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A NEUMANN BOUNDARY VALUE PROBLEM FOR A CLASS OF GRADIENT SYSTEMS

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Abstract. In this paper we study a class of two-point boundary value systems. Using very recent critical points theorems, we establish the existence of one non-trivial solution and infinitely many solutions of this problem, respectively.

Keywords: Neumann problems, weak solutions, critical points, $(p_1, \ldots, p_n)$-Laplacian.

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1. INTRODUCTION

In this paper, we study the Neumann boundary value problems:

$$\begin{cases} -(|u_1'(x)|^{p_1-2}u_1'(x))' + |u_1(x)|^{p_1-2}u_1(x) = \lambda F_{u_1}(x, u_1, \ldots, u_m), & x \in (a,b), \\ -(|u_2'(x)|^{p_2-2}u_2'(x))' + |u_2(x)|^{p_2-2}u_2(x) = \lambda F_{u_2}(x, u_1, \ldots, u_m), & x \in (a,b), \\ \ldots \\ -(|u_m'(x)|^{p_m-2}u_m'(x))' + |u_m(x)|^{p_m-2}u_m(x) = \lambda F_{u_m}(x, u_1, \ldots, u_m), & x \in (a,b), \\ u_i'(a) = u_i'(b) = 0, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $p_i > 1$ are constants, for $1 \le i \le m$, λ is a positive parameter, $F : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ is a function such that $F(., t_1, \ldots, t_m)$ is measurable in $[a,b]$ for all $(t_1, \ldots, t_m) \in \mathbb{R}^m$, $F(x, ., \ldots, .)$ is C^1 in $\mathbb{R}^m$ for every $x \in [a,b]$ and for every $\varrho > 0$,

$$\sup_{|(t_1,\ldots,t_m)| \le \varrho} \sum_{i=1}^{m} |F_{t_i}(x, t_1, \ldots, t_m)| \in L^1([a,b]),$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \le i \le m$.

In the last decade or so, many authors applied variational methods to study the existence or multiplicity solutions of the Neumann problem of its variations; see, for example, [6, 7, 9–13] and the references therein. We note that the main tools in these cited papers are several critical point theorems due to Bonanno [3], Bonanno and Bisci [4], Bonanno and Marano [8]. A Neumann boundary value problem for a class of gradient systems has already been studied by Afrouzi, Hadjian and Heidarkhani [1] and Hedarkhani and Tian [14] in the ODE case and Afrouzi, Heidarkhani and O'Regan [2] in the PDE case. In that papers at least three solutions are established. The aim of this article is to prove the existence of at least one non-trivial solution and infinitely many solutions for $(\mathcal{P}_\lambda)$ for appropriate values of the parameter λ belonging to a precise real interval. Our motivation comes from the recent paper [4, 10]. We want to systematically study a class of gradient systems under a Neumann boundary using Bonanno's critical point theorems. For basic notation and definitions, and also for a thorough account of the subject, we refer the reader to [15, 16].

2. PRELIMINARIES AND BASIC NOTATION

First we recall Bonanno's critical point theorems which is our main tool to transfer the question of existence of weak solutions of $(\mathcal{P}_\lambda)$ to the existence of critical points of the Euler functional.

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following two functions:

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}((r_1, r_2))} \frac{\sup_{u \in \Phi^{-1}((r_1, r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}((r_1, r_2))} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.1 ([3, Theorem 5.1])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in \left(\frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right)$ there is $u_{0,\lambda} \in \Phi^{-1}((r_1, r_2))$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for each $u \in \Phi^{-1}((r_1, r_2))$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([4, Theorem 2.1])**.** *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\left(\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r).$$

Under the above assumptions if $\gamma < +\infty$ *then, for each* $\lambda \in \left(0, \frac{1}{\gamma}\right)$*, the following alternative holds:*
either
(b_1) I_λ *possesses a global minimum,*
or
(b_2) *there is a sequence* $\{u_n\}$ *of critical points* (*local minima*) *of* I_λ *such that* $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

Let us introduce notation that will be used later. Let Y_i be the Sobolev space $W^{1,p_i}([a,b])$ endowed with the norm

$$\|u\|_{p_i} := \left(\int_a^b |u'(x)|^{p_i} dx + \int_a^b |u(x)|^{p_i} dx\right)^{1/p_i},$$

and let

$$k_i = 2^{(p_i-1)/p_i} \max\{(b-a)^{-1/p_i}, (b-a)^{(p_i-1)/p_i}\},$$

we recall the following inequality which we use in the sequel

$$|u(x)| \le k_i \|u\|_{p_i} \tag{2.1}$$

for all $u \in Y_i$, and for all $x \in [a,b]$. Let $K = \max\{k_i^{p_i}\}$, for $1 \le i \le m$. Here and in the sequel, $X := Y_1 \times \cdots \times Y_m$.

We say that $u = (u_1, \ldots, u_m)$ is a weak solution to the $(\mathcal{P}_\lambda)$ if $u = (u_1, \ldots, u_m) \in X$ and

$$\sum_{i=1}^m \int_a^b \left(|u_i'(x)|^{p_i-2} u_i'(x) v_i'(x) + |u_i(x)|^{p_i-2} u_i(x) v_i(x)\right) dx -$$

$$- \lambda \sum_{i=1}^m \int_a^b F_{u_i}(x, u_1, \ldots, u_m) v_i(x) dx = 0$$

for every $v = (v_1, \ldots, v_m) \in X$. For $\gamma > 0$ we denote the set

$$\Theta(\gamma) = \left\{(t_1, \ldots, t_m) \in \mathbb{R}^m : \sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i} \le \frac{\gamma}{\Pi_{i=1}^m p_i}\right\}. \tag{2.2}$$

Let

$$\Phi(u) = \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \tag{2.3}$$

$$\Psi(u) = \int_a^b F(x, u_1(x), \ldots, u_m(x)) dx. \tag{2.4}$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_m) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \sum_{i=1}^{m} \int_a^b \left(|u_i'(x)|^{p_i-2} u_i'(x) v_i'(x) + |u_i(x)|^{p_i-2} u_i(x) v_i(x) \right) dx,$$

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^{m} F_{u_i}(x, u_1(x), \dots, u_m(x)) v_i(x) dx$$

for every $v = (v_1, \dots, v_m) \in X$, respectively. Moreover, Φ is sequentially weakly lower semicontinuous, Φ' admits a continuous inverse on X^* as well as Ψ is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1, \dots, u_m) \in X$, let $(u_{1n}, \dots, u_{mn}) \to (u_1, \dots, u_m)$ weakly in X as $n \to +\infty$, then we have $(u_{1n}, \dots, u_{mn})$ converges uniformly to $(u_1, \dots, u_m)$ on $[a, b]$ as $n \to +\infty$ (see [16]). Since $F(x, ., \dots, .)$ is C^1 in $\mathbb{R}^m$ for every $x \in [a, b]$, the derivatives of F are continuous in $\mathbb{R}^m$ for every $x \in [a, b]$, so for $1 \le i \le m$, $F_{u_i}(x, u_{1n}, \dots, u_{mn}) \to F_{u_i}(x, u_1, \dots, u_m)$ strongly as $n \to +\infty$ which follows $\Psi'(u_{1n}, \dots, u_{mn}) \to \Psi'(u_1, \dots, u_m)$ strongly as $n \to +\infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [16].

3. RESULTS

Before our proof, we first list nonlinear term F which satisfies the following hypotheses, where μ_1, μ_2 and ν are some constants.

(H1) $F(x, 0, \dots, 0) = 0$ for a.e. $x \in [a, b]$,

(H2) $a_\nu(\mu_2) < a_\nu(\mu_1)$, where

$$a_\nu(\mu) := K\Pi_{i=1}^m p_i \frac{\int_a^b \sup_{(t_1,\dots,t_m)\in\Theta(\mu)} F(x, t_1, \dots, t_m) dx - \int_a^b F(x, \nu, \dots, \nu) dx}{\mu - K \sum_{i=1}^m (\Pi_{j=1, j\neq i}^m p_j) \nu^{p_i}},$$

(H3)

$$\frac{\int_a^b \sup_{(t_1,\dots,t_m)\in\Theta(\mu)} F(x, t_1, \dots, t_m) dx}{\mu} < \frac{\int_a^b F(x, \nu, \dots, \nu) dx}{K \sum_{i=1}^m (\Pi_{j=1, j\neq i}^m p_j) \nu^{p_i}},$$

(H4)

$$\liminf_{\mu\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_m)\in\Theta(\mu)} F(x, t_1, \dots, t_m) dx}{\mu} <$$

$$< \frac{1}{K\Pi_{i=1}^m p_i (b-a)} \limsup_{|t_1|\to+\infty,\dots,|t_m|\to+\infty} \frac{\int_a^b F(x, t_1, \dots, t_m) dx}{\sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i}}.$$

3.1. ONE NONTRIVIAL SOLUTION

We formulate our main result as follows:

Theorem 3.1. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with*

$$c_1 < K(b-a)\sum_{i=1}^{m}(\Pi_{j=1,j\neq i}^{m} p_j)d^{p_i} < c_2$$

such that (H1) *and* (H2) *are satisfies. Then, for each $\lambda \in (\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)})$, system $(\mathcal{P}_\lambda)$ admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0m}) \in X$ such that*

$$\frac{c_1}{K\Pi_{i=1}^{m} p_i} < \sum_{i=1}^{m}\frac{\|u_{0i}\|_{p_i}^{p_i}}{p_i} < \frac{c_2}{K\Pi_{i=1}^{m} p_i}.$$

Proof. To apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$, as (2.3) and (2.4). Moreover, Φ is sequentially weakly lower semicontinuous, Φ' admits a continuous inverse on X^* as well as $\Psi' : X \to X^*$ is a compact operator. Set $w(x) = (w_1(x), \dots, w_m(x))$ such that for $1 \le i \le m$,

$$w_i(x) = d$$

$r_1 = \frac{c_1}{K\Pi_{i=1}^{m} p_i}$ and $r_2 = \frac{c_2}{K\Pi_{i=1}^{m} p_i}$. It is easy to verify that $w = (w_1, \dots, w_m) \in X$, and in particular, one has

$$\|w_i\|_{p_i}^{p_i} = (b-a)d^{p_i}$$

for $1 \le i \le m$. So, from the definition of Φ, we have

$$\Phi(w) = (b-a)\sum_{i=1}^{m}\frac{d^{p_i}}{p_i}.$$

From the conditions $c_1 < K\sum_{i=1}^{m}(\Pi_{j=1,j\neq i}^{m} p_j)(b-a)d^{p_i} < c_2$, we obtain

$$r_1 < \Phi(w) < r_2.$$

Moreover, from (2.1) one has

$$\sup_{x\in[a,b]} |u_i(x)|^{p_i} \le k_i^{p_i}\|u_i\|_{p_i}^{p_i}$$

and

$$\sup_{x\in[a,b]} |u_i(x)|^{p_i} \le K\|u_i\|_{p_i}^{p_i}$$

for each $u = (u_1, \dots, u_m) \in X$, so from the definition of Φ, we observe that

$$\begin{aligned}\Phi^{-1}((-\infty, r_2)) &= \{(u_1, \dots, u_n) \in X \colon \Phi(u_1, \dots, u_n) < r_2\} = \\ &= \Big\{(u_1, \dots, u_n) \in X \colon \sum_{i=1}^{m}\frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_2\Big\} \subseteq \\ &\subseteq \left\{(u_1, \dots, u_n) \in X \colon \sum_{i=1}^{m}\frac{|u_i(x)|^{p_i}}{p_i} \le \frac{c_2}{\Pi_{i=1}^{m} p_i} \text{ for all } x \in [a,b]\right\},\end{aligned}$$

from which it follows

$$\sup_{(u_1,\ldots,u_m)\in\Phi^{-1}((-\infty,r_2))}\Psi(u)=\sup_{(u_1,\ldots,u_m)\in\Phi^{-1}((-\infty,r_2))}\int\limits_a^b F(x,u_1(x),\ldots,u_m(x))dx\le$$

$$\le\int\limits_a^b\sup_{(t_1,\ldots,t_m)\in\Theta(c_2)}F(x,t_1,\ldots,t_m)dx.$$

Since for $1\le i\le m$, for each $x\in[a,b]$, the condition (A1) ensures that

$$\beta(r_1,r_2)\le\frac{\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)-\Psi(w)}{r_2-\Phi(w)}\le$$

$$\le\frac{\int\limits_a^b\sup_{(t_1,\ldots,t_m)\in\Theta(c_2)}F(x,t_1,\ldots,t_m)dx-\Psi(w)}{r_2-\Phi(w)}\le a_d(c_2).$$

On the other hand, by similar reasoning as before, one has

$$\rho(r_1,r_2)\ge\frac{\Psi(w)-\sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(u)}{\Phi(w)-r_1}\ge$$

$$\ge\frac{\Psi(w)-\int\limits_a^b\sup_{(t_1,\ldots,t_m)\in\Theta(c_1)}F(x,t_1,\ldots,t_m)dx}{\Phi(w)-r_1}\ge a_d(c_1).$$

Hence, from Assumption (A2), one has $\beta(r_1,r_2)<\rho(r_1,r_2)$. Therefore, from Theorem 2.1, taking into account that the weak solutions of the system $(\mathcal{P}_\lambda)$ are exactly the solutions of the equation $\Phi'(u)-\lambda\Psi'(u)=0$, we have the conclusion. □

Now we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Suppose that there exist two positive constants c and d with*

$$c>K(b-a)\sum_{i=1}^m(\Pi_{j=1,j\ne i}^m p_j)d^{p_i}$$

such that (H1) *and* (H3) *hold. Then, for each*

$$\lambda\in\left(\frac{K(b-a)\sum_{i=1}^m(\Pi_{j=1,j\ne i}^m p_j)d^{p_i}}{\int\limits_a^b F(x,d,\ldots,d)dx},\ \frac{c}{\int\limits_a^b\sup_{(t_1,\ldots,t_m)\in\Theta(c)}F(x,t_1,\ldots,t_m)dx}\right),$$

system $(\mathcal{P}_\lambda)$ *admits at least one non-trivial weak solution* $u_0=(u_{01},\ldots,u_{0n})\in X$ *such that*

$$\sum_{i=1}^m\frac{\|u_{0i}\|_\infty^{p_i}}{p_i}<\frac{c}{K\Pi_{i=1}^m p_i}.$$

Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to our assumptions, one has

$$a_d(c_2) = K\Pi_{i=1}^m p_i \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(c)} F(x,t_1,\ldots,t_m)dx - \int_a^b F(x,d,\ldots,d)dx}{c - K(b-a)\sum_{i=1}^m(\Pi_{j=1,j\neq i}^m p_j)d^{p_i}} \leq$$

$$\leq K\Pi_{i=1}^m p_i \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(c)} F(x,t_1,\ldots,t_m)dx - \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(c)} F(x,t_1,\ldots,t_m)dx}{\frac{c}{K\sum_{i=1}^m(\Pi_{j=1,j\neq i}^m p_j)d^{p_i}}}}{c - K(b-a)\sum_{i=1}^m(\Pi_{j=1,j\neq i}^m p_j)d^{p_i}} =$$

$$= \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(c)} F(x,t_1,\ldots,t_m)dx}{c}.$$

On the other hand, taking Assumption (A1) into account, one has

$$\frac{\int_a^b F(x,d,\ldots,d)dx}{K(b-a)\sum_{i=1}^m(\Pi_{j=1,j\neq i}^m p_j)d^{p_i}} = a_d(c_1).$$

Moreover, since

$$\sup_{x\in[a,b]} |u_i(x)|^{p_i} \leq K\|u_i\|_{p_i}^{p_i}$$

for each $u = (u_1,\ldots,u_m) \in X$, an easy computation ensures that

$$\sum_{i=1}^m \frac{\|u_{0i}\|_\infty^{p_i}}{p_i} < \frac{c}{K\Pi_{i=1}^m p_i}$$

whenever $\Phi(u) < r_2$. Now, owing to Assumption (A3), it is sufficient to invoke Theorem 3.1 to conclude the proof. □

3.2. INFINITY MANY SOLUTIONS

Theorem 3.3. *Assume that* (H1) *and* (H4) *hold. Then, for every* $\lambda \in \Lambda := (\lambda_1, \lambda_2)$, *where*

$$\lambda_1 = \frac{(b-a)}{\limsup_{|t_1|\to+\infty,\ldots,|t_m|\to+\infty} \frac{\int_a^b F(x,t_1,\ldots,t_m)dx}{\sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i}}}$$

and

$$\lambda_2 = \frac{1}{K\Pi_{i=1}^m \liminf_{\mu\to+\infty} \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(\mu)} F(x,t_1,\ldots,t_m)dx}{\mu}},$$

the problem $(\mathcal{P}_\lambda)$ *admits an unbounded sequence of weak solutions which is unbounded in* X.

Proof. Our goal is to apply Theorem 2.2. Now, as has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Theorem 2.2. Let $\{c_n\}$ be a real sequence such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\liminf_{n\to+\infty} \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(c_n)} F(x,t_1,\ldots,t_m)dx}{c_n} = \mathcal{A}. \tag{3.1}$$

Taking into account (2.1) for every $u \in X$ one has

$$|u(x)| \leq K\|u\|_{p_i}.$$

Also note

$$\sum_{i=1}^{m} \frac{|u_i(x)|^{p_i}}{p_i} \leq K\left(\sum_{i=1}^{m} \frac{\|u_i(x)\|_{p_i}^{p_i}}{p_i}\right).$$

Hence, an easy computation ensures that $\sum_{i=1}^m u \leq c_n$ when ever $u \in \Phi^{-1}((-\infty, r_n))$, where

$$r_n = \frac{1}{K}\frac{c_n}{\Pi_{i=1}^m p_i}.$$

Taking into account $\left\|u_i^0\right\|_{p_i} = 0$ (where $u_i^0(x) = 0$ for every $x \in [a,b]$) and that $\int_a^b F(t,0,\ldots,0)\,\mathrm{d}x = 0$ for all $x \in [a,b]$, for every n large enough, one has

$$\varphi(r_n) = \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} =$$

$$= \inf_{\sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_n} \frac{\sup_{\sum_{i=1}^m \frac{\|v_i\|_{p_i}^{p_i}}{p_i} < r_n} \int_a^b F(t,v_1(x),\ldots,v_m(x))\,\mathrm{d}x - \int_a^b F(t,u_1(x),\ldots,u_m)\,\mathrm{d}x}{r_n - \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}} \leq$$

$$\leq \frac{\sup_{\sum_{i=1}^m \frac{\|v_i\|_{p_i}^{p_i}}{p_i} < r_n} \int_a^b F(t,v_1(x),\ldots,v_m(x))\,\mathrm{d}x}{r_n} \leq$$

$$\leq K\Pi_{i=1}^m p_i \liminf_{n\to+\infty} \frac{\int_a^b \sup_{(t_1,\ldots,t_m)\in\Theta(c_n)} F(x,t_1,\ldots,t_m)dx}{c_n}.$$

Therefore, since from assumption (H4) one has $\mathcal{A} < +\infty$, we obtain

$$\gamma = \liminf_{n\to+\infty} \varphi(r_n) \leq K\Pi_{i=1}^m p_i \mathcal{A} < +\infty. \tag{3.2}$$

Now, fix $\lambda \in (\lambda_1, \lambda_2)$ and let us verify that the functional I_λ is unbounded from below. Let $\{\xi_{i,n}\}$ be m be positive real sequences such that $\lim_{n\to+\infty} \sqrt{\sum_{i=1}^m \xi_{i,n}^2} = +\infty$, and

$$\limsup_{n\to+\infty} \frac{\int_a^b F(x, \xi_{1,n}, \ldots, \xi_{m,n})\, \mathrm{d}x}{\sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}} = \mathcal{B}. \tag{3.3}$$

For each $n \in \mathbb{N}$ define

$$w_{i,n}(x) := \xi_{i,n}$$

and put $w_n := (w_{1,n}, \ldots, w_{m,n})$.

We easily get that

$$\|w_{i,n}\|_{p_i}^{p_i} = (b-a)|\xi_{i,n}|^{p_i}.$$

At this point, bearing in mind (i), we infer

$$\Phi(w_n) - \lambda\Psi(w_n) = \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda \int_a^b F(x, \xi_{1,n}, \ldots, \xi_{m,n})dx, \quad n \in \mathbb{N}.$$

If $\mathcal{B} < +\infty$, let $\epsilon \in \left(\frac{1}{\lambda\mathcal{B}}, 1\right)$. By (3.3), there exists v_ϵ such that

$$\int_a^b F(x, \xi_{1,n}, \ldots, \xi_{m,n})dx > \epsilon\mathcal{B} \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.$$

Moreover,

$$\Phi(w_n) - \lambda\Psi(w_n) \le \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda\epsilon\mathcal{B} \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.$$

Taking into account the choice of ϵ, one has

$$\lim_{n\to+\infty} [\Phi(w_n) - \Psi(w_n)] = -\infty.$$

If $\mathcal{B} = +\infty$, let us consider $M > \frac{1}{\lambda}$. By (3.3), there exist v_m such that

$$\int_a^b F(x, \xi_{1,n}, \ldots, \xi_{m,n})dx > M \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_m.$$

Moreover,

$$\Phi(w_n) - \lambda\Psi(w_n) \le \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda M \sum_{i=1}^m \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.$$

Taking into account the choice of M, also in this case, one has

$$\lim_{n\to+\infty}[\Phi(w_n) - \Psi(w_n)] = -\infty.$$

Applying Theorem 2.2, we deduce that the functional $\Phi - \lambda\Psi$ admits a sequence of critical points which is unbounded in X. Hence, our claim is proved and the conclusion is achieved. □

Remark 3.4. If

$$\liminf_{\mu\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_m)\in\Theta(\mu)} F(x,t_1,\dots,t_m)dx}{\mu} = 0$$

and

$$\limsup_{|t_1|\to+\infty,\dots,|t_m|\to+\infty} \frac{\int_a^b F(x,t_1,\dots,t_m)dx}{\sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i}} = +\infty,$$

clearly, hypothesis (H4) is verified and Theorem 3.3 guarantees the existence of infinitely many weak solutions for problem $(\mathcal{P}_\lambda)$, for every $\lambda \in (0,+\infty)$, the main result ensures the existence of infinitely many weak solutions for problem $(\mathcal{P}_\lambda)$.

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