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ON THE ALMOST UNIFORM CONVERGENCE

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Abstract

Uniform convergence for continuous real functions sequences preserves continuity of the limit of such sequences. There are weaker types of convergence which have similar properties. We consider such types of convergence for functions from one topological space into another one.

1. Introduction

It is known that uniform convergence of a sequence of continuous functions defined in a topological space X with values in $\mathbb{R}$ preserves continuity of its limit. It occurs that there exist weaker types of convergence of functional sequences than uniform ones which also preserve continuity of their limits. M. Predoi proved this fact for quasi-uniformly convergent sequences in the sense of Arzelá in [7], J. Ewert in [4] established this for almost-uniformly convergent sequences and A. Sochaczewska in [8] showed it for strong quasi-uniformly convergent sequences of functions. The same result was also obtained in [2] for quasi-uniformly convergent sequence of mappings from a topological space X into a regular topological space Y.

Many authors applied the notion of almost uniform convergence for function from a topological space into $\mathbb{R}$.

In such a case the definition of almost uniform convergence would read as follows:

Definition 1. *Let all functions of the sequence be define in a topological space $(X, \mathcal{T})$ and take values in a metric space (Y, ϱ). A pointwise convergent sequence $(f_n)_{n\in\mathbb{N}}$ of functions defined on a topological space X is called almost uniformly convergent to a function $f\colon X \to Y$ if*

$$\forall_{x\in X}\forall_{\varepsilon>0}\forall_{n\in\mathbb{N}}\exists_{n_x\geq n}\exists_{U_x\in\mathscr{B}_x}\forall_{t\in U_x}\left(\varrho\left(f_{n_x}(t), f(t)\right) < \varepsilon\right) \tag{1}$$

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where $\mathscr{B}_x$ *denotes the system of all open neighborhoods of the point* x.

It is possible, repeating all proofs, to consider functions defined in a topological space with values in a metric space. In that case one can obtain the following theorems:

Theorem 1. [6], [7] *If a sequence* $(f_n)_{n\in\mathbb{N}}$ *of continuous functions defined in a topological space* X *with values in a metric space* (Y,ϱ) *is almost uniformly convergent to a function* $f\colon X\to Y$ *then this function* f *is continuous as well.*

Theorem 2. [6], [7] *If a sequence* $(f_n)_{n\in\mathbb{N}}$ *of continuous functions defined in a topological space* X *with values in a metric space* (Y,ϱ) *is pointwise convergent to a continuous function* $f\colon X\to Y$ *then this convergence is almost uniform.*

Before we start with some generalization of the convergence, we shall remind additional notions. The notion of a (open) cover of a topological space X is meant a class of open sets for which its union equals X.

By a star of a point x in a topological space X with respect to an open cover Ω of the space X we mean the union of the class of all sets from the cover Ω which contain the point x. This set is usually denoted by the symbol $\mathrm{St}(x,\Omega)$. Symbolically this set can be defined in the following way:

$$\mathrm{St}\,(x,\Omega)=\bigcup\{U\in\Omega\colon x\in U\}\,.$$

Denotations and terminology which will be used later we understand as in [3]. Modifying the notion of almost-uniform convergence of sequences of real functions we state:

Definition 2. [2] *Let* X *and* Y *be topological spaces,* f_n, $n\in\mathbb{N}$, *and* f *be functions defined in* X *with values in* Y. *We say that the sequence* $(f_n)_{n=1}^{\infty}$ *is quasi-uniformly convergent to* f *if:*

(2) $$(f_n)_{n=1}^{\infty} \text{ converges pointwise to} f;$$

for each open cover Ω *of* Y *and each* $n\in\mathbb{N}$ *exist* $k\in\mathbb{N}$ *and* $n_1,\dots,n_k\geq n$ *such that if* $t\in X$ *then*

(3) $$f_{n_1}(t)\in\mathrm{St}(f(t),\Omega)\vee f_{n_2}(t)\in\mathrm{St}(f(t),\Omega)\vee\ldots\vee f_{n_k}(t)\in\mathrm{St}(f(t),\Omega).$$

Definition 3. *Let* X, Y *be topological spaces,* f_σ *and* f *be functions from* X *into* Y. *It is said that the sequence* $(f_\sigma)_{\sigma\in\Sigma}$ *converges almost-uniformly to* f *if:*

(4) $(f_\sigma)_{\sigma\in\Sigma}$ *is pointwise convergent to* f;

(5) *for every $x \in X$ and for every open cover Ω of Y and for each $\sigma \in \Sigma$ there exists $\sigma_x \geq \sigma$ and a neighborhood U of x such that*

$$f_{\sigma_x}(t) \in \mathrm{St}(f(t), \Omega)$$

for each t in U.

Let us remark that this definition generalizes almost uniform convergence of (usual) sequences of continuous functions from a topological space into a metric space. In fact:

Theorem 3. *Suppose, $(f_n)_{n\in\mathbb{N}}$ is a usual sequence of functions from a topological space $(X, \mathcal{T})$ into a metric space (Y, ϱ). If this sequence is almost uniformly convergent in view of Definition 3 to a function $f\colon X \to Y$ then this sequence is almost uniformly convergent in view of Definition 1.*

Proof. It follows, that $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to f.

For each positive real number r the class $\Omega_r := \{B(y, r) : y \in Y\}$ of open balls in Y is an open cover of that space. Remark that $\mathrm{St}(y_0, \Omega_r) \subset B(y_0, 2r)$. In view of the second condition of almost uniform convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to a function f we state, that for each $x \in X$, each positive integer n and each $\varepsilon > 0$ there exist a positive integer $n_x > n$ and open neighborhood U_x of the point x such that

$$f_{n_x}(t) \in \mathrm{St}(f(t), \Omega_\varepsilon),$$

which means that

$$\varrho\,(f_{n_x}(t), f(t)) < 2\varepsilon)$$

for each $t \in U_x$. □

Theorem 4. *Suppose, $(f_n)_{n\in\mathbb{N}}$ is a usual sequence of continuous functions from a topological space $(X, \mathcal{T})$ into a metric space (Y, ϱ). If this sequence is almost uniformly convergent in view of Definition 1 to a function $f\colon X \to Y$ then this sequence is almost uniformly convergent in view of Definition 3.*

Proof. First notice that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to f.

Fix $x \in X$, $n \in \mathbb{N}$ and an open cover Ω of the space Y. For the point $f(x)$ there exist a positive number ε and V in Ω such that

$$B(f(x), 3\varepsilon) \subset V.$$

Then there exists a positive integer n_0 such that $n_0 > n$ and for each $k > n_0$

$$\varrho\,(f_k(x), f(x)) < \varepsilon.$$

From Definition 1 we infer that there exist a positive integer $n_x > n_0$ and a neighborhood U_1 of x such that

$$f_{n_x}(t) \in B(f(t), \varepsilon) \quad \text{if} \quad t \in U_1$$

and there exists a neighborhood U_2 of x such that

$$f_{n_x}(t) \in B(f_{n_x}(x), \varepsilon) \quad \text{if} \quad t \in U_2$$

since f_{n_x} is a continuous function.

Let $t \in U_1 \cap U_2$. Therefore

$$\varrho\left(f_{n_x}(t), f(x)\right) \leq \varrho\left(f_{n_x}(t), f_{n_x}(x)\right) + \varrho\left(f_{n_x}(x), f(x)\right) < 2\varepsilon$$

and

$$\varrho\left(f(t), f(x)\right) \leq \varrho\left(f(t), f_{n_x}(t)\right) + \varrho\left(f_{n_x}(t), f_{n_x}(x)\right) + \varrho\left(f_{n_x}(x), f(x)\right) < 3\varepsilon.$$

Hence

$$f_{n_x}(t) \in B(f(x), 2\varepsilon), \quad B(f(x), 2\varepsilon) \subset V \quad \text{and} \quad V \in \Omega,$$

and

$$f(t) \in B(f(x), 3\varepsilon), \quad B(f(x), 3\varepsilon) \subset V \quad \text{and} \quad V \in \Omega.$$

It follows that both of the elements $f(t)$ and $f_{n_x}(t)$ belong to the same set V from the cover Ω. Thus

$$f_{n_x}(t) \in \operatorname{St}\left(f(t), \Omega\right).$$

In this way we have proved that almost uniform convergence (in the sense of metric spaces) implies new kind of almost convergence in the class of continuous functions. □

This Theorem completes the proof, that almost uniform convergence of nets of functions with values in a topological space is a generalization of almost uniform convergence of (usual) sequences of continuous functions with values in a metric space.

2. Main results

One can ask whether there is some connection between almost-uniform convergence and quasi-uniform convergence in the sense of Definition 2. The answer is no, which is shown by the following two examples:

Example 1. Let $f_n\colon [0,1) \to [0,1)$ be defined as follows $f_n(x) = x^n$, $n \in \mathbb{N}$. Then $(f_n)_{n=1}^{\infty}$ converges almost-uniformly to zero function, but it is not quasi-uniformly convergent to that function.

Example 2. Let $(p_n)_{n=1}^{\infty}$ be the increasing sequence of all prime numbers. Now let functions $f_n\colon [0,1] \longrightarrow [0,1]$ be defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if} \quad x = \dfrac{1}{p_n^k},\ k \in \mathbb{N} \\ \\ 0, & \text{if} \quad x \neq \dfrac{1}{p_n^k},\ k \in \mathbb{N}. \end{cases}$$

One can check that $(f_n)_{n=1}^{\infty}$ is quasi-uniformly convergent sequence to zero function in $[0,1]$ but it is not almost-uniformly convergent to that function.

Theorem 5. *Let X be any topological space, Y be a regular topological space, $(f_n)_{n=1}^{\infty}$ be a sequence of functions from X into Y and f be a function from X into Y. If each function f_n is continuous at a point x_0 from X and $(f_n)_{n=1}^{\infty}$ is almost-uniformly convergent to f then f is continuous at x_0 as well.*

Proof. Let $x_0 \in X$ and U be an open neighborhood of $f(x_0)$. Since Y is a regular space, one can find a neighborhood V of $f(x_0)$ such that

$$f(x_0) \in V \qquad \text{and} \qquad V \subset \overline{V} \subset U.$$

The assumption of almost-uniform convergence of $(f_n)_{n=1}^{\infty}$ implies pointwise convergence of the sequence, hence there exists n_0 such that

$$f_n(x_0) \in V \qquad \text{if} \qquad n \geq n_0. \tag{5}$$

The class of sets $\{U, Y \setminus \overline{V}\}$ forms open cover Ω of Y, so by the second condition of almost-uniform convergence it is possible to find n_x not less than n_0 and a neighborhood W_1 of x_0 such that

$$f_{n_x}(t) \in \mathrm{St}(f(t), \Omega) \tag{6}$$

for each $t \in W_1$. Since f_{n_x} is continuous at x_0, then there exists a neighborhood W_2 of x_0 such that

$$f_{n_x}(W_2) \subset V. \tag{7}$$

Now, if $W = W_1 \cap W_2$ then W is a neighborhood of x_0 and conditions (6) and (7) imply that for each $t \in W$ we have:

$$f_{n_x}(t) \in \mathrm{St}(f(t), \Omega) \qquad \text{and} \qquad f_{n_x}(t) \in V.$$

By above we infer that $f(t) \in U$ for each $t \in W$, i.e. $f(W) \subset U$. Hence f is continuous at x_0 and we are done. $\square$

Note that if X is a topological space and Y is a regular topological space then by Theorem 5 the set $D(f)$ of points of discontinuity of f is contained in the set $\bigcup_{n=1}^{\infty} D(f_n)$, where $D(f_n)$ is the set at which f_n is discontinuous. By this remark if we consider a proper σ-ideal $\mathscr{I}$ of subsets of X and assume that $D(f_n)$ belongs to $\mathscr{I}$ for each $n \in \mathbb{N}$ then $D(f)$ also belongs to $\mathscr{I}$. In other words one can say that:

Corollary 1. *If X is a topological space, Y is a regular topological space, $\mathscr{I}$ is a proper σ-ideal of subsets of X and $(f_n)_{n=1}^{\infty}$ is a sequence of functions from X to Y such that $D(f_n)$ belongs to $\mathscr{I}$ for each $n \in \mathbb{N}$ then the set $D(f)$ belongs to $\mathscr{I}$ as well.*

Corollary 2. *Let X be a topological space, Y be a regular one, $(f_n)_{n=1}^{\infty}$ be a sequence of functions from X to Y for $n \in \mathbb{N}$ and f be a function from X to Y. If f_n is continuous for each $n \in \mathbb{N}$ and $(f_n)_{n=1}^{\infty}$ converges almost-uniformly to f then f is continuous as well.*

Next theorem is analogous to theorem that was proved in [2]. However, the difference is that X is an arbitrary topological space here while X was assumed to be a compact one (for details see Theorem 1, p. 20 of [2]).

Theorem 6. *Let X and Y be topological spaces. If $f_n \colon X \to Y$, $(n \in \mathbb{N})$, $f \colon X \to Y$ are continuous functions and $(f_n)_{n=1}^{\infty}$ is pointwise convergent to the function f then $(f_n)_{n=1}^{\infty}$ is almost-uniformly convergent to f.*

Proof. Let $x_0 \in X$ and Ω be an open cover of Y. Let W be a neighborhood of $f(x_0)$ such that $W \in \Omega$ and n be an arbitrary positive integer. Pointwise convergence of $(f_n)_{n=1}^{\infty}$ implies the existence of positive integer n_x not less than n such that $f_{n_x}(x_0) \in W$. By continuity of f_{n_x} one can find a neighborhood U_1 of x_0 such that $f_{n_x}(U_1) \subset W$.

Similarly, there exists a neighborhood U_2 of x_0 such that $f(U_2) \subset W$. Now, let $U_0 = U_1 \cap U_2$. It is obvious that U_0 forms a neighborhood of x_0 and

$$f_{n_x}(t) \in \mathrm{St}(f(t), \Omega)$$

for each t in U_0. And so we are done. □

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