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Exponential decay of transient values in positive nonlinear systems

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Abstract. The exponential decay of transient values in nonlinear continuous-time standard and fractional orders with linear dynamical positive feedback systems and of positive linear parts is investigated. Sufficient conditions for the exponential decay of transient values in this class of positive nonlinear systems are established. Procedures for the computation of gains characterizing the class of nonlinear elements are given and illustrated in simple examples.

Keywords: exponential decay; transient value; fractional order; positive nonlinear dynamical feedback system.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1–6]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models, and electrical circuits. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology, medicine, etc. An overview of the state of the art in positive systems theory is given in the monographs [1–4].

Mathematical fundamentals of the fractional calculus are given in the monographs [7, 8]. The positive fractional linear systems are investigated in [3, 4, 6, 9–16]. Positive linear systems with different fractional orders are addressed in [6, 14, 15]. Linear positive electrical circuits are investigated in [4]. The global stability of nonlinear systems with positive feedback and positive stable linear parts is investigated in [11–13, 17], and the stability of discrete-time systems with delays in [18].

In this paper, the exponential decay of transient values of nonlinear standard and fractional positive systems with dynamical positive feedback will be addressed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the positive standard and fractional orders linear systems are recalled. The main results of the paper are given in Section 3 where sufficient conditions for the exponential decay of transient values in the positive nonlinear systems are established and procedures for computation of the gains characterizing the class of characteristics of nonlinear elements are given. In Section 4 the results of Section 3 are extended to fractional nonlinear positive systems. The procedures are illustrated by numerical examples. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. POSITIVE INTEGER AND DIFFERENT FRACTIONAL ORDERS LINEAR SYSTEMS

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input, and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 1. [3, 4] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 1. [3, 4] The continuous-time linear system (1) is positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \tag{2}$$

Definition 2. [3, 4] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for any} \quad x(0) \in \Re_+^n. \tag{3}$$

Theorem 2. [3, 4] The positive continuous-time linear system (1) for $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

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1. All coefficients of the characteristic polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad (4)$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2. There exists a strictly positive vector $\lambda^T = \begin{bmatrix}\lambda_1 & \cdots & \lambda_n\end{bmatrix}^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0 \quad \text{or} \quad \lambda^T A < 0. \quad (5)$$

If the matrix A is nonsingular then we can choose $\lambda = A^{-1}c$, where $c \in \Re^n$ is strictly positive.

In this paper, the following Caputo definition of the fractional derivative of α order will be used [3, 4]

$$\begin{aligned} {}_0D_t^\alpha f(t) &= \frac{d^\alpha f(t)}{dt^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{f}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \quad 0 < \alpha < 1, \end{aligned} \quad (6)$$

where $\dot{f}(\tau) = \frac{df(\tau)}{d\tau}$ and $\Gamma(z) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$, $\mathrm{Re}(x) > 0$ is the Euler gamma function.

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad (7a)$$

$$y(t) = Cx(t), \quad (7b)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input, and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 3. [3, 4] The fractional system (7) is called (internally) positive if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 3. [3, 4] The fractional system (7) is positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}. \quad (8)$$

Definition 4. [3, 4] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for any } x(0) \in \Re_+^n. \quad (9)$$

Theorem 4. [3, 4] The positive linear system (1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad (10)$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2. There exists a strictly positive vector $\lambda^T = \begin{bmatrix}\lambda_1 & \cdots & \lambda_n\end{bmatrix}^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0 \quad \text{or} \quad \lambda^T A < 0. \quad (11)$$

Theorem 5. The positive system (1) (and (7)) is asymptotically stable if the sum of entries of each column (row) of the matrix A is negative.

Proof. Using (11) we obtain

$$\begin{aligned} A\lambda &= \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} \lambda_1 + \ldots + \begin{bmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{bmatrix} \lambda_n < \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \quad (12)$$

and the sum of entries of each column of the matrix A is negative since $\lambda_k > 0$, $k = 1, \ldots, n$. The proof for rows is similar. □

3. EXPONENTIAL DECAY OF TRANSIENT VALUES IN NONLINEAR SYSTEMS WITH POSITIVE DYNAMICAL FEEDBACK

Consider the nonlinear feedback system shown in Fig. 1 which consists of the positive linear part, the nonlinear element with characteristic $u = f(e)$ and positive dynamical feedback. The linear part is described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \quad (13a)$$

with interval matrices

$$\underline{A} \leq A \leq \overline{A}, \quad \underline{B} \leq B \leq \overline{B}, \quad \underline{C} \leq C \leq \overline{C}, \quad (13b)$$

where $x = x(t) \in \Re_+^{n_1}$, $u = u(t) \in \Re_+$, $y = y(t) \in \Re_+$ is the state, input, and output vectors of the system (13), and $A \in M_{n_1}$, $B \in \Re_+^{n_1\times 1}$, $C \in \Re_+^{1\times n_1}$. It is assumed that the matrix A of (13a) has all eigenvalues s_k with real parts smaller than

$$\mathrm{Re}\, s_k < -\gamma, \quad \text{i.e.} \quad \gamma > 0, \quad k = 1, \ldots, n. \quad (13c)$$

The characteristic of the nonlinear element is shown in Fig. 2 and it satisfies the condition

$$0 \leq \frac{f(e)}{e} \leq k < \infty. \quad (14)$$

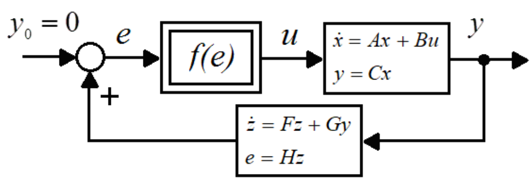

Fig. 1. The nonlinear feedback system

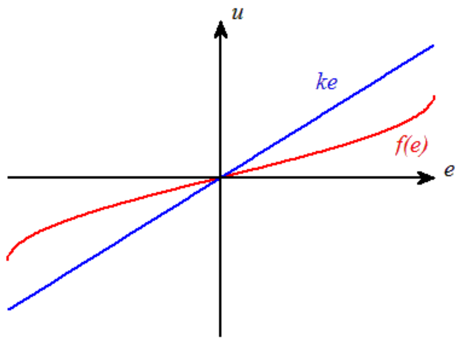

Fig. 2. The characteristic of the nonlinear element

The positive feedback system is described by the equations

$$\begin{aligned}\dot{z} &= Fz + Gy,\\ e &= Hz,\end{aligned} \tag{15a}$$

with interval matrices

$$\underline{F} \le F \le \overline{F}, \quad \underline{G} \le G \le \overline{G}, \quad \underline{H} \le H \le \overline{H}, \tag{15b}$$

where $z = z(t) \in \Re_+^{n_2}$, $e = e(t) \in \Re_+$ are the state vector and output vectors.

It is assumed that the matrix $F + I_n\gamma \in M_{n_2}$ is also asymptotically stable.

From (13) and (15) we have

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \hat{A}\begin{bmatrix}x\\ z\end{bmatrix} + \hat{B}u, \tag{16a}$$

where

$$\hat{A} = \begin{bmatrix}A & 0\\ GC & F\end{bmatrix} \in M_n, \quad \hat{B} = \begin{bmatrix}B\\ 0\end{bmatrix} \in \Re_+^{n_1\times 1}, \quad n = n_1 + n_2. \tag{16b}$$

The following theorem gives sufficient conditions for the exponential decay of transient values in the positive feedback nonlinear system faster than $e^{-\gamma t}$.

Theorem 6. The state variables of the nonlinear system consisting of the positive linear part (13), the nonlinear element satisfying the condition (14), and the positive asymptotically stable dynamical feedback system (15) decay exponentially faster than $e^{-\gamma t}$ if

$$\begin{bmatrix}\underline{A} + I_{n_1}\gamma & k\overline{B}\overline{H}\\ \overline{G}\overline{C} & \underline{F} + I_{n_2}\gamma\end{bmatrix} \in M_n \tag{17}$$

is asymptotically stable.

Proof. The proof will be accomplished using the Lyapunov method [19, 20]. As the Lyapunov function $V(x,z)$ we choose

$$V(x,z) = \lambda^T\begin{bmatrix}x\\ z\end{bmatrix} \ge 0 \quad \text{for} \quad x \in \Re_+^{n_1}, \ z \in \Re_+^{n_2}, \tag{18}$$

where $\lambda \in \Re_+^n$ is a strictly positive vector, i.e. $\lambda_k > 0$, $k = 1, \ldots, n$.

It is well-known that if the matrix $A \in M_n$ is asymptotically stable then state variables of the system $\dot{x} = (A + I_n\gamma)x$ exponentially decay faster than $e^{-\gamma t}$.

Using (18), (13) and (15) we obtain

$$\begin{aligned}\dot{V}(x,z) &= \lambda^T\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \lambda^T\left\{\begin{bmatrix}A & 0\\ GC & F\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}u\right\}\\ &= \lambda^T\left\{\begin{bmatrix}A & 0\\ GC & F\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\text{kHz}\right\}\\ &= \lambda^T\begin{bmatrix}A & kBH\\ GC & F\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} \le 0\end{aligned} \tag{19}$$

since $Bu = Bf(e) \le kBHz$ by the condition (14). □

From (19) it follows that $\dot{V}(x,z) \le 0$ if the condition (17) is satisfied then the state variables exponentially decay faster than $e^{-\gamma t}$.

Theorem 6 can be applied to solve the following two problems:

Problem 1. Given matrices A, B, C and F, G, H of the positive systems (13), (15) and the nonlinear characteristic $u = f(e)$ of the nonlinear element. Knowing the value of k satisfying the condition (14) check if the transient values in the nonlinear system decay faster than $e^{-\gamma t}$.

Problem 2. Given matrices A, B, C and F, G, H of the positive systems (13), (15) and the nonlinear characteristic $u = f(e)$ of the nonlinear element. Find the maximal value of k for which the characteristic $u = f(e)$ of the nonlinear element satisfies the condition (14) and the transient values of the nonlinear system decay faster than $e^{-\gamma t}$.

The Problem 1 can be solved using the following:

Procdure 1.

Step 1. Knowing the characteristic $u = f(e)$ find the minimal value of k satisfying the condition (14).

Step 2. Using Theorem 6 find the sum of entries of each column (row) of the matrix (17). If all these sums are negative, then the transient values in the nonlinear system decay faster than $e^{-\gamma t}$.

The Problem 2 can be solved using the following:

Procdure 2.

Step 1. Using Theorem 6 find the sum of entries of each column (row) of the matrix (17).

Step 2. Find the maximal value of $k_c(k_r)$ for which the sums of entries of all columns (rows) of (17) are negative.

Step 3. Find $k_{\max} = \min(k_c,\ k_r)$.

In this case, the transient values in the nonlinear system decrease faster than $e^{-\gamma t}$ for all nonlinear characteristics $u = f(e)$ satisfying the condition

$$0 < f(e) < k_{\max}e. \tag{20}$$

Remark 1. The value of $k_{\max}$ depends only on the first n_1 rows and of the last n_2 columns of the matrix (17).

Example 1. Consider the nonlinear system shown in Fig. 1 with linear positive parts described by (13) and (15) with

$$\underline{A}=\begin{bmatrix}-6.5 & 1\\ 2 & -6.2\end{bmatrix},\quad \overline{A}=\begin{bmatrix}-7 & 1.5\\ 2.3 & -6.5\end{bmatrix},$$
$$\underline{B}=\begin{bmatrix}0.5\\ 0.7\end{bmatrix},\qquad \overline{B}=\begin{bmatrix}0.5\\ 0.7\end{bmatrix}, \tag{21}$$
$$\underline{C}=\begin{bmatrix}0.4 & 0.5\end{bmatrix},\qquad \overline{C}=\begin{bmatrix}0.5 & 0.6\end{bmatrix}$$

and

$$\underline{F}=\begin{bmatrix}-6 & 2\\ 1.6 & -7\end{bmatrix},\quad \overline{F}=\begin{bmatrix}-6.2 & 2.2\\ 1.8 & -7.3\end{bmatrix},$$
$$\underline{G}=\begin{bmatrix}0.8\\ 0.6\end{bmatrix},\qquad \overline{G}=\begin{bmatrix}1\\ 0.8\end{bmatrix}, \tag{22}$$
$$\underline{H}=\begin{bmatrix}0.4 & 0.2\end{bmatrix},\quad \overline{H}=\begin{bmatrix}0.5 & 0.4\end{bmatrix},$$

respectively and the nonlinear element with characteristics satisfying the condition (14).

Case 1. Using $k = 1$ check the global stability of the nonlinear system for $\gamma = -2$.

In this case using (17), (21), and (22) we obtain

$$\begin{bmatrix}\underline{A}+I_{n_1}\gamma & k\overline{B}\overline{H}\\ \overline{G}\overline{C} & \underline{F}+I_{n_2}\gamma\end{bmatrix}=\begin{bmatrix}-4.5 & 1 & 0.3 & 0.24\\ 2 & -4.2 & 0.4 & 0.32\\ 0.5 & 0.6 & -4 & 2\\ 0.4 & 0.48 & 1.6 & -5\end{bmatrix}. \tag{23}$$

The sums of the entries of columns of the matrix (23) are: column 1: $= -1.6$,, column 2: $= -2.12$, column 3: $= -1.7$, column 4: $= -2.44$. Therefore, by Theorem 6 the nonlinear system is globally stable.

Case 2. Find the maximal value of $k_{\max}$ satisfying the condition (14) for which the transient values in the nonlinear system decrease faster than $e^{-\gamma t}$.

Using Procedure 2 we obtain the following:

Step 1. The sums of entries of each column (row) of the matrix

$$\begin{bmatrix}\underline{A}+I_{n_1}\gamma & k\overline{B}\overline{H}\\ \overline{G}\overline{C} & \underline{F}+I_{n_2}\gamma\end{bmatrix}=\begin{bmatrix}-4.5 & 1 & 0.3k & 0.24k\\ 2 & -4.2 & 0.4k & 0.32k\\ 0.5 & 0.6 & -4 & 2\\ 0.4 & 0.48 & 1.6 & -5\end{bmatrix} \tag{24}$$

are: column 1: $= -1.6$, column 2: $= -2.12$, column 3: $= 0.7k - 2.4$, column 4: $= 0.56k - 3$, row 1: $= -2.96$, row 2: $= -1.48$, row 3: $= 0.54k - 3.5$, row 4: $= 0.72k - 2.2$.

Step 2. From Theorem 6 we have: for column 3: $k < 3.428$, and for column 4: $k < 5.357$, and for row 1: $k < 6.482$, row 2: $k < 3.0555$.

Step 3. The desired value of k is $k_{\max} = \min(k_c, k_r) = 3.0555$.

Therefore, the transient values in the nonlinear system with characteristics satisfying the condition (14) with $k < 3.0555$ decrease faster than $e^{-\gamma t}$.

Remark 2. From the matrix (17) and the computation procedure, it follows that the k depends only on the matrices F, G, H and is independent of the matrices A, C, G.

4. EXPONENTIAL DECAY IN FRACTIONAL POSITIVE NONLINEAR FEEDBACK SYSTEMS

Consider the fractional nonlinear feedback system with a similar structure as shown in Fig. 1, which consists of the fractional positive linear part, the nonlinear element with characteristics shown in Fig. 2, and the dynamical positive feedback element.

The fractional positive linear part is described by equations (7a)–(7b) and the fractional positive feedback element by equations

$$\frac{d^\beta z}{dt^\beta} = Fz + Gy, \tag{25a}$$
$$e = Hz, \tag{25b}$$

where $z = z(t) \in \Re_+^{n_2}$, $y = y(t) \in \Re_+$, $e = e(t) \in \Re_+$ are the state, input, and output vectors, and the fractional derivative is defined by (6).

It is assumed that the fractional positive linear systems (7) and (25) are asymptotically stable and the nonlinear characteristic $u = f(e)$ satisfies the condition (14).

From (7) and (25) we obtain

$$\begin{bmatrix}\dfrac{d^\alpha x}{dt^\alpha}\\ \dfrac{d^\beta z}{dt^\beta}\end{bmatrix} = \hat{A}\begin{bmatrix}x\\ z\end{bmatrix} + \hat{B}u, \tag{26}$$

where the matrices $\hat{A}$ and $\hat{B}$ are defined by (16b).

Definition 5. The fractional nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $x(0) \in \Re_+^{n_1}$.

The following theorem gives sufficient conditions for the global stability of the fractional positive nonlinear system.

Theorem 7. The fractional nonlinear system consisting of the positive linear part (7), the nonlinear element satisfying the condition (14), and the positive fractional dynamical feedback (25) is globally stable if the matrix

$$\begin{bmatrix}\underline{A}+I_{n_1}\gamma & k\overline{B}\overline{H}\\ \overline{G}\overline{C} & \underline{F}+I_{n_2}\gamma\end{bmatrix} \in M_n \tag{27}$$

is asymptotically stable.

Proof. The proof will be accomplished using the Lyapunov method [19, 20]. As the Lyapunov function $V(x,z)$ we choose the scalar function defined by (18).

Using (18) and (26) we obtain

$$\begin{bmatrix} \dfrac{d^\alpha V(x,z)}{dt^\alpha} \\ \dfrac{d^\beta V(x,z)}{dt^\beta} \end{bmatrix} = \lambda^T \begin{bmatrix} \dfrac{d^\alpha x}{dt^\alpha} \\ \dfrac{d^\beta z}{dt^\beta} \end{bmatrix} = \lambda^T \left\{ \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \right\}$$
$$= \lambda^T \left\{ \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \text{kHz} \right\}$$
$$= \lambda^T \begin{bmatrix} A & kBH \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq 0 \tag{28}$$

since $Bu = Bf(e) \leq kBHz$.

From (28) it follows that the fractional derivatives of the Lyapunov function are negative if the condition (27) is satisfied and the fractional nonlinear system is globally stable. □

Note that if the condition (27) is satisfied then the transient values in the nonlinear system decrease faster than $e^{-\gamma t}$.

For the fractional nonlinear feedback systems, we can also formulate and solve similar two problems as for the standard nonlinear systems presented in Section 3.

Example 2. Consider the nonlinear fractional positive system shown in Fig. 1 described by (13) and (15a) with the matrices

$$A = \begin{bmatrix} -4 & 1 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, \tag{29a}$$

and

$$F = \begin{bmatrix} -5 & 1 \\ 1 & -4 \end{bmatrix}, \quad G = \begin{bmatrix} 0.6 \\ 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix}, \tag{29b}$$

respectively. Find the value of k satisfying the condition (14) for which all transient values in the nonlinear system decrease faster than $e^{-\gamma t}$ for $\gamma = -2$.

Using (27) and (29) we obtain

$$\begin{bmatrix} A + I_2\gamma & kBH \\ GC & \underline{F} + I_2\gamma \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0.1k & 0.2k \\ 1 & -3 & 0.08k & 0.16k \\ 0.12 & 0.24 & -3 & 1 \\ 0.1 & 0.2 & 1 & -2 \end{bmatrix}. \tag{30}$$

The sums of entries of the columns of the matrix (30) are: column 1: $= -0.78$, column 2: $= -1.56$, column 3: $= 0.18k - 2$, column 4: $= 0.36k - 1$. By Theorem 6 we have for the column 3:$-k = \dfrac{2}{0.18} = 11.11$ and for column 4: $-k = \dfrac{1}{0.36} = 2.777$. Therefore, the transient values in the nonlinear system decrease faster than e^{-2t} for $k < 2.777$.

5. CONCLUDING REMARKS

The exponential decay of transient values in nonlinear continuous-time standard and fractional orders with linear dynamical positive feedback systems and of positive linear parts with interval matrices is investigated. Sufficient conditions for the exponential decay of transient values in this class of positive nonlinear systems are established (Theorem 6). The main result is extended to different fractional orders nonlinear positive systems (Theorem 7). Procedures for the computation of gains characterizing the class of nonlinear elements are given and illustrated in simple examples.

The considerations can be extended to nonlinear discrete-time fractional systems with interval matrices of positive linear parts. An open problem is an extension of the considerations to nonlinear different-order fractional systems with interval matrices of their positive linear parts.

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REFERENCES

[1] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, 1994.

[2] L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*, New York: J. Wiley, 2000.

[3] T. Kaczorek, *Selected Problems of Fractional Systems Theory*, Berlin: Springer 2011.

[4] T. Kaczorek and K. Rogowski, *Fractional Linear Systems and Electrical Circuits*, Cham: Springer 2015.

[5] W. Mitkowski, "Dynamical properties of Metzler systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 56, no. 4, pp. 309–312, 2008.

[6] L. Sajewski, "Stabilization of positive descriptor fractional discrete-time linear systems with two different fractional orders by decentralized controller," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 65, no. 5, pp. 709714, 2017, doi: 10.1515/bpasts-2017-0076.

[7] P. Ostalczyk, *Discrete Fractional Calculus*, River Edgle, NJ: World Scientific, 2016.

[8] I. Podlubny, *Fractional Differential Equations*, San Diego, USA: Academic Press, 1999.

[9] T. Kaczorek, "Absolute stability of a class of fractional positive nonlinear systems," *Int. J. Appl. Math. Comput. Sci.*, vol. 29, no. 1, pp. 93–98, 2019, doi: 10.2478/amcs-2019-0007.

[10] T. Kaczorek, "Analysis of positivity and stability of fractional discrete-time nonlinear systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 64, no. 3, pp. 491–494, 2016, doi: 10.1515/bpasts-2016-0054.

[11] T. Kaczorek, "Global stability of nonlinear feedback systems with fractional positive linear parts," *Int. J. Appl. Math. Comput. Sci.*, vol.30, no.3, pp. 493–499, 2020, doi: 10.34768/amcs-2020-0036.

[12] T. Kaczorek, "Global stability of positive standard and fractional nonlinear feedback systems," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 68, no.2, pp. 285–288, 2020, doi: 10.24425/bpasts.2020.133112.

[13] T. Kaczorek, "Global stability of nonlinear feedback systems with positive linear parts," *Int. J. Nonlinear Sci. Num. Simul.*, vol. 20, no 5, pp. 575–579, 2019, doi: 10.1515/ijnsns-2018-0189.

[14] T. Kaczorek, "Positive linear systems with different fractional orders," *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 58, no. 3, pp. 453–458, 2010, doi: 10.2478/v10175-010-0043-1.

[15] T. Kaczorek, "Positive linear systems consisting of n subsystems with different fractional orders, *IEEE Trans. Circ. Syst.*, vol. 58, no. 6, pp. 1203–1210, 2011, doi: 10.1109/TCSI.2010.2096111.

[16] T. Kaczorek, "Stability of fractional positive nonlinear systems," *Arch. Control Sci.*, vol. 25, no. 4, pp. 491–496, 2015, doi: 10.1515/acsc-2015-0031.

[17] T. Kaczorek and A. Ruszewski, "Global stability of discrete-time nonlinear systems with descriptor standard and fractional positive linear parts and scalar feedbacks," *Arch. Control Sci.*, vol. 30, no. 4, pp. 667–681, 2020, doi: 10.24425/acs.2020.135846.

[18] A. Ruszewski, "Stability of discrete-time fractional linear systems with delays," *Arch. Control Sci.*, vol. 29, no. 3, pp. 549–567, 2019, doi: 10.24425/acs.2019.130205.

[19] A.M. Lyapunov, *Obscaja zadaca ob ustoicivosti dvizenija*, Gostechizdat, Moskwa, 1963 (in Russian).

[20] H. Leipholz, *Stability Theory*, New York Academic Press, 1970.