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SELECTED GEOMETRICAL AND PHYSICAL ASPECTS OF CONFORMAL TRANSFORMATIONS

Abstract

Introduction and aims: The conformal transformations play the crucial role in the analysis of the global structure of the space-time. The main goal of this article is to present the consequences of the conformal transformation of the metric like the creation of the energy and momentum for the gravitational field or the creation of the matter.
Material and methods: In the paper have been proved the transformation formulas for some geometrical and physical objects. The analytical methods have been used in the paper.
Results: In the paper there were proved the transformation formulas of the affine connection, the covariant derivative, the geodesics, the curvature tensor, the Ricci tensor, the curvature scalar and the Weyl tensor. The Einstein's equations were proved not to be invariant under the conformal rescaling of the metric, as well as the Landau-Lifshitz pseudotensor.
Conclusion: The article shows that the conformal rescaling of the metric creates a new matter and an additional energy and momentum of the gravity. There is also possibility of creation of the Friedman universes from the vacuum.

Keywords: Conformal transformation of the metric, general theory of relativity, space-time, Riemannian manifold (pseudo-Riemannian manifold, Lorentzian manifold), Einstein's equations, Landau-Lifshitz pseudotensor of the energy-momentum.
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WYBRANE GEOMETRYCZNE I FIZYCZNE ASPEKTY TRANSFORMACJI KONFOREMNYCH

Streszczenie

Wstęp i cele: *Transformacje konforemne odgrywają istotną rolę w analizie globalnej struktury czasoprzestrzeni. Głównym celem tego artykułu było przedstawienie konsekwencji konforemnego przeskalowania metryki takich jak kreacja energii i pędu dla pola grawitacyjnego oraz kreacja materii.*
Materiał i metody: *W pracy zostały udowodnione wzory transformacyjne dla niektórych obiektów geometrycznych i fizycznych. Metody analityczne zostały wykorzystane w tym artykule.*
Wyniki: *W pracy zostały udowodnione prawa transformacyjne dla koneksji afinicznej, pochodnej kowariantnej, linii geodezyjnych, tensora krzywizny, tensora Ricciego, skalara krzywizny i tensora Weyla. Udowodniono, że równania Einsteina jak również tensor Landaua-Lifszyca nie są konforemnie niezmiennicze w wyniku konforemnego przeskalowania metryki.*
Wniosek: *Artykuł pokazuje, że konforemne przeskalowanie metryki kreuje nową materię i dodatkową energię oraz pęd grawitacji. Istnieje również możliwość tworzenia wszechświatów Friedmana z próżni*
Słowa kluczowe: *Transformacja konforemna metryki, ogólna teoria względności, czasoprzestrzeń, rozmaitość riemannowska (pseudoriemmanowska, lorentzowska), równania Einsteina, pseudotensor energii-pędu Landaua-Lifszyca.*
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1. Introduction

The conformal transformations play the crucial role in the analysis of the global structure of the space-time. The main application of the conformal transformations was done by Roger Penrose, the British mathematician and mathematical physicist. He showed that with the help of conformal transformations the infinitely distant regions of space-time can be shrinked to finity and then studied in the ordinary way. The one of the purposes of this paper is to check which geometrical and physical objects are invariants under the conformal rescaling of the metric.

2. The information about the actual mathematical model of the physical space-time

The theory of relativity stated by Albert Einstein concerns the structure of the space-time. It consists of the special theory of relativity (SR) and the general theory of relativity (GR). The former theory is valid for inertial reference frames and does not take into account the gravitation. In the later one are described events in the non-inertial reference frames, so it means that the gravity is taken into consideration. Both of them give the certain mathematical model of the space-time. This model is called shortly the space-time.
It is defined as a set of all possible physical events. According to the GR the space-time is a 4-dimentional connected manifold $M_4$of the class C^∞, with the Hausdorf's topology, orientable, having Lorentzian structure, non-extendable and time-oriented. The metric tensor of that space-time satisfies the Einstein's equations.

Definition2.1.

The conformal rescaling or the conformal transformation of the metric g on the Riemannian manifold (pseudoriemannian, Lorentzian) is the following transformation (in the established coordinates)

$$\hat{g}_{ab}(x)=\Omega^2(x)g_{ab}(x)$$

where $\Omega(x)$is a smooth and positive-definite function called the conformal factor.

Fact2.1.

Lets denote by g^{ab} the components of the tensor $(2,0)$which is inverse to the metric tensor. Then we have

$$\hat{g}^{ab} = \Omega^{-2}(x)g^{ab}.$$

Proof.
From the definition we have $g^{ab}g_{bd} = \delta^a_d$, where $\delta^a_d = \begin{cases} 1 \ if \ a = d \\ 0 \ if \ a \neq d \end{cases}$.

The symbol $\delta^a_d$is Kronecker's delta or the unit tensor. Because

$$\hat{g}_{bd} = \Omega^2 g_{bd}$$

and

$$\hat{g}_{bd}\hat{g}^{ab} = \Omega^2 g_{bd}\Omega^x g^{ab} = \delta^a_d$$

so to preserve an equality x must be equal -2.■

Fact 2.2.

The length of a vector changes under the conformal rescalling of the metric but the ratio of the vectors is preserved.

Proof.

The length of the vector in the metric $g_{ab}$is given by the formula

$$\|\vec{v}\| = \sqrt{(\vec{v}|\vec{v})} = \sqrt{g_{ik}v^i v^k}.$$

In the new metric $\hat{g}_{ab}$it has the following form

$$\|\vec{v}\| = \sqrt{\hat{g}_{ik}v^i v^k} = \sqrt{\Omega^2 g_{ik}v^i v^k} = \Omega\|\vec{v}\|.$$

Therefore the ratio of the vectors is not changed.

$$\frac{\|\vec{\hat{u}}\|}{\|\vec{\hat{v}}\|} = \frac{\|\vec{u}\|}{\|\vec{v}\|}. \blacksquare$$

Fact 2.3.

The angle between two vectors is an invariant of the conformal rescaling of the metric.

Proof.
The angle between two vectors we calculate following the formula

$$\cos(\vec{u}|\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}.$$

Then in the new coordinates it has the form

$$\cos(\vec{\hat{u}}|\vec{\hat{v}}) = \frac{\vec{\hat{u}}\cdot\vec{\hat{v}}}{\|\vec{\hat{u}}\|\|\vec{\hat{v}}\|} = \frac{\Omega^2\vec{u}\cdot\vec{v}}{\Omega^2\|\vec{u}\|\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}. \blacksquare$$

The last two facts justify the name “the conformal transformation” or “the conformal rescaling of the metric” from the first definition.

Fact 2.4.

The line element changes under the conformal rescaling of the metric as follows

$$d\hat{s}^2 = \Omega^2(x)ds^2.$$

Proof.

The line element is given by the formula

$$ds^2 = g_{ik}dx^i dx^k.$$

After the conformal rescaling of the metric the line element has the form

$$d\hat{s}^2 = \hat{g}_{ik}(x)dx^i dx^k = \Omega^2(x)g_{ik}(x)dx^i dx^k = \Omega^2(x)ds^2. \blacksquare$$

3. Selected geometrical aspects of conformal transformations

Definition 3.1.

Affine connection (in other words Christoffel symbols of the second kind or Christoffel’s connection) is defined as [2],[5]:

$$\Gamma^a_{bc} = \frac{1}{2}g^{ae}\left(\frac{\partial g_{be}}{\partial x^c} + \frac{\partial g_{ce}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^e}\right) = \frac{1}{2}g^{ae}\left(g_{be,c} + g_{ce,b} - g_{bc,e}\right).$$

Affine connection is an geometrical object, which permits us parallel transport of vectors and tensors on a manifold and develop tensor analysis on it.

Fact 3.1.

Christoffel symbols change under the conformal transformation of the metric as follows

$$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc} + P^a_{bc},$$

where $P_{bc}^{a} = \Omega^{-1}\left(\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e}\right)$.

Proof.

In the proof below we will write the Christoffel symbols in the new gauge and use the transformation law for the metric tensor.

$$\begin{aligned}
\hat{\Gamma}_{bc}^{a} &= \frac{1}{2}\hat{g}^{ae}\left(\hat{g}_{be,c} + \hat{g}_{ce,b} - \hat{g}_{bc,e}\right) \\
&= \frac{1}{2}\Omega^{-2} g^{ae}\left[(\Omega^2 g_{be})_{,c} + (\Omega^2 g_{ce})_{,b} - (\Omega^2 g_{bc})_{,e}\right] \\
&= \frac{1}{2}\Omega^{-2} g^{ae}\left(2\Omega\Omega_{,c} g_{be} + \Omega^2 g_{be,c} + 2\Omega\Omega_{,b} g_{ce} + \Omega^2 g_{ce,b} - 2\Omega\Omega_{,e} g_{bc} - \Omega^2 g_{bc,e}\right) \\
&= \frac{1}{2} g^{ae}\left(g_{be,c} + g_{ce,b} - g_{bc,e}\right) + \Omega^{-1} g^{ae}\left(\Omega_{,c} g_{be} + \Omega_{,b} g_{ce} - \Omega_{,e} g_{bc}\right) \\
&= \frac{1}{2} g^{ae}\left(g_{be,c} + g_{ce,b} - g_{bc,e}\right) + \Omega^{-1}\left(\delta_b^a \Omega_{,c} + \delta_c^a \Omega_{,b} - g^{ae} g_{bc} \Omega_{,e}\right) \\
&= \Gamma_{bc}^{a} + P_{bc}^{a}. \blacksquare
\end{aligned}$$

If $\Omega(x) = const$, then $\hat{\Gamma}_{bc}^{a} = \Gamma_{bc}^{a}$.

Definition 3.2.

Covariant derivative is defined as [4][1]

$$\nabla_i v^k \coloneqq \partial_i v^k + \Gamma_{ic}^{k} v^c$$

for the contravariant vector field i.e. for the tensor of type $(1,0)$ and

$$\nabla_i v_k \coloneqq \partial_i v_k + \Gamma_{ic}^{k} v_c$$

for the covariant vector field (covector or the 1-form or the tensor of type $(0,1)$).

Fact 3.2.

Covariant derivative changes under the conformal transformation of the metric in the following way

$$\hat{\nabla}_i v^k \coloneqq \nabla_i v^k + P_{ic}^{k} v_c,$$

and

$$\hat{\nabla}_i v_k \coloneqq \nabla_i v_k - P_{ic}^{k} v_c$$

where $P_{ic}^{k} = \Omega^{-1}\left(\delta_i^k \Omega_{,c} + \delta_c^k \Omega_{,i} - g_{ic} g^{ke} \Omega_{,e}\right)$.

Proof.

Writing down the covariant derivative in the new gauge in accordance with the definition 3 and developing it we get

$$\hat{\nabla}_i v^k = \partial_i v^k + \hat{\Gamma}_{ic}^{k} v^c = \partial_i v^k + \Gamma_{ic}^{k} v^c + P_{ic}^{k} v^c = \nabla_i v^k + P_{ic}^{k} v^c$$
$$\hat{\nabla}_i v^k = \nabla_i v^k \Leftrightarrow \Omega(x) = const$$
$$\hat{\nabla}_i v_k = \partial_i v_k = \hat{\Gamma}_{ik}^{c} v_c = \partial_i v^k - \Gamma_{ik}^{c} v_c - P_{ik}^{c} v_c = \nabla_i v_k - P_{ic}^{k} v_c$$
$$\hat{\nabla}_i v_k = \nabla_i v_k \Leftrightarrow \Omega(x) = const. \blacksquare$$

Fact 3.3.

Conformal transformation preserves the metricity of the connection.

Proof.

$$\begin{aligned}
\hat{\nabla}_c \hat{g}_{ab} &= \partial_c \hat{g}_{ab} - \hat{\Gamma}_{ac}^{d} \hat{g}_{db} - \hat{\Gamma}_{bc}^{d} \hat{g}_{ad} \\
&= (\Omega^2 g_{ab})_{,c} - (\Gamma_{ac}^{d} + P_{ac}^{d})\Omega^2 g_{db} - \left(\Gamma_{bc}^{d} + P_{bc}^{d}\right)\Omega^2 g_{ad} \\
&= \Omega_{,c}^2 g_{ab} + \Omega^2 g_{ab,c} - \Gamma_{ac}^{d}\Omega^2 g_{db} - \Gamma_{bc}^{d}\Omega^2 g_{ad} - P_{ac}^{d}\Omega^2 g_{db} - P_{bc}^{d}\Omega^2 g_{ad}.
\end{aligned}$$

Because $g_{ab,c} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} d_{ad} = 0$so we obtain

$$\hat{\nabla}_c \hat{g}_{ab} = \Omega^2_{,c} g_{ab} - P^d_{ac} \Omega^2 g_{db} - P^d_{bc} \Omega^2 g_{ad}$$
$$= 2\Omega\Omega_{,c} g_{ab} - \Omega^{-1}\left(\delta^d_a \Omega_{,c} + \delta^d_c \Omega_{,a} - g_{ac} g^{de} \Omega_{,e}\right)\Omega^2 g_{db}$$
$$= \Omega^{-1}\left(\delta^d_b \Omega_{,c} + \delta^d_c \Omega_{,b} - g_{bc} g^{de} \Omega_{,e}\right)\Omega^2 g_{ad}$$
$$= 2\Omega\Omega_{,c} g_{ab} - \Omega\left(g_{ab}\Omega_{,c} + g_{cb}\Omega_{,a} - g_{ac}\Omega_{,b}\right) - \Omega\left(g_{ab}\Omega_{,c} + g_{ac}\Omega_{,b} - g_{bc}\Omega_{,a}\right) = 0. \blacksquare$$

Hence the metricity of the connection is preserved.

Definition 3.3.

A geodesic is a curve on a Riemannian manifold (pseudoremannian or Lorentzian) which gives theextremum of the distance between any two neighbouring points [1]. Such a curve is a generalization of a straight line and satisfies the equations in so-called affine parameter u

$$\frac{d^2 x^i}{du^2} + \Gamma^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = 0.$$

Fact 3.4.

Under the conformal rescaling of the metric the equations of the geodesic develop as follows

$$\frac{d^2 x^i}{du^2} + \hat{\Gamma}^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = P^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du}.$$

Proof.

In the proof below we start with the equation of the geodesic

$$\frac{d^2 x^i}{du^2} + \Gamma^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = 0.$$

Then we use the Christoffel symbol after the conformal rescaling of the metric

$$\hat{\Gamma}^i_{kl} = \Gamma^i_{kl} + P^i_{kl}.$$

Replacing Γ^i_{kl} by the expression$\hat{\Gamma}^i_{kl} - P^i_{kl}$ in the initial equation we obtain

$$\frac{d^2 x^i}{du^2} + \left(\hat{\Gamma}^i_{kl} - P^i_{kl}\right) \frac{dx^k}{du} \frac{dx^l}{du} = 0. \blacksquare$$

As we can see it is not the equation of the geodesic $\hat{\Gamma}^i_{kl}$, except the case of $\Omega(x) = const.$
Generally one gets

$$\frac{d^2 x^i}{du^2} + \hat{\Gamma}^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = \Omega^{-1}\left(\delta^i_k \Omega_{,l} + \delta^i_l \Omega_{,k} - g_{kl} g^{ie} \Omega_{,e}\right) \frac{dx^k}{du} \frac{dx^l}{du}$$
$$= \Omega^{-1}\left[\Omega_{,k} \frac{dx^i}{du} \frac{dx^k}{du} + \Omega_{,k} \frac{dx^k}{du} \frac{dx^i}{du} - g_{kl} \frac{dx^k}{du} \frac{dx^l}{du} g^{ie} \Omega_{,e}\right].$$

If the geodesic was isotropic, then $g_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = 0$, and hence

$$\frac{d^2 x^i}{du^2} + \hat{\Gamma}^i_{kl} \frac{dx^k}{du} \frac{dx^l}{du} = \left(2\Omega^{-1}\Omega_{,k} \frac{dx^k}{du}\right) \frac{dx^i}{du} = h(u) \frac{dx^i}{du}.$$

This is an equation of the geodesic line with non-affine parameter u.
Then we have

Fact 3.5.

Under the conformal transformation of the metric only the null geodesics are preserved, but the affine parameter loses its affine character.

However we can introduce a new parameter $\lambda = \lambda(u)$ on these lines, which will be affine. Namely we have

Fact 3.6.

The new parameter λ defined by the equation $\frac{d\lambda}{du} = c\Omega^2$ where $c = const$ is an affine parameter.

Proof.

$$\frac{d^2x^i}{du^2} + \hat{\Gamma}^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du} = h(u)\frac{dx^i}{du}$$

where $h(u) = 2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}$.

Lets write $x^i = x^i[\lambda(u)]$.Then

$$\frac{dx^i}{du} = \frac{dx^i}{d\lambda}\frac{d\lambda}{du}$$

$$\frac{d^2x^i}{du^2} = \frac{d}{du}\left(\frac{dx^i}{d\lambda}\frac{d\lambda}{du}\right) = \frac{d^2x^i}{d\lambda^2}\left(\frac{d\lambda}{du}\right)^2 + \frac{dx^i}{d\lambda}\frac{d^2\lambda}{du^2}$$

Replacing $\frac{d^2x^i}{du^2}$ by the above expression in the initial equation we obtain

$$\frac{d^2x^i}{d\lambda^2}\left(\frac{d\lambda}{du}\right)^2 + \frac{dx^i}{d\lambda}\frac{d^2\lambda}{du^2} + \hat{\Gamma}^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}\left(\frac{d\lambda}{du}\right)^2 = h(u)\frac{dx^i}{d\lambda}\frac{d\lambda}{du}$$

or

$$\left(\frac{d^2x^i}{d\lambda^2} + \hat{\Gamma}^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda}\right)\left(\frac{d\lambda}{du}\right)^2 = h(u)\frac{dx^i}{d\lambda}\frac{d\lambda}{du} - \frac{d^2\lambda}{du^2}\frac{dx^i}{d\lambda}. \qquad (1)$$

The right side of the equation will be zero if

$$h(u)\frac{d\lambda}{du} - \frac{d^2\lambda}{du^2} = 0$$

or

$$2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}\frac{d\lambda}{du} - \frac{d^2\lambda}{du^2} = 0. \qquad (2)$$

On the other hand, if we differentiate $\frac{d\lambda}{du} = c\Omega^2$ then we have

$$\frac{d^2\lambda}{du^2} = 2c\Omega\Omega_{,k}\frac{dx^k}{du}.$$

Putting $\frac{d^2\lambda}{du^2}$ into the equation (2) we obtain (reminding that $\frac{d\lambda}{du} = c\Omega^2$)

$$2\Omega^{-1}\Omega_{,k}\frac{dx^k}{du}\frac{d\lambda}{du} - 2c\Omega\Omega_{,k}\frac{dx^k}{du} = 0\ ,$$

and the equations (1) take the form

$$\frac{d^2x^i}{d\lambda^2} + \hat{\Gamma}^i_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0\ .$$

Therefore the new parameter λ which satisfies the equation $\frac{d\lambda}{du} = c\Omega^2$, where $c = const$, is the affine parameter.■

Definition 3.4.

The curvature tensor (the Riemann-Christoffel tensor or the Riemann tensor) is defined as follows [1]

$$R^a_{.bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{nc}\Gamma^n_{bd} - \Gamma^a_{nd}\Gamma^n_{bc}\ .$$

The curvature tensor is the tensor of type $(1,3)$ and its components are defined by the metric tensor and its derivatives. It is a basic tool used in the differential geometry bacause it is a measure of the local curvature. Let us notice that for a flat manifold the Christoffel's symbols are not already equal zero in the curvilinear coordinates but the curvature tensor $R^a_{.bcd}$ is equal zero.

Fact 3.7.

The Riemann curvature changes under the conformal transformation of the metric as follows

$$\hat{R}^a_{.bcd} = R^a_{.bcd} + \frac{\Omega^2}{2}\left(\delta^a_{[c}\Omega_{d]b} - g_{b[c}\Omega^a_{d]}\right) = R^a_{.bcd} + \frac{\Omega^2}{4}\left(\delta^a_c\Omega_{db} - \delta^a_d\Omega_{cb} - g_{bc}\Omega^a_{.d} - g_{bd}\Omega^a_{.c}\right).$$

Proof.

The idea of our proof is to show that the both sides of the equation are equal.
From the definition we have

$$\begin{aligned}
\hat{R}^a_{.bcd} &= \hat{\Gamma}^a_{bd,c} - \hat{\Gamma}^a_{bc,d} + \hat{\Gamma}^a_{nc}\hat{\Gamma}^n_{bd} - \hat{\Gamma}^a_{nd}\hat{\Gamma}^n_{bc}\\
&= \Gamma^a_{bd,c} + P^a_{bd,c} - \Gamma^a_{bc,d} - P^a_{bc,d} + (\Gamma^a_{nc} + P^a_{nc})(\Gamma^n_{bd} + P^n_{bd}) - (\Gamma^a_{nd} + P^a_{nd})(\Gamma^n_{bc} + P^n_{bc})\\
&= \Gamma^a_{bd,c} + P^a_{bd,c} - \Gamma^a_{bc,d} - P^a_{bc,d} + \Gamma^a_{nc}\Gamma^n_{bd} + \Gamma^a_{nc}P^n_{bd} + P^a_{nc}\Gamma^n_{bd} + P^a_{nc}P^n_{bd} - \Gamma^a_{nd}\Gamma^n_{bc} - \Gamma^a_{nd}P^n_{bc}\\
&\qquad - P^a_{nd}\Gamma^n_{bc} - P^a_{nd}P^n_{bc}\\
&= \left(\Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{nc}\Gamma^n_{bd} - \Gamma^a_{nd}\Gamma^n_{bc}\right) + \left(P^a_{bd,c} - P^a_{bc,d}\right)\\
&\quad + (\Gamma^a_{nc}P^n_{bd} + P^a_{nc}\Gamma^n_{bd} + P^a_{nc}P^n_{bd} - \Gamma^a_{nd}P^n_{bc} - P^a_{nd}\Gamma^n_{bc} - P^a_{nd}P^n_{bc})\\
&= R^a_{.bcd} + \left(P^a_{bd,c} - P^a_{bc,d} + \Gamma^a_{nc}P^n_{bd} + P^a_{nc}\Gamma^n_{bd} + P^a_{nc}P^n_{bd} - \Gamma^a_{nd}P^n_{bc} - P^a_{nd}\Gamma^n_{bc} - P^a_{nd}P^n_{bc}\right)
\end{aligned}$$

where$R^a_{.bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{nc}\Gamma^n_{bd} - \Gamma^a_{nd}\Gamma^n_{bc}$.

To make easier further calculations we will divide the equation into three parts.

Lets denote

$$\begin{aligned}
I_1 &= \Gamma^a_{nc}P^n_{bd} + P^a_{nc}\Gamma^n_{bd} - \Gamma^a_{nd}P^n_{bc} - P^a_{nd}\Gamma^n_{bc}\,,\\
I_2 &= P^a_{nc}P^n_{bd} - P^a_{nd}P^n_{bc}\,,\\
I_3 &= P^a_{bd,c} - P^a_{bc,d}\,.
\end{aligned}$$

Then we compute I_1, I_2 and I_3.

$$\begin{aligned}
I_1 &= \Omega^{-1}\left[\Gamma^a_{nc}\left(\delta^n_b\Omega_{,d} + \delta^n_d\Omega_{,b} - g^{ne}g_{bd}\Omega_{,e}\right) + \Gamma^n_{bd}\left(\delta^a_n\Omega_{,c} + \delta^a_c\Omega_{,n} - g^{ae}g_{nc}\Omega_{,e}\right)\right]\\
&\quad -\Omega^{-1}\left[\Gamma^a_{nd}\left(\delta^n_b\Omega_{,c} + \delta^n_c\Omega_{,b} - g^{ne}g_{bc}\Omega_{,e}\right) + \Gamma^n_{bc}\left(\delta^a_n\Omega_{,d} + \delta^a_d\Omega_{,n} - g^{ae}g_{nd}\Omega_{,e}\right)\right]\\
&= \Omega^{-1}\left[\Gamma^a_{bc}\Omega_{,d} + \Gamma^a_{cd}\Omega_{,b} - \Gamma^a_{nc}g^{ne}g_{bd}\Omega_{,e} + \Gamma^a_{bd}\Omega_{,c} + \Gamma^n_{bd}\delta^a_c\Omega_{,n} - \Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}\right]\\
&\quad -\Omega^{-1}\left[\Gamma^a_{bd}\Omega_{,c} + \Gamma^a_{cd}\Omega_{,b} - \Gamma^a_{nd}g^{ne}g_{bc}\Omega_{,e} + \Gamma^a_{bc}\Omega_{,d} + \Gamma^n_{bc}\delta^a_d\Omega_{,n} - \Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}\right]\\
&= \Omega^{-1}\left(\Gamma^n_{bd}\delta^a_c\Omega_{,n} - \Gamma^a_{nc}g^{ne}g_{bd}\Omega_{,e} - \Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}\right)\\
&\quad +\Omega^{-1}\left(\Gamma^a_{nd}g^{ne}g_{bc}\Omega_{,e} - \Gamma^n_{bc}\delta^a_d\Omega_{,n} + \Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}\right)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \Omega^{-2}\left(\delta^a_n\Omega_{,c} + \delta^a_c\Omega_{,n} - g^{ae}g_{nc}\Omega_{,e}\right)\left(\delta^n_b\Omega_{,d} + \delta^n_d\Omega_{,b} - g^{nf}g_{bd}\Omega_{,f}\right)\\
&\quad -\Omega^{-2}\left(\delta^a_n\Omega_{,d} + \delta^a_d\Omega_{,n} - g^{ae}g_{nd}\Omega_{,e}\right)\left(\delta^n_b\Omega_{,c} + \delta^n_c\Omega_{,b} - g^{nf}g_{bc}\Omega_{,f}\right)\\
&= \Omega^{-2}\left(\delta^a_n\Omega_{,c}\delta^n_b\Omega_{,d} + \delta^a_n\Omega_{,c}\delta^n_d\Omega_{,b} - \delta^a_n\Omega_{,c}g^{ne}g_{bd}\Omega_{,e} + \delta^a_c\Omega_{,n}\delta^n_b\Omega_{,d}\right)\\
&\quad +\Omega^{-2}\left(\delta^a_c\Omega_{,n}\delta^n_d\Omega_{,b} - \delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e} - \delta^n_b\Omega_{,d}g^{ae}g_{nc}\Omega_{,e} - g^{ae}g_{nc}\Omega_{,e}\delta^n_d\Omega_{,b}\right)\\
&\quad +\Omega^{-2}\left(g^{ae}g_{nc}\Omega_{,e}g^{nf}g_{bd}\Omega_{,f} - \delta^a_n\Omega_{,d}\delta^n_b\Omega_{,c} - \delta^a_n\Omega_{,d}\delta^n_c\Omega_{,b} + g^{ne}g_{bc}\Omega_{,e}\delta^a_n\Omega_{,d}\right)\\
&\quad -\Omega^{-2}\left(\delta^a_d\Omega_{,n}\delta^n_b\Omega_{,c} + \delta^a_d\Omega_{,n}\delta^n_c\Omega_{,b} - \delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e} - g^{ae}g_{nd}\Omega_{,e}\delta^n_b\Omega_{,c}\right)\\
&\quad +\Omega^{-2}\left(g^{ae}g_{nd}\Omega_{,e}\delta^n_c\Omega_{,b} - g^{ae}g_{nd}\Omega_{,e}g^{nf}g_{bc}\Omega_{,f}\right)\\
&= \Omega^{-2}\left(\delta^a_b\Omega_{,c}\Omega_{,d} + \delta^a_d\Omega_{,c}\Omega_{,b} - \Omega_{,c}g^{ae}g_{bd}\Omega_{,e} + \delta^a_c\Omega_{,b}\Omega_{,d} + \delta^a_c\Omega_{,b}\Omega_{,d}\right)
\end{aligned}$$

$$-\Omega^{-2}\left(\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}+g^{ae}g_{bc}\Omega_{,e}\Omega_{,d}+\Omega_{,b}g^{ae}g_{dc}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,e}\delta^f_c\Omega_{,f}+\delta^a_b\Omega_{,d}\Omega_{,c}\right)$$
$$-\Omega^{-2}\left(\delta^a_c\Omega_{,d}\Omega_{,b}-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}+\delta^a_d\Omega_{,b}\Omega_{,c}+\delta^a_d\Omega_{,c}\Omega_{,b}-g^{ne}g_{bc}\Omega_{,e}\delta^a_d\Omega_{,n}\right)$$
$$+\Omega^{-2}\left(g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}+g^{ae}g_{cd}\Omega_{,e}\Omega_{,b}-g^{ae}g_{bc}\delta^f_d\Omega_{,e}\Omega_{,f}\right)$$
$$=\Omega^{-2}\left(\delta^a_c\Omega_{,b}\Omega_{,d}-\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}-\delta^a_d\Omega_{,c}\Omega_{,b}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}+g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}\right.$$
$$\left.-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}\right)$$

$$I_3=\Omega^{-1}\left(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e}\right)_{,c}-\Omega^{-1}\left(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e}\right)_{,d}$$
$$=\Omega^{-1}\left(\delta^a_b\Omega_{,dc}+\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec}\right)$$
$$-\Omega^{-1}\left(\delta^a_b\Omega_{,cd}+\delta^a_c\Omega_{,bd}-g^{ae}_{,d}g_{bc}\Omega_{,e}-g^{ae}g_{bc,d}\Omega_{,e}-g^{ae}g_{bc}\Omega_{,ed}\right)$$
$$+\Omega^{-1}_{,c}\left(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e}\right)-\Omega^{-1}_{,d}\left(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e}\right)$$
$$=\Omega^{-1}\left(\delta^a_b\Omega_{,cd}+\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec}\right)$$
$$-\Omega^{-1}\left(\delta^a_b\Omega_{,cd}+\delta^a_c\Omega_{,bd}-g^{ae}_{,d}g_{bc}\Omega_{,e}-g^{ae}g_{bc,d}\Omega_{,e}-g^{ae}g_{bc}\Omega_{,ed}\right)$$
$$-\frac{\Omega_{,c}}{\Omega^2}\left(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e}\right)+\frac{\Omega_{,d}}{\Omega^2}\left(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e}\right)$$
$$=\Omega^{-1}\left(\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec}\right)$$
$$-\Omega^{-1}\left(\delta^a_c\Omega_{,bd}-g^{ae}_{,d}g_{bc}\Omega_{,e}-g^{ae}g_{bc,d}\Omega_{,e}-g^{ae}g_{bc}\Omega_{,ed}\right)$$
$$-\frac{\Omega_{,c}}{\Omega^2}\left(\delta^a_b\Omega_{,d}+\delta^a_d\Omega_{,b}-g^{ae}g_{bd}\Omega_{,e}\right)+\frac{\Omega_{,d}}{\Omega^2}\left(\delta^a_b\Omega_{,c}+\delta^a_c\Omega_{,b}-g^{ae}g_{bc}\Omega_{,e}\right)$$

Summing I_1, I_2 and I_3 we obtain

$$\Omega^{-1}\left(\Gamma^n_{bd}\delta^a_c\Omega_{,n}-\Gamma^a_{nc}g^{ne}g_{bd}\Omega_{,e}-\Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}+\Gamma^a_{nd}g^{ne}g_{bc}\Omega_{,e}-\Gamma^n_{bc}\delta^a_d\Omega_{,n}\right)$$
$$\Omega^{-1}\left(\Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}+\delta^a_d\Omega_{,bc}-g^{ae}_{,c}g_{bd}\Omega_{,e}-g^{ae}g_{bd,c}\Omega_{,e}-g^{ae}g_{bd}\Omega_{,ec}\right)$$
$$-\Omega^{-1}\left(\delta^a_c\Omega_{,bd}-g^{ae}_{,d}g_{bc}\Omega_{,e}-g^{ae}g_{bc,d}\Omega_{,e}-g^{ae}g_{bc}\Omega_{,ed}\right)$$
$$+\Omega^{-2}\left(\delta^a_c\Omega_{,b}\Omega_{,d}-\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}-\delta^a_d\Omega_{,c}\Omega_{,b}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}\right)$$
$$+\Omega^{-2}\left(g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}-\delta^a_b\Omega_{,c}\Omega_{,d}-\delta^a_d\Omega_{,c}\Omega_{,b}\right)$$
$$+\Omega^{-2}\left(g^{ae}g_{bd}\Omega_{,e}\Omega_{,c}-g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}+\delta^a_c\Omega_{,b}\Omega_{,d}+\delta^a_b\Omega_{,c}\Omega_{,d}\right)$$

Using the equations $g_{bd,c}=\Gamma^n_{bc}g_{nd}+\Gamma^n_{dc}g_{bn}$ and $g_{bc,d}=\Gamma^n_{bd}g_{nc}+\Gamma^n_{cd}g_{bn}$ we obtain

$$\Omega^{-1}\left(g_{bc,d}g^{ae}\Omega_{,e}-g_{bd,c}g^{ae}\Omega_{,e}-\Gamma^n_{bd}g^{ae}g_{nc}\Omega_{,e}+\Gamma^n_{bc}g^{ae}g_{nd}\Omega_{,e}\right)=0$$

Lets take the equation $g^{ae}{}_{;c}=g^{ae}{}_{,c}+\Gamma^a_{nc}g^{ne}+\Gamma^e_{nc}g^{an}$ and $g^{ae}{}_{;d}=g^{ae}{}_{,d}+\Gamma^a_{nd}g^{ne}+\Gamma^e_{nd}g^{an}$ then the formula reduces as follows

$$\Omega^{-1}\left(\Gamma^n_{bd}\delta^a_c\Omega_{,n}+\Gamma^e_{nc}g^{an}g_{bd}\Omega_{,e}-\Gamma^n_{bc}\delta^a_d\Omega_{,n}+\delta^a_d\Omega_{,bc}\right)$$
$$-\Omega^{-1}\left(\delta^a_c\Omega_{,bd}+g^{ae}g_{bd}\Omega_{,ec}-g^{ae}g_{bc}\Omega_{,ed}+\Gamma^e_{nd}g^{an}g_{bc}\Omega_{,e}\right)$$
$$+\Omega^{-2}\left(2\delta^a_c\Omega_{,b}\Omega_{,d}-2\delta^a_d\Omega_{,c}\Omega_{,b}+2\Omega_{,c}g^{ae}g_{bd}\Omega_{,e}+\delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}\right)$$
$$-\Omega^{-2}\left(2g^{ae}g_{bc}\Omega_{,d}\Omega_{,e}+\delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}\right)$$

After extending the right side of the initial equation we get the following form

$$R^a_{.bcd}+\frac{\Omega^2}{2}\left(\delta^a_{[c}\Omega_{d]b}-g_{b[c}\Omega^a_{.d]}\right)$$
$$=R^a_{.bcd}+\frac{\Omega^2}{2}\left[\frac{1}{2}\left(\delta^a_c\Omega_{db}-\delta^a_d\Omega_{cb}\right)-\frac{1}{2}\left(g_{bc}\Omega^a_{.d}-g_{bd}\Omega^a_{.c}\right)\right]$$
$$=R^a_{.bcd}+\frac{\Omega^2}{4}\left(\delta^a_c\Omega_{db}-\delta^a_d\Omega_{cb}-g_{bc}\Omega^a_{.d}+g_{bd}\Omega^a_{.c}\right)$$

Lets denote by I_4 the expression $\frac{\Omega^2}{4}\left(\delta^a_c\Omega_{db}-\delta^a_d\Omega_{cb}-g_{bc}\Omega^a_{.d}+g_{bd}\Omega^a_{.c}\right)$. Putting into

$$\Omega_{db} = g_{ad}\Omega^{a}_{.b}$$
$$= g_{ad}\left[4(\Omega^{-1})(\Omega^{-1})_{;be}g^{ae} - 2(\Omega^{-1})_{,p}(\Omega^{-1})_{,t}g^{pt}\delta^{a}_{b}\right]$$
$$= 4(\Omega^{-1})(\Omega^{-1})_{;be}g_{ad}g^{ae} - 2(\Omega^{-1})_{,p}(\Omega^{-1})_{,t}g_{ad}g^{pt}\delta^{a}_{b}$$
$$= 4(\Omega^{-1})(\Omega^{-1})_{;bd} - 2(\Omega^{-1})_{,p}(\Omega^{-1})_{,t}g_{bd}g^{pt}$$

and developing in a similar way Ω_{cb}, $\Omega^{a}_{.d}$ and $\Omega^{a}_{.c}$ we have

$$I_4 = \frac{\Omega^2}{4}\left(\frac{-4\delta^a_c\Omega_{,bd}}{\Omega^3} + \frac{8\delta^a_c\Omega_{,b}\Omega_{,d}}{\Omega^4} + \frac{4\delta^a_c\Gamma^e_{bd}\Omega_{,e}}{\Omega^3} - \frac{4\delta^a_c\Omega_{,p}\Omega_t g^{pt}g_{db}}{\Omega^4}\right)$$
$$+\frac{\Omega^2}{4}\left(\frac{4\delta^a_d\Omega_{,bc}}{\Omega^3} - \frac{8\delta^a_d\Omega_{,b}\Omega_{,c}}{\Omega^4} - \frac{4\delta^a_d\Gamma^e_{bc}\Omega_{,e}}{\Omega^3} + \frac{4\delta^a_d\Omega_{,p}\Omega_t g^{pt}g_{bc}}{\Omega^4}\right)$$
$$+\frac{\Omega^2}{4}\left(\frac{4\Omega_{,de}g^{ae}g_{bc}}{\Omega^3} - \frac{8\Omega_{,d}\Omega_{,e}g^{ae}g_{bc}}{\Omega^4} - \frac{4\Gamma^p_{de}\Omega_{,p}g_{bc}g^{ae}}{\Omega^3}\right)$$
$$-\frac{\Omega^2}{4}\left(\frac{4\Omega_{,ce}g^{ae}g_{bd}}{\Omega^3} - \frac{8\Omega_{,c}\Omega_{,e}g^{ae}g_{bd}}{\Omega^4} - \frac{4\Gamma^n_{ce}\Omega_{,n}g_{bd}g^{ae}}{\Omega^3}\right)$$
$$= \Omega^{-1}\left(-\delta^a_c\Omega_{,bd} + \Gamma^e_{bd}\delta^a_c\Omega_{,e} + \delta^a_d\Omega_{,bc} - \delta^a_d\Gamma^e_{bc}\Omega_{,e} + \Omega_{,de}g^{ae}g_{bc}\right)$$
$$-\Omega^{-1}\left(\Gamma^p_{de}\Omega_{,p}g_{bc}g^{ae} + \Omega_{,ce}g_{bd}g^{ae} - \Gamma^n_{ce}\Omega_{,n}g^{ae}g_{bd}\right)$$
$$+\Omega^{-2}\left(2\delta^a_c\Omega_{,b}\Omega_{,d} - \delta^a_c\Omega_{,p}\Omega_{,t}g^{pt}g_{bd} - 2\delta^a_d\Omega_{,b}\Omega_{,c}\right)$$
$$+\Omega^{-2}\left(\delta^a_d g^{pt}g_{bc}\Omega_{,p}\Omega_{,t} - 2\Omega_{,d}\Omega_{,e}g^{ae}g_{bc} + 2g^{ae}g_{bd}\Omega_{,c}\Omega_{,e}\right)$$

After changing some of the indexes we finally obtain that the left side of the previous equation equals the right side.

$$\Omega^{-1}\left(\Gamma^n_{bd}\delta^a_c\Omega_{,n} + \Gamma^e_{cn}g^{an}g_{bd}\Omega_{,e} - \Gamma^n_{bc}\delta^a_d\Omega_{,n} + \delta^a_d\Omega_{,bc}\right)$$
$$-\Omega^{-1}\left(\delta^a_c\Omega_{,bd} + g^{ae}g_{bd}\Omega_{,ce} - g^{ae}g_{bc}\Omega_{,de} + \Gamma^e_{nd}g^{an}g_{bc}\Omega_{,e}\right)$$
$$+\Omega^{-2}\left(2\delta^a_c\Omega_{,b}\Omega_{,d} - 2\delta^a_d\Omega_{,c}\Omega_{,b} + 2\Omega_{,c}g^{ae}g_{bd}\Omega_{,e} + \delta^a_d\Omega_{,n}g^{ne}g_{bc}\Omega_{,e}\right)$$
$$-\Omega^{-2}\left(2g^{ae}g_{bc}\Omega_{,d}\Omega_{,e} + \delta^a_c\Omega_{,n}g^{ne}g_{bd}\Omega_{,e}\right)$$

If $\Omega(x) = const$, then $\hat{R}^{a}_{.bcd} = R^{a}_{.bcd}$.■

Definition 3.5.

The Ricci tensor is defined as [2]

$$R^{b}_{.g} = R^{.b}_{g} = g^{ac}R^{.b..}_{a.cg} = R^{cb...}_{..cg}\ .$$

Fact 3.8.

The Ricci tensor changes under the conformal transformation of the metric as follows

$$\hat{R}^{b}_{.g} = \Omega^{-2}R^{b}_{.g} + \frac{\Omega^{b}_{.g}}{2} + \frac{\delta^{b}_{g}}{4}\Omega^{a}_{.a}\ .$$

Proof.

In the proof we are using the transformation laws for the curvature tensor and for the metric tensor.

$$\hat{R}^{b}_{.g} = \hat{g}^{al}\hat{R}^{b}_{.agl} = \Omega^{-2}g^{al}\left[R^{b}_{.agl} + \frac{\Omega^2}{4}\left(\delta^b_g\Omega_{la} - g_{ag}\Omega^{b}_{.l} - \delta^b_l\Omega_{ga} + g_{al}\Omega^{b}_{.g}\right)\right]$$
$$= \Omega^{-2}R^{b}_{.g} + \frac{1}{4}\left(\delta^b_g\Omega^{a}_{.a} - \delta^l_g\Omega^{b}_{.l} - g^{ab}\Omega_{ga} + \delta^a_a\Omega^{b}_{.g}\right)$$
$$= \Omega^{-2}R^{b}_{.g} + \frac{1}{4}\left(\delta^b_g\Omega^{a}_{.a} - \Omega^{b}_{.g} - \Omega^{b}_{.g} + 4\Omega^{b}_{.g}\right)$$
$$= \Omega^{-2}R^{b}_{.g} + \frac{\Omega^{b}_{.g}}{2} + \frac{\delta^b_g\Omega^{a}_{.a}}{4}$$

After lowering the upper index we obtain the form of the Ricci tensor as below

$$\hat{R}_{ag} = \hat{g}_{ab}\hat{R}^{b}_{.g} = \Omega^2 g_{ab}\Omega^{-2}R^{b}_{g} + \Omega^2 g_{ab}\frac{\Omega^{b}_{.g}}{2} + \Omega^2 g_{ab}\frac{\delta^{b}_{g}\Omega^{a}_{a}}{4}$$
$$= R_{ag} + \frac{\Omega^2}{2}\Omega_{ag} + \frac{\Omega^2}{4}g_{ab}\delta^{b}_{g}\Omega^{a}_{a}$$
$$= R_{ag} + \frac{\Omega^2}{2}\Omega_{ag} + \frac{\Omega^2}{4}g_{ab}\Omega^{a}_{.a}$$

If $\Omega(x) = const$, then $\hat{R}^{b}_{g} = \Omega^{-2}R^{b}_{g}$ and $\hat{R}_{ag} = R_{ag}$.■

Definition 3.6.

The curvature scalar we define as follows [2]

$$R = g^{ab}R_{ab} = R^{b}_{.b}$$

Fact 3.9.

The curvature scalar changes under the conformal transformation of the metric in the following way

$$\hat{R} = \Omega^{-2}R + \tfrac{3}{2}\Omega^{a}_{.a}\ .$$

Proof.

One has $R = R^{i}_{.i}$.

In our proof we are using the transformation law for the Ricci tensor.

$$\hat{R} = \hat{R}^{i}_{.i} = \Omega^{-2}R^{i}_{.i} + \frac{\Omega^{i}_{.i}}{2} + \frac{\delta^{i}_{i}}{4}\Omega^{a}_{.a} = \Omega^{-2}R^{i}_{.i} + \frac{\Omega^{i}_{.i}}{2} + \frac{4}{4}\Omega^{a}_{.a} = \Omega^{-2}R^{i}_{.i} + \frac{3}{2}\Omega^{a}_{.a}$$

If $\Omega(x) = const$, then $\hat{R} = \Omega^{-2}R$.■

Definition 3.7.

The Weyl tensor is defined as [1]

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} - g_{ac}R_{db} + g_{bc}R_{da} - g_{bd}R_{ca}) + \frac{1}{6}R(g_{ac}g_{db} - g_{ad}g_{cb})$$

In GR the Weyl tensor describes the free gravitational field, i.e. the gravitational field independent of matter.

Fact 3.10.

The Weyl tensor of the conformal curvature $C^{e}_{.bcd}$ is an invariant of the conformal transformation

$$\hat{C}^{a}_{.bcd} = C^{a}_{.bcd}\ .$$

Proof.

Using the formula for C_{abcd} we calculate at first the Weyl tensor $C^{a}_{.bcd}$.

$$C^{e}_{.bcd} = g^{ae}C_{abcd}$$
$$= g^{ae}R_{abcd} + \frac{1}{2}g^{ae}(g_{ad}R_{cb} - g_{ac}R_{db} + g_{bc}R_{da} - g_{bd}R_{ca}) + \frac{1}{6}g^{ae}R(g_{ac}g_{db} - g_{ad}g_{cb})$$
$$= R^{e}_{.bcd} + \frac{1}{2}g^{ae}(\delta^{e}_{d}R_{cb} - \delta^{e}_{c}R_{db} + g_{bc}R^{e}_{.d} - g_{bd}R^{e}_{.c}) + \frac{1}{6}R(\delta^{e}_{c}g_{db} - \delta^{e}_{d}g_{cb})$$

After the conformal rescaling of the metric one has

$$\hat{C}^{a}_{.bcd} = \hat{R}^{a}_{.bcd} + \frac{1}{2}\left(\delta^{a}_{d}\hat{R}_{cb} - \delta^{a}_{c}\hat{R}_{db} + \hat{g}_{bc}\hat{R}^{a}_{.d} - \hat{g}_{bd}\hat{R}^{a}_{.c}\right) + \frac{1}{6}\hat{R}(\delta^{a}_{c}\hat{g}_{db} - \delta^{a}_{d}\hat{g}_{cb})$$

$$= R^{a}_{.bcd} + \frac{\Omega^2}{4}(\delta^a_c \Omega_{db} - \delta^a_d \Omega_{cb} - g_{bc}\Omega^a_{.d} + g_{bd}\Omega^a_{.c})$$
$$+\frac{1}{2}\delta^a_d\left[R_{cb} + \frac{\Omega^2}{2}\left(\Omega_{cb} + \frac{g_{cb}}{2}\Omega^e_{.e}\right)\right] - \frac{1}{2}\delta^a_c\left[R_{db} + \frac{\Omega^2}{2}\left(\Omega_{db} + \frac{g_{db}}{2}\Omega^e_{.e}\right)\right]$$
$$+\frac{1}{2}\Omega^2 g_{bc}\left(\Omega^{-2}R^a_{.d} + \frac{\Omega^a_{.d}}{2} + \frac{\delta^a_d}{4}\Omega^e_{.e}\right) - \frac{1}{2}\Omega^2 g_{bd}\left(\Omega^{-2}R^a_{.c} + \frac{\Omega^a_{.c}}{2} + \frac{\delta^a_c}{4}\Omega^e_{.e}\right)$$
$$+\frac{1}{6}\left(\Omega^{-2}R + \frac{3}{2}\Omega^e_{.e}\right)(\delta^a_c\Omega^2 g_{bd} - \delta^a_d\Omega^2 g_{cb})$$
$$= R^{a}_{.bcd} + \frac{\Omega^2}{4}(\delta^a_c \Omega_{db} - \delta^a_d \Omega_{cb} - g_{bc}\Omega^a_{.d} + g_{bd}\Omega^a_{.c})$$
$$+\frac{1}{2}\left(\delta^a_d R_{cb} + \frac{\Omega^2}{2}\delta^a_d\Omega_{cb} + \frac{\Omega^2}{4}g_{cb}\delta^a_d\Omega^e_{.e} - \delta^a_c R_{db} - \frac{\Omega^2}{2}\delta^a_c\Omega_{db} - \frac{\Omega^2}{4}\delta^a_c g_{db}\Omega^e_{.e}\right)$$
$$+\frac{1}{2}\left(g_{bc}R^a_{.d} + \frac{\Omega^2}{2}g_{bc}\Omega^a_{.d} + \frac{\Omega^2}{4}g_{bc}\delta^a_d\Omega^e_{.e} - g_{bd}R^a_{.c} - \frac{\Omega^2}{2}g_{bd}\Omega^a_{.c} - \frac{\Omega^2}{4}\delta^a_c g_{bd}\Omega^e_{.e}\right)$$
$$+\frac{1}{6}R(\delta^a_c g_{bd} - \delta^a_d g_{cb}) + \frac{1}{4}\delta^a_c\Omega^2 g_{bd}\Omega^e_{.e} - \frac{1}{4}\delta^a_d\Omega^2 g_{cb}\Omega^e_{.e}$$
$$= R^{a}_{.bcd} + \tfrac{1}{2}(\delta^a_d R_{cb} - \delta^a_c R_{db} + g_{bc}R^a_{.d} - g_{bd}R^a_{.c}) + \tfrac{1}{6}R(\delta^a_c g_{db} - \delta^a_d g_{cb}) = C^a_{.bcd}. \blacksquare$$

If we change the location of the indexes in the Weyl tensor, then the above invariance is missed, e.g.

$$\hat{C}^{ab}_{..cd} = \hat{g}^{be}\hat{C}^a_{.ecd} = \Omega^{-2}g^{be}C^a_{.ecd} = \Omega^{-2}C^{ab}_{..cd}\ .$$

Definition 3.8.

The Einstein equations (in the other words the field equations in GR or the equations of the gravitational field) have the following form [1]

$$G^{.k}_{i.} = \kappa T^{.k}_{i.}\ ,$$

where$\kappa = \frac{8\pi G}{c^4}$.

If we use the geometrized units $G = c = 1$, then $\kappa = 8\pi$ and the Einstein's equation has the simpler form $G^{.k}_{i.} = 8\pi T^{.k}_{i.}$. The left side of the equation represents the geometry of the space-time. On the other hand the right side describes matter and energy which fills the space-time. The Einstein's equation can be summarized as:
the distribution of the matter and the energy in the space-time determines its local geometry.
The tensor $G^{.k}_{i.}$ has the following structure

$$G^{.k}_{i.} = R^{.k}_{i.} - \tfrac{1}{2}\delta^k_i R\ .$$

and it is called the Einstein tensor. Raising or lowering one of the indexes we obtain the Einstein tensor in the covariant or contravariant form.

$$G_{ik} = R_{ik} - \tfrac{1}{2}g_{ik}R\ ,$$
$$G^{ik} = R^{ik} - \tfrac{1}{2}g^{ik}R\ .$$

Fact 3.11.

The Einstein tensor changes under the conformal transformation of metric in the following way

$$\hat{G}^{.k}_{i.} = \Omega^{-2}G^{.k}_{i.} + \tfrac{1}{2}\left(\Omega^{.k}_{i.} - \delta^k_i\Omega^a_{.a}\right)\ .$$

Proof.

$$\hat{G}_{i.}^{.k} = \hat{R}_{i.}^{.k} - \frac{1}{2}\delta_i^k \hat{R} = \Omega^{-2} R_{i.}^{.k} + \frac{\Omega_{i.}^{;k}}{2} + \frac{\delta_i^k}{4}\Omega_{.a}^{a} - \frac{1}{2}\delta_i^k\left(\Omega^{-2}R + \frac{3}{2}\Omega_{.a}^{a}\right)$$
$$= \Omega^{-2}R_{i.}^{.k} - \frac{1}{2}\Omega^{-2}\delta_i^k R + \frac{\Omega_{i.}^{;k}}{2} + \frac{\delta_i^k}{4}\Omega_{.a}^{a} - \frac{3}{4}\delta_i^k\Omega_{.a}^{a}$$
$$= \Omega^{-2}G_{i.}^{.k} + \frac{\Omega_{i.}^{;k}}{2} - \frac{1}{2}\delta_i^k\Omega_{.a}^{a} = \Omega^{-2}G_{i.}^{.k} + \frac{1}{2}\left(\Omega_{i.}^{;k} - \delta_i^k\Omega_{.a}^{a}\right). \blacksquare$$

As we can see the Einstein tensor is not conformally invariant.

4. Selected physical aspects of conformal transformations

In the previous chapter we have calculated the formula for the Einstein tensor under the conformal rescaling of the metric. Now we will show the transformation law for the right side of the Einstein equations, the matter tensor $T_{i.}^{.k}$, which is responsible for the curvature of the space-time.

Definition 4.1.

The matter tensor of the macroscopic matter (the matter energy-momentum tensor or the stress tonsor) we define as follows [1]

$$T_{i.}^{.k} = (\rho + p)u_i u^k - p\delta_i^k,$$

where ρ denotes a density of the matter and p pressure.

Fact 4.1.

The matter tensor changes under the conformal transformation of the metric in the following way

$$\hat{T}_{i.}^{.k} = \Omega^{-4} T_{i.}^{.k}.$$

Proof.

In the proof we will use the formula for the conformal rescaling of the metric g_{ik} and also the formula for mass of the matter contained in the volume element $d^4\Omega = \sqrt{|g|}d^4x$ of 4-dimentional space-time , where $g = det[g_{ik}]$. Hence

$$M = \sqrt{|g|}\rho d^4x,$$

where ρ is proper density of matter. After the conformal rescaling of the metric $d^4\Omega$ transforms into

$$d^4\hat{\Omega} = \sqrt{|\hat{g}|}d^4x.$$

The conformal rescaling of the metric does not change the mass included in the volume element $d^4\Omega$. It changes only into its density $\rho \to \hat{\rho}$. As a consequence one has

$$M = \hat{\rho} d^4\hat{\Omega} = \sqrt{|\hat{g}|}\hat{\rho} d^4x$$

where M indicates the mass of the matter contained in the element $\sqrt{|\hat{g}|}d^4x$.
Because this mass does not change under the conformal rescaling of the metric, we can compare both values and hence we get

$$\sqrt{|\hat{g}|}\hat{\rho} d^4x = \sqrt{|g|}\rho d^4x. \tag{3}$$

Using the formula for $\hat{g} = \Omega^8 g$ and putting the expression $\sqrt{|\hat{g}|} = \Omega^4\sqrt{|g|}$ in (3) we obtain the transformation law for the density of the matter under the conformal rescaling of the metric.

$$\Omega^4\sqrt{|g|}\hat{\rho}d^4x = \sqrt{|g|}\rho d^4x$$
$$\hat{\rho} = \Omega^{-4}\rho.$$

Returning to the initial equation which describes the matter tensor and using the transformtion formula for the density of the matter we obtain

$$\hat{T}_{i.}^{.k} = \Omega^{-4}\rho u_i u^k + \hat{p}\hat{u}_i\hat{u}^k - \hat{p}\delta_i^k.$$

In order $\hat{T}_{i.}^{.k}$ to be connected with $T_{i.}^{.k}$ by the transformation rule $\hat{T}_{i.}^{.k} = \Omega^x T_{i.}^{.k}$, the pressure must change under the conformal rescaling of the metric following the formula $\hat{p} = \Omega^{-4}p$, because $u_i u^k = \hat{u}_i\hat{u}^k$, $\delta_i^k = \hat{\delta}_i^k$.
If this condition is satisfied, then the transformation formula has the following form

$$\hat{T}_{i.}^{.k} = \Omega^{-4}T_{i.}^{.k}.$$

Raising or lowering one of the indexes we obtain the other useful transformation rules

$$\hat{T}^{ik} = \hat{g}^{il}\hat{T}_{i.}^{.k} = \Omega^{-6}T^{ik}$$

and

$$\hat{T}_{ik} = \hat{g}_{kl}\hat{T}_{i.}^{.k} = \Omega^{-2}T_{ik}.\blacksquare$$

Fact 4.2.

Einstein's equations are not conformally invariant.

Proof.

In the proof we use the formula for the Einstein tensor after the conformal transformation of the metric.
Using the Einstein's equations in the metric g_{ik} and replacing the Einstein tensor by the matter tensor we get

$$\hat{G}_{i.}^{.k} = \kappa\Omega^{-2}T_{i.}^{.k} + \frac{1}{2}\left(\Omega_{i.}^{.k} - \delta_i^k\Omega_{.a}^{a}\right). \quad (4)$$

Converting the relation between the matter tensor in the initial gauge and the new one we obtain

$$\hat{T}_{i.}^{.k} = \Omega^{-4}T_{i.}^{.k} \Longrightarrow T_{i.}^{.k} = \Omega^4\hat{T}_{i.}^{.k}. \quad (5)$$

Replacing $T_{i.}^{.k}$ in (4) by (5) one has

$$\hat{G}_{i.}^{.k} = \kappa\Omega^2\hat{T}_{i.}^{.k} + \frac{1}{2}\left(\Omega_{i.}^{.k} - \delta_i^k\Omega_{.a}^{a}\right)$$

or

$$\hat{G}_{i.}^{.k} = \kappa\Omega^2\hat{T}_{i.}^{.k} + \kappa\tilde{T}_{i.}^{.k}$$

where $\kappa\tilde{T}_{i.}^{.k} = \frac{1}{\kappa}\frac{1}{2}\left(\Omega_{i.}^{.k} - \delta_i^k\Omega_{.a}^{a}\right).\blacksquare$

The final formula shows that Einstein's equations are not conformally invariant, i.e. we do not have the formula of the form $\hat{G}_{i.}^{.k} = \kappa\hat{T}_{i.}^{.k}$. As a consequence there is a new additional matter described by the tensor $\tilde{T}_{i.}^{.k}$ apart from the matter which had already existed before the conformal rescaling of the metric and satisfied Einstein's equations $G_{i.}^{.k} = \kappa T_{i.}^{.k}$.

This fact was used in the articles “An interesting property of the Friedman universes” [6], “On Energy of the Friedman Universes in Conformally Flat Coordinates” [7] and “Superenergy, conformal transformations, and Friedman universes” [8] written by J. Garecki.

Namely, he used the conformal rescaling of the metric to create the Friedman dust universes from the empty Minkowskian space-time. One should emphasize in this context that Friedman universes are actually the best mathematical models of the real universe.

As a consequence of Einstein's equivalence principle the gravitional field does not have an energy-momentum tensor. It has only the energy-momentum pseudotensors. One such pseudotensor is given in the book "Teoria pola" by L. Landau and E.M. Lifshitz.

Definition 4.2.

The Landau-Lifshitzpseudotensor of the energy-momentum of the gravitational field is defined as[1]

$$t^{ik} = \frac{1}{16\pi}\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km} - g^{ik}g^{lm}\right)$$
$$+\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{lm} - \Gamma^k_{lm}\Gamma^p_{np}\right)$$
$$+\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}\right)$$
$$+\frac{1}{16\pi}g^{lm}g^{np}\left(\Gamma^i_{ln}\Gamma^k_{mp} - \Gamma^i_{lm}\Gamma^k_{np}\right).$$

Fact 4.3.

The Landau-Lifshitzpseudotensor of the energy-momentum of the gravitational field changes under the conformal transformation of the metric in the following way

$$\hat{t}^{ik} = \frac{1}{16\pi}\left(2\hat{\Gamma}^n_{lm}\hat{\Gamma}^p_{np} - \hat{\Gamma}^n_{lp}\hat{\Gamma}^p_{mn} - \hat{\Gamma}^n_{ln}\hat{\Gamma}^p_{mp}\right)\left(\hat{g}^{il}\hat{g}^{km} - \hat{g}^{ik}\hat{g}^{lm}\right)$$
$$+\frac{1}{16\pi}\hat{g}^{il}\hat{g}^{mn}\left(\hat{\Gamma}^k_{lp}\hat{\Gamma}^p_{mn} + \hat{\Gamma}^k_{mn}\hat{\Gamma}^p_{lp} - \hat{\Gamma}^k_{np}\hat{\Gamma}^p_{lm} - \hat{\Gamma}^k_{lm}\hat{\Gamma}^p_{np}\right)$$
$$+\frac{1}{16\pi}\hat{g}^{kl}\hat{g}^{mn}\left(\hat{\Gamma}^i_{lp}\hat{\Gamma}^p_{mn} + \hat{\Gamma}^i_{mn}\hat{\Gamma}^p_{lp} - \hat{\Gamma}^i_{np}\hat{\Gamma}^p_{lm} - \hat{\Gamma}^i_{lm}\hat{\Gamma}^p_{np}\right)$$
$$+\frac{1}{16\pi}\hat{g}^{lm}\hat{g}^{np}\left(\hat{\Gamma}^i_{ln}\hat{\Gamma}^k_{mp} - \hat{\Gamma}^i_{lm}\hat{\Gamma}^k_{np}\right)$$
$$= \Omega^{-4}t^{ik} + \Omega^{-5}\frac{1}{16\pi}\left(I^{ik}_1 + I^{ik}_2\right) + \Omega^{-6}\frac{1}{16\pi}I^{ik}_3.$$

The analytical structures of I^{ik}_1, I^{ik}_2 and I^{ik}_3 are given in the proof.

Proof.

$$\hat{t}^{ik} = \frac{1}{16\pi}\left(2\hat{\Gamma}^n_{lm}\hat{\Gamma}^p_{np} - \hat{\Gamma}^n_{lp}\hat{\Gamma}^p_{mn} - \hat{\Gamma}^n_{ln}\hat{\Gamma}^p_{mp}\right)\left(\hat{g}^{il}\hat{g}^{km} - \hat{g}^{ik}\hat{g}^{lm}\right)$$
$$+\frac{1}{16\pi}\hat{g}^{il}\hat{g}^{mn}\left(\hat{\Gamma}^k_{lp}\hat{\Gamma}^p_{mn} + \hat{\Gamma}^k_{mn}\hat{\Gamma}^p_{lp} - \hat{\Gamma}^k_{np}\hat{\Gamma}^p_{lm} - \hat{\Gamma}^k_{lm}\hat{\Gamma}^p_{np}\right)$$
$$+\frac{1}{16\pi}\hat{g}^{kl}\hat{g}^{mn}\left(\hat{\Gamma}^i_{lp}\hat{\Gamma}^p_{mn} + \hat{\Gamma}^i_{mn}\hat{\Gamma}^p_{lp} - \hat{\Gamma}^i_{np}\hat{\Gamma}^p_{lm} - \hat{\Gamma}^i_{lm}\hat{\Gamma}^p_{np}\right)$$
$$+\frac{1}{16\pi}\hat{g}^{lm}\hat{g}^{np}\left(\hat{\Gamma}^i_{ln}\hat{\Gamma}^k_{mp} - \hat{\Gamma}^i_{lm}\hat{\Gamma}^k_{np}\right)$$

Putting $\hat{\Gamma}^n_{lm} = \Gamma^n_{lm} + P^n_{lm}$ in the equation, where $P^n_{lm} = \Omega^{-1}\left(\delta^n_l\Omega_{,m} + \delta^n_m\Omega_{,l} - g_{lm}g^{ne}\Omega_{,e}\right)$ and $\hat{g}^{mn} = \Omega^{-2}g^{mn}$ we obtain

$$= \Omega^{-4}\frac{1}{16\pi}\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km} - g^{ik}g^{lm}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{lm} - \Gamma^k_{lm}\Gamma^p_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}\right)$$

$$+\Omega^{-4}\frac{1}{16\pi}g^{lm}g^{np}\left(\Gamma^i_{ln}\Gamma^k_{mp}-\Gamma^i_{lm}\Gamma^k_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}\left(2\Gamma^n_{lm}P^p_{np}-P^n_{lm}\Gamma^p_{np}-\Gamma^n_{lp}P^p_{mn}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)$$
$$-\Omega^{-4}\frac{1}{16\pi}\left(P^n_{lp}\Gamma^p_{mn}+\Gamma^n_{ln}P^p_{mp}+P^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{lp}P^p_{mn}+P^k_{lp}\Gamma^p_{mn}+\Gamma^k_{mn}P^p_{lp}+P^k_{mn}\Gamma^p_{lp}\right)$$
$$-\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(\Gamma^k_{np}P^p_{lm}+P^k_{np}\Gamma^p_{lm}-\Gamma^k_{lm}P^p_{np}-P^k_{lm}\Gamma^p_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{lp}P^p_{mn}+P^i_{lp}\Gamma^p_{mn}+\Gamma^i_{mn}P^p_{lp}+P^i_{mn}\Gamma^p_{lp}\right)$$
$$-\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(\Gamma^i_{np}P^p_{lm}+P^i_{np}\Gamma^p_{lm}+\Gamma^i_{lm}P^p_{np}+P^i_{lm}\Gamma^p_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{lm}g^{np}\left(\Gamma^i_{ln}P^k_{mp}+P^i_{ln}\Gamma^k_{mp}-\Gamma^i_{lm}P^k_{np}-P^i_{lm}\Gamma^k_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}\left(2P^n_{lm}P^p_{np}-P^n_{lp}P^p_{mn}-P^n_{ln}P^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{il}g^{mn}\left(P^k_{lp}P^p_{mn}+P^k_{mn}P^p_{lp}-P^k_{np}P^p_{lm}-P^k_{lm}P^p_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{kl}g^{mn}\left(P^i_{lp}P^p_{mn}+P^i_{mn}P^p_{lp}-P^i_{np}P^p_{lm}-P^i_{lm}P^p_{np}\right)$$
$$+\Omega^{-4}\frac{1}{16\pi}g^{lm}g^{np}\left(P^i_{ln}P^k_{mp}-P^i_{lm}P^k_{np}\right).$$

Hence the above equation we can write in the form

$$\hat{t}^{ik}=\Omega^{-4}t^{ik}+\Omega^{-5}\frac{1}{16\pi}\left(I^{ik}_1+I^{ik}_2\right)+\Omega^{-6}\frac{1}{16\pi}I^{ik}_3,$$

where

$$\begin{aligned}I^{ik}_1=&\left(2\Gamma^n_{lm}\tilde{P}^p_{np}-\Gamma^n_{lp}\tilde{P}^p_{mn}-\Gamma^n_{ln}\tilde{P}^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\&+g^{il}g^{mn}\left(\Gamma^k_{lp}\tilde{P}^p_{mn}+\Gamma^k_{mn}\tilde{P}^p_{lp}-\Gamma^k_{np}\tilde{P}^p_{lm}-\Gamma^k_{lm}\tilde{P}^p_{np}\right)\\&+g^{kl}g^{mn}\left(\Gamma^i_{lp}\tilde{P}^p_{mn}+\Gamma^i_{mn}\tilde{P}^p_{lp}-\Gamma^i_{np}\tilde{P}^p_{lm}-\Gamma^i_{lm}\tilde{P}^p_{np}\right)\\&+g^{lm}g^{np}\left(\Gamma^i_{ln}\tilde{P}^k_{mp}-\Gamma^i_{lm}\tilde{P}^k_{np}\right)\\I^{ik}_2=&\left(2\tilde{P}^n_{lm}\Gamma^p_{np}-\tilde{P}^n_{lp}\Gamma^p_{mn}-\tilde{P}^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\&+g^{il}g^{mn}\left(\tilde{P}^k_{lp}\Gamma^p_{mn}+\tilde{P}^k_{mn}\Gamma^p_{lp}-\tilde{P}^k_{np}\Gamma^p_{lm}-\tilde{P}^k_{lm}\Gamma^p_{np}\right)\\&+g^{kl}g^{mn}\left(\tilde{P}^i_{lp}\Gamma^p_{mn}+\tilde{P}^i_{mn}\Gamma^p_{lp}-\tilde{P}^i_{np}\Gamma^p_{lm}-\tilde{P}^i_{lm}\Gamma^p_{np}\right)\\&+g^{lm}g^{np}\left(\tilde{P}^i_{ln}\Gamma^k_{mp}-\tilde{P}^i_{lm}\Gamma^k_{np}\right)\\I^{ik}_3=&\left(2\tilde{P}^n_{lm}\tilde{P}^p_{np}-\tilde{P}^n_{lp}\tilde{P}^p_{mn}-\tilde{P}^n_{ln}\tilde{P}^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)\\&+g^{il}g^{mn}\left(\tilde{P}^k_{lp}\tilde{P}^p_{mn}+\tilde{P}^k_{mn}\tilde{P}^p_{lp}-\tilde{P}^k_{np}\tilde{P}^p_{lm}-\tilde{P}^k_{lm}\tilde{P}^p_{np}\right)\\&+g^{kl}g^{mn}\left(\tilde{P}^i_{lp}\tilde{P}^p_{mn}+\tilde{P}^i_{mn}\tilde{P}^p_{lp}-\tilde{P}^i_{np}\tilde{P}^p_{lm}-\tilde{P}^i_{lm}\tilde{P}^p_{np}\right)\\&+g^{lm}g^{np}\left(\tilde{P}^i_{ln}\tilde{P}^k_{mp}-\tilde{P}^i_{lm}\tilde{P}^k_{np}\right)\end{aligned}$$

and $\tilde{P}^p_{mn}=\left(\delta^p_m\Omega_{,n}+\delta^p_n\Omega_{,m}-g_{mn}g^{pe}\Omega_{,e}\right)$.■

As we can see, the structures of I^{ik}_1, I^{ik}_2 and I^{ik}_3 are the same as the structure of t^{ik} with the change $\Gamma\Gamma\to\Gamma P$ in I^{ik}_1, $\Gamma\Gamma\to P\Gamma$ in I^{ik}_2 and $\Gamma\Gamma\to PP$ in I^{ik}_3.

The conformal rescaling of the metric creates an additional energy and momentum for the gravitation. It is represented by the expressions with Ω^{-5} and Ω^{-6} in the transformation law of the Landau-Lifshitz pseudotensor $t^{ik}=t^{ki}$.

5. Conclusions

- I have proved the number of the transformation formulas for various geometrical objects under the conformal rescaling of the metric on the Riemannian manifold (or pseudoriemannian).
- Some of the formulas I have obtained in the simpler form than the formulas given in standard books. One can easily check this fact by comparing our formulas with the ones given e.g. in “The Large Structure of Space-Time” by S. Hawking and G.F.R. Ellis.
- The basic equations of the general theory of relativity - the Einstein's equations- are not conformally invariant. This fact means that there is a creation of the new matter and there is also possibility of creation of the Friedman universes from the vaccum.
- The new result of this paper is the transformation law for the Landau-Lifshitz pseudotensor of the energy-momentum of the gravitational field under the conformal transformation of the metric. It is easily seen from this law that the conformal rescaling of the metric creates an additional energy and momentum for gravitation.

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