+}\left(F\cap\bigcup_{n=1}^{\infty}I_n,x_0\right)+\overline{d}^{+}\left(F\setminus\bigcup_{n=1}^{\infty}I_n,x_0\right)\leq\\ &\leq\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}I_n\setminus E,x_0\right)+\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}(K_n\setminus I_n),x_0\right)=\\ &=\overline{d}^{+}(H\setminus E,x_0)\leq\gamma-c(1-\gamma)<\varrho.\end{aligned}$$

It follows that $f+g$ is not weakly ϱ-upper continuous at x_0. Hence $f+g\notin u\mathcal{UC}_\varrho$, which completes the proof. □

The proof of the following lemma is identical to the proof of Lemma 2.2 in [3] and we omit it.

Lemma 3. *Let $f\colon I\to\mathbb{R}$, $g\colon I\to\mathbb{R}$ be weakly ϱ-upper continuous at some point $x\in I$, where $0<\varrho<1$. If at least one of those functions is approximately continuous at x, then $f+g$ and $f\cdot g$ are weakly ϱ-upper continuous at x.*

Corollary 9. *Let $g\colon I \to \mathbb{R}$ be weakly ϱ-upper continuous at some point $x \in I$, where $0 < \varrho < 1$. If $f\colon I \to \mathbb{R}$ is approximately continuous at x, then $f + g$ and $f \cdot g$ are weakly ϱ-upper continuous at x.*

Corollary 10. *Let $f\colon I \to \mathbb{R}$, $g\colon I \to \mathbb{R}$ be weakly ϱ-upper continuous in I, where $0 < \varrho < 1$. If $D_{ap}(f) \cap D_{ap}(g) = \emptyset$, where $D_{ap}(f)$ denotes the set of all points at which f is not approximately continuous, then $f + g$ and $f \cdot g$ are weakly ϱ-upper continuous in I.*

Theorem 4. *If $0 < \varrho < 1$, then $\mathcal{M}_a(u\mathcal{UC}_\varrho) = \mathcal{A}$.*

Proof. By Theorem 3, we have $u\mathcal{UC}_\varrho \cap \mathcal{M}_a(u\mathcal{UC}_\varrho) \subset \mathcal{A}$. By Remark 1, we have the inclusion $\mathcal{M}_a(u\mathcal{UC}_\varrho) \subset u\mathcal{UC}_\varrho$. Therefore $\mathcal{M}_a(u\mathcal{UC}_\varrho) \subset \mathcal{A}$. Finally, by Lemma 9, we have $\mathcal{A} \subset \mathcal{M}_a(u\mathcal{UC}_\varrho)$. □

4. Maximal multiplicative family

Definition 7. *If $\mathcal{F}$ is any family of real valued functions defined on an open interval I, then the set $\{g\colon \forall_{f\in\mathcal{F}}\ f \cdot g \in \mathcal{F}\}$ is called a maximal multiplicative family for $\mathcal{F}$ and is denoted by $\mathcal{M}_m(\mathcal{F})$.*

Remark 2. *If a constant function equalled to 1 is a member of a family of functions $\mathcal{F}$, then $\mathcal{M}_m(\mathcal{F}) \subset \mathcal{F}$.*

Lemma 4. *If $0 < \varrho < 1$ and a measurable function $f\colon I \to \mathbb{R}$ is not approximately continuous at some point x_0 from I for which $f(x_0) \neq 0$, then there exists $g\colon I \to \mathbb{R}$ such that $g \in u\mathcal{UC}_\varrho$ and $f \cdot g \notin u\mathcal{UC}_\varrho$.*

Proof. Without loss of generality we may assume that f is not approximately continuous from right side at x_0. Then we can find a positive ε such that $\varepsilon < |f(x_0)|$ and

$$\overline{d}^+\big(\{x \in I\colon |f(x) - f(x_0)| \geq \varepsilon\}, x_0\big) = c > 0.$$

Put $E = \{x \in I\colon |f(x) - f(x_0)| \geq \varepsilon\}$. Take γ such that

$$\varrho \leq \gamma < 1, \quad c < \gamma \quad \text{and} \quad \gamma - c(1-\gamma) < \varrho.$$

By Lemma 2, there exists a measurable set H such that

$$E \subset H, \quad \overline{d}^+(H, x_0) \geq \gamma \quad \text{and} \quad \overline{d}^+(H \setminus E, x_0) \leq \gamma - c(1-\gamma).$$

Similarly as in proof of Lemma 3.1 in [3] we can find a sequence $([a_n, b_n])_{n\geq 1}$ of closed intervals such that $x_0 < b_{n+1} < a_n < b_n$ for each $n \geq 1$ and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n] \setminus H, x_0\right) = \overline{d}^+\left(H \setminus \bigcup_{n=1}^{\infty}[a_n, b_n], x_0\right) = 0.$$

Then $\overline{d}^+\left(\bigcup_{n=1}^{\infty}[a_n, b_n], x_0\right) = \overline{d}^+(H, x_0) \geq \gamma \geq \varrho$.

Let $([c_n, d_n])_{n\geq 1}$ be a sequence of pairwise disjoint closed intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for all n and $\overline{d}^+\left(\bigcup_{n=1}^{\infty}([c_n, d_n] \setminus [a_n, b_n]), x_0\right) = 0$. Denote now $I_n = [a_n, b_n]$, $K_n = [c_n, d_n]$ for each $n \geq 1$. Define a function $g\colon (a, b) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} I_n \cup [b_1, b), \\ 0 & \text{if } x \in (a, x_0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on each interval } [c_n, a_n], [b_{n+1}, d_{n+1}], n \geq 1. \end{cases}$$

Then g is continuous except at x_0. Moreover,

$$\overline{d}\left(\{x \in I\colon g(x) = 1\}, x_0\right) \geq \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, x_0\right) = \overline{d}^+\left(H, x_0\right) \geq \gamma \geq \varrho$$

and g restricted to $\{x \in I\colon g(x) = 1\}$ is continuous at x_0. It follows that g is weakly ϱ-upper continuous at x_0. Therefore $g \in w\mathcal{UC}_\varrho$.

Moreover, $(f \cdot g)(x_0) = f(x_0)$ and

$$\{x\colon\ |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\} \cap \left((a, x_0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n]\right) = \emptyset.$$

Then

$$\begin{aligned} &\overline{d}\left(\{x \in I\colon\ |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\}, x_0\right) \leq \\ &\leq \overline{d}^+\left(\left\{x \in \bigcup_{n=1}^{\infty} K_n\colon\ |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\right\}, x_0\right) = \\ &= \overline{d}^+\left(\left\{x \in \bigcup_{n=1}^{\infty} I_n\colon\ |f(x) - f(x_0)| < \varepsilon\right\}, x_0\right) = \\ &= \overline{d}^+\left(\{x \in H\colon\ |f(x) - f(x_0)| < \varepsilon\}, x_0\right) = \\ &= \overline{d}^+\left(\{x \in H \setminus E\colon\ |f(x) - f(x_0)| < \varepsilon\}, x_0\right) \leq \gamma - c(1 - \gamma) < \varrho. \end{aligned}$$

It implies that $f \cdot g$ is not weakly ϱ-upper continuous at x_0 i.e. $fg \notin w\mathcal{UC}_\varrho$. $\square$

Definition 8. *If $0 < \varrho < 1$, then by $\mathcal{W}(\varrho)$ we shall denote the family of all measurable functions $f\colon I \to \mathbb{R}$ such that at each $x_0 \in D_{ap}(f)$ the following two conditions hold*

$(W1)$ $f(x_0) = 0$ (*in other words $D_{ap}(f) \subset N_f$, where $N_f = \{x\colon f(x) = 0\}$*);

(W2) *for each* $\varepsilon > 0$ *and for each measurable set* F *such that* $F \supset N_f$ *and* $\overline{d}(F, x_0) \geq \varrho$ *we have*

$$\overline{d}(F \cap \{x \in I \colon |f(x) - f(x_0)| < \varepsilon\}, x_0) \geq \varrho.$$

Theorem 5. $\mathcal{M}_m(u\mathcal{UC}_\varrho) = \mathcal{W}(\varrho)$ *for each* ϱ *such that* $0 < \varrho < 1$.

Proof. Fix ϱ from the interval $(0, 1)$. Let $f \in \mathcal{W}(\varrho)$ and $g \in u\mathcal{UC}_\varrho$. Take any $x_0 \in I$. Then we can find a measurable set E such that $x_0 \in E$, $\overline{d}(E, x_0) \geq \varrho$ and $g|_E$ is continuous at x_0.

First, we assume that f is approximately continuous at x_0. Then, by Lemma 9, $f \cdot g$ is weakly ϱ-upper continuous at x_0.

Now, we assume that $x_0 \in D_{ap}(f)$. By condition $(W1)$, we obtain $f(x_0) = 0$. Since $g|_E$ is continuous at x_0, there exist positive numbers r and M such that $|g(x)| < M$ for $x \in E \cap [x_0 - r, x_0 + r]$. Put $F = E \cup N_f$. Then $N_f \subset F$ and $\overline{d}(F, x_0) \geq \varrho$. Let $\varepsilon > 0$. Then

$$\{x \in I \colon |(f \cdot g)(x)| < \varepsilon\} \cap [x_0 - r, x_0 + r] \supset$$
$$\supset F \cap \left\{x \in I \colon |f(x)| < \tfrac{\varepsilon}{M}\right\} \cap [x_0 - r, x_0 + r].$$

By condition $(W2)$, we have

$$\overline{d}\big(\{x \colon |(f \cdot g)(x)| < \varepsilon\}, x_0\big) \geq \overline{d}\left(\left\{x \colon |f(x)| < \tfrac{\varepsilon}{M}\right\} \cap F, x_0\right) =$$
$$= \overline{d}\big(\{x \colon |f(x)| < \varepsilon'\} \cap F, x_0\big) \geq \varrho,$$

where $\varepsilon' = \frac{\varepsilon}{M}$. By Theorem 1, $f \cdot g$ is weakly ϱ-upper continuous at x_0. Hence $f \cdot g \in u\mathcal{UC}_\varrho$. In this way, we have proven that $\mathcal{W}(\varrho) \subset \mathcal{M}_m(u\mathcal{UC}_\varrho)$.

Finally, assume that $f \in \mathcal{M}_m(u\mathcal{UC}_\varrho)$. If $x_0 \in D_{ap}(f)$, then, by Lemma 4, we obtain $f(x_0) = 0$. Therefore f satisfies the condition $(W1)$. Take any measurable set F such that $N_f \subset F$ and $\overline{d}(F, x) \geq \varrho$. Identically as in the proof of Theorem 3.1 in [3] we can find sequences $([a_n, b_n])_{n\geq 1}$, $([c_n, d_n])_{n\geq 1}$, $([a'_n, b'_n])_{n\geq 1}$, $([c'_n, d'_n])_{n\geq 1}$, $(\alpha_n)_{n\geq 1}$, $(\alpha'_n)_{n\geq 1}$ that satisfy conditions listed in that proof.

Define a function $g \colon (a, b) \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1, & \text{if } x \in \bigcup\limits_{n=1}^{\infty} [a_n, b_n] \cup \bigcup\limits_{n=1}^{\infty} [a'_n, b'_n] \cup (a, a'_1] \cup [b_1, b) \cup \{x_0\}, \\ \alpha_n, & \text{if } x \in [d_{n+1}, c_n],\ n = 1, 2, \ldots, \\ \alpha'_n, & \text{if } x \in [d'_n, c'_{n+1}],\ n = 1, 2, \ldots, \\ \text{linear on each } [c_n, a_n], [b_{n+1}, d_{n+1}], [c'_{n+1}, a'_{n+1}], [b'_n, d'_n], n \geq 1. \end{cases}$$

It follows directly from the definition of g, that g is continuous at each point except at x_0. Since

$$\overline{d}\left(\bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [a'_n, b'_n], x_0\right) = \overline{d}(F, x_0)$$

and g restricted to the set $\bigcup_{n=1}^{\infty}[a_n, b_n] \cup \bigcup_{n=1}^{\infty}[a'_n, b'_n] \cup \{x_0\}$ is constant, g is weakly ϱ-upper continuous at x_0. Thus $g \in u\mathcal{UC}_\varrho$. Hence $f \cdot g \in u\mathcal{UC}_\varrho$. Moreover, $(f \cdot g)(x_0) = 0$. Put

$$E_\varepsilon = \{x \in I \colon |(f \cdot g)(x) - (f \cdot g)(x_0)| < \varepsilon\} = \{x \in I \colon |(f \cdot g)(x)| < \varepsilon\}$$

if $0 < \varepsilon < 1$. Since $f \cdot g \in u\mathcal{UC}_\varrho$, $\overline{d}(E_\varepsilon, x_0) \geq \varrho$. On the other hand, in the same way as in mentioned proof, we obtain

$$\overline{d}^{+}(E_\varepsilon, x_0) \leq \overline{d}^{+}\big(\{x \in F \colon |f(x)| < \varepsilon\}, x_0\big),$$
$$\overline{d}^{-}(E_\varepsilon, x_0) \leq \overline{d}^{-}\big(\{x \in F \colon |f(x)| < \varepsilon\}, x_0\big)$$

if $0 < \varepsilon < 1$. Thus $\overline{d}\big(\{x \in F \colon |f(x)| < \varepsilon\}, x_0\big) \geq \overline{d}(E_\varepsilon, x_0) \geq \varrho$. It follows that the condition $(W2)$ is satisfied and $f \in \mathcal{W}(\varrho)$. □

Corollary 11. *If a measurable function $f \colon I \to \mathbb{R}$ satisfies the following conditions:*

(1) $x_0 \in D_{ap}(f)$,
(2) $\overline{d}(N_f, x_0) \geq \varrho$,
(3) $f(x_0) = 0$

for some $x_0 \in I$ and $\varrho \in (0, 1)$, then $f \in \mathcal{W}(\varrho)$.

Corollary 12. $\mathcal{M}_a(u\mathcal{UC}_\varrho) = \mathcal{A} \subsetneq \mathcal{M}_m(u\mathcal{UC}_\varrho)$.

References

[1] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, Vol. **659**, Springer-Verlag Berlin Heidelberg New York, 1978.
[2] S. Kowalczyk, K. Nowakowska, *A note on ϱ-upper continuous functions*, Tatra Mt. Math. Publ. **44** (2009), 153-158.
[3] S. Kowalczyk, K. Nowakowska, *Maximal classes for ϱ-upper continuous functions*, Real Anal. Exchange **36** (2010/2011), 307-324.
[4] K. Nowakowska, *Algebraic properties of ϱ-upper continous and $[\lambda, \varrho]$-continuous functions*, Ph.D. thesis (in Polish), Technical University in Łódź, 2012.

Received: September 2014

Katarzyna Nowakowska
Pomeranian University in Słupsk,
Institute of Mathematics,
ul. Arciszewskiego 22d, 76-200 Słupsk, Poland
E-mail address: nowakowska_k@go2.pl

Małgorzata Turowska
Pomeranian University in Słupsk,
Institute of Mathematics,
ul. Arciszewskiego 22d, 76-200 Słupsk, Poland
E-mail address: malgorzata.turowska@apsl.edu.pl