ARCHIVES OF MECHANICS
Arch. Mech. **72** (5), 465–481, 2020, DOI: 10.24423/aom.3612

Formulas for the slowness of Stoneley waves with sliding contact

P. T. H. GIANG[1)], P. C. VINH[2)], V. T. N. ANH[2)]

[1)] *Faculty of Civil Engineering, Hanoi Architectural University, Km 10 Nguyen Trai Str., Hanoi, Vietnam*

[2)] *Faculty of Mathematics, Mechanics and Informatics, VNU Hanoi University of Science, 334, Nguyen Trai Str., Thanh Xuan, Hanoi, Vietnam, e-mail: pcvinh@vnu.edu.vn*

The main aim of this paper is to derive formulas for the slowness of Stoneley waves traveling along the sliding interface of two isotropic elastic half-spaces. These formulas have been obtained by employing the complex function method. From the derivation of them, it is shown that if a Stoneley wave exists, it is unique. Based on the obtained formulas, it is proved that a Stoneley wave is always possible for two isotropic elastic half-spaces with the same bulk wave velocities. This result leads to the fact that a Stoneley wave is always possible for two elastic half-spaces satisfying the Wiechert condition, a condition that plays an important role in acoustic analyses. The obtained formulas are of theoretical interest and they will be useful in practical applications, especially in nondestructive evaluations.

Key words: Stoneley waves, sliding contact, complex function method, formulas for the Stoneley wave slowness.

1. Introduction

Stoneley waves that propagate along an interface of two dissimilar elastic half-spaces and decay from the interface were first investigated by Stoneley [1] in 1924 for the case when the half-spaces are isotropic and in welded contact. He derived the existence condition (secular equation) for a Stoneley wave and based on it he showed that such an interfacial wave does not always exist. Following him, Sezawa and Kanai [2] and Scholte [3, 4] investigated domains of existence and indicated that the restrictions on material constants that permit the existence of Stoneley waves are rather severe. However, it was proved by Vinh *et al.* [5] that a Stoneley wave is always possible for two isotropic elastic half-spaces with the same bulk wave velocities. Considering the propagation of Stoneley waves in the vicinity of Wiechert condition [6], Ilyashenko [7] asserted that the existence domain of Stoneley waves is not simply connected. As at the Wiechert condition two bulk wave velocities of two isotropic elastic half-spaces are equal to each other, there always exists a Stoneley wave propagating along the interface of two Wiechert-half-spaces, according to Vinh *et al.* [5]. Stoneley waves in generally anisotropic solids were investigated by Stroh [8], Barnett *et al.* [9]. Barnett *et al.* [9] established a definite existence criterion and proved the uniqueness of Stoneley waves using the surface impedance matrix method. The propagation of Stoneley waves in pre-stressed elastic half-spaces was studied by Chadwick and Jarvis [10, 11], Dasgupta [12], Dunwoody [13], Dowaikh and Ogden [14], Vinh and Giang [15]. In the above mentioned investigations, the contact of two half-spaces is perfectly bonded. Stoneley waves propagating along sliding interfaces were investigated by Murty [16], Vinh and Giang [17] for the isotropic case, by Barnett *et al.* [18] for the anisotropic case. Barnett *et al.* [18] showed that for the isotropic elastic half-spaces, if a Stoneley wave exists, then it is unique. However, for the anisotropic half-spaces, a new slip-wave mode is possible.

For Stoneley waves, their velocity and slowness are fundamental quantities which interest researchers in seismology, geophysics, material sciences and other fields of physics. Since Green's function for many elastodynamic problems for two dissimilar elastic half-space involves the solution of the secular equation of Stoneley waves [19], formulas for the Stoneley wave velocity and slowness are of practical as well as theoretical interest. They are also a powerful tool for solving the inverse problems: determining material parameters of elastic half-spaces from measured values of the wave velocity or slowness of Stoneley waves. By using the complex function method, formulas for the velocity and slowness were derived by Vinh *et al.* [5] for Stoneley waves traveling along the welded interface between two isotropic elastic half-spaces with the same bulk wave velocities, by Vinh [20] for Stoneley waves propagating along the interface between an isotropic elastic half-space and a fluid half-space (Scholte waves). The formulas for the velocity of Stoneley waves in two isotropic elastic half-spaces with sliding contact were obtained by Vinh and Giang [17]. However, those for the wave slowness have not appeared in the literature.

In this paper, we establish formulas for the slowness of Stoneley waves traveling along the sliding interface of two isotropic elastic half-spaces. These formulas have been derived by employing the complex function method. First, the (transcendental) real secular equation for the wave slowness is written in the complex form. Then, this equation is proved to be equivalent to a (complex) quadratic equation using the complex function method. The formulas for the wave slowness is derived by solving this quadratic equation. It is noted that, in order to obtain the formula for the wave velocity, Vinh and Giang [17] have to solve a complex cubic equation.

From the derivation of the wave slowness formulas, it is proved that if a Stoneley wave exists, then it is unique. Applying the obtained formulas, it has been proved that for two isotropic elastic half-spaces with sliding contact and having the same bulk wave velocities, a Stoneley wave is always possible. This result is in accordance with the conclusion made by Barnett *et al.* [18]. It also provides the fact that a Stoneley wave is always possible for two elastic half-spaces with sliding contact and satisfying the Wiechert condition [6], a condition that plays an important role in acoustic analyses [7].

2. Secular equation

In this section we show briefly the derivation of the secular equation of Stoneley waves propagating along the sliding interface of two isotropic elastic half-spaces. For details, the reader is referred to the paper by Murty [16].

Let us consider two isotropic elastic solids Ω and Ω^* occupying the half-space $x_2 \geq 0$ and $x_2 \leq 0$, respectively. The same quantities related to Ω and Ω^* have the same symbol but are systematically distinguished by an asterisk if pertaining to Ω^*. Suppose that these two elastic half-spaces are in sliding contact with each other at the plane $x_2 = 0$ (see [16, 21]). In particular, the normal component of the particle displacement vector and the normal component of the stress tensor are continuous, while the shearing stress vanishes across the interface $x_2 = 0$, i.e.:

$$u_2 = u_2^*, \quad \sigma_{22} = \sigma_{22}^*, \quad \sigma_{12} = \sigma_{12}^* = 0 \quad \text{at } x_2 = 0. \tag{2.1}$$

Consider the propagation of a plane wave, traveling with velocity c (> 0) and wave number k (> 0) in the x_1-direction, being mostly confined to the neighbourhood of the interface $x_2 = 0$. Then the displacement components u_1, u_2, u_3 (corresponding to Ω) are given by [17]:

$$\begin{aligned} u_1 &= (Q_1 e^{-kb_1 x_2} + Q_2 e^{-kb_2 x_2}) e^{ik(x_1 - ct)}, \\ u_2 &= \left(-\frac{b_1}{i} Q_1 e^{-kb_1 x_2} + \frac{i}{b_2} Q_2 e^{-kb_2 x_2}\right) e^{ik(x_1 - ct)}, \\ u_3 &\equiv 0 \end{aligned} \tag{2.2}$$

where Q_1, Q_2 are constants to be determined, and:

$$b_1 = \sqrt{1 - \frac{c^2}{c_L^2}}, \qquad b_2 = \sqrt{1 - \frac{c^2}{c_T^2}},$$

$c_L = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_T = \sqrt{\mu/\rho}$ denote speed of the longitudinal wave and the transverse wave, respectively, of the half-space Ω, λ and μ are usual Lame constants. Similarly, for Ω^* the displacement components u_1^*, u_2^*, u_3^* are expressed by [17]:

$$
\begin{aligned}
u_1^* &= (Q_1^* e^{kb_1^* x_2} + Q_2^* e^{kb_2^* x_2}) e^{ik(x_1 - ct)},\\
u_2^* &= \left(\frac{b_1^*}{i} Q_1^* e^{kb_1^* x_2} - \frac{i}{b_2^*} Q_2^* e^{kb_2^* x_2}\right) e^{ik(x_1 - ct)},\\
u_3^* &\equiv 0
\end{aligned} \tag{2.3}
$$

where Q_1^*, Q_2^* are constants to be determined, and:

$$
b_1^* = \sqrt{1 - \frac{c^2}{c_L^{*2}}}, \qquad b_2^* = \sqrt{1 - \frac{c^2}{c_T^{*2}}},
$$

$c_L^* = \sqrt{(\lambda^* + 2\mu^*)/\rho^*}$ and $c_T^* = \sqrt{\mu^*/\rho^*}$ denote speed of the longitudinal wave and the transverse wave, respectively, of the half-space Ω^*. It is not difficult to check that:

PROPOSITION 1. *If a Stoneley wave exists, then its velocity is subject to:*

$$
0 < c < \min\{c_T, c_T^*\}. \tag{2.4}
$$

The inequalities (2.4) ensure that b_k, b_k^* $(k = 1, 2)$ are strictly positive, the decay conditions are therefore satisfied. Introducing (2.2), (2.3) into (2.1) with noting that (commas indicate the differentiation with respect to x_k):

$$
\begin{aligned}
\sigma_{12} &= \mu(u_{1,2} + u_{2,1}), \qquad \sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu u_{2,2},\\
\sigma_{12}^* &= \mu^*(u_{1,2}^* + u_{2,1}^*), \qquad \sigma_{22}^* = \lambda^*(u_{1,1}^* + u_{2,2}^*) + 2\mu^* u_{2,2}^*
\end{aligned}
$$

yields a system of four homogeneous linear equations for Q_1, Q_2, Q_1^*, Q_2^*, and making zero the determinant of matrix of coefficients of this system gives:

$$
c_T^4 \frac{\left[4\sqrt{1 - \frac{c^2}{c_T^2}}\sqrt{1 - \frac{c^2}{c_L^2}} - (2 - \frac{c^2}{c_T^2})^2\right]}{\sqrt{1 - \frac{c^2}{c_L^2}}} + \frac{\rho^*}{\rho} c_T^{*4} \frac{\left[4\sqrt{1 - \frac{c^2}{c_L^{*2}}}\sqrt{1 - \frac{c^2}{c_T^{*2}}} - \left(2 - \frac{c^2}{c_T^{*2}}\right)^2\right]}{\sqrt{1 - \frac{c^2}{c_L^{*2}}}} = 0. \tag{2.5}
$$

Equation (2.5) is the secular equation of the Stoneley wave that coincides with the one derived by MURTY [16, 21]. It is clear that (2.4) and (2.5) are the necessary condition for the existence of a Stoneley wave. It is not difficult to show that they are also the sufficient condition. Thus we have:

PROPOSITION 2. *For a Stoneley wave to exist, it is necessary and sufficient that* (2.4) *and* (2.5) *are both satisfied.*

3. Formulas for the slowness of Stoneley waves

In this section we establish formulas for the Stoneley wave slowness $y = c_T^2/c^2$. Without loss of generality we can suppose that $c_T \leq c_T^*$. We introduce dimensionless parameters:

$$B = \frac{c_T^2}{c_T^{*2}}, \quad D = \frac{\rho}{\rho^*}, \quad F = \frac{c_T^2}{c_L^{*2}}, \quad E = \frac{c_T^2}{c_L^2}. \tag{3.1}$$

With the assumption $c_T \leq c_T^*$ ($\Rightarrow B \leq 1$) and the facts $c_T < c_L$ ($\Rightarrow E < 1$) and $c_T^* < c_L^*$ ($\Rightarrow F < B$), we have 3 different cases (of relative order of 1, B, E, F):

Case 1: $1 \geq B \geq E \geq F > 0$.

Case 2: $1 \geq B > F \geq E > 0$.

Case 3: $1 > E \geq B > F > 0$.

In terms of dimensionless parameters (3.1), Eq. (2.5) is of the form:

$$\begin{aligned}(2y-1)^2(y-F)^{1/2} - 4y(y-1)^{1/2}(y-E)^{1/2}(y-F)^{1/2}\\ + \frac{1}{DB^2}\left[(2y-B)^2(y-E)^{1/2} - 4y(y-B)^{1/2}(y-F)^{1/2}(y-E)^{1/2}\right] = 0.\end{aligned} \tag{3.2}$$

Due to (2.4) we have:

$$y > 1. \tag{3.3}$$

Suppose that there exists a Stoneley wave. Then, Eq. (3.2) has a real root that satisfies the inequality (3.3). In order to obtain formulas for the Stoneley wave slowness, we have to find analytical expressions of real roots bigger than 1 of Eq. (3.2). Since to this end we will apply the complex function method, we now consider Eq. (3.2) in the complex plane $\mathbf{C}$:

$$\begin{aligned}(2z-1)^2\sqrt{z-F} - 4z\sqrt{z-1}\sqrt{z-E}\sqrt{z-F}\\ + \frac{1}{DB^2}\Big[(2z-B)^2\sqrt{z-E} - 4z\sqrt{z-B}\sqrt{z-F}\sqrt{z-E}\Big] = 0\end{aligned} \tag{3.4}$$

where $\sqrt{z-1}$, $\sqrt{z-B}$, $\sqrt{z-E}$, $\sqrt{z-F}$ are chosen as the principal branches of the corresponding square roots. Equation (3.4) coincides with Eq. (3.2) for real values of z bigger than 1. Our task now is to find real roots bigger than 1 of the complex Eq. (3.4). Multiplying two sides of (3.4) by $\sqrt{z-1}$ yields equation:

$$\begin{aligned}f(z) =: (2z-1)^2\sqrt{z-1}\sqrt{z-F} + 4z(1-z)\sqrt{z-F}\sqrt{z-E}\\ + \frac{1}{DB^2}\Big[(2z-B)^2\sqrt{z-1}\sqrt{z-E} - 4z\sqrt{z-1}\sqrt{z-F}\sqrt{z-B}\sqrt{z-E}\Big] = 0.\end{aligned} \tag{3.5}$$

As Eq. (3.5) is equivalent to Eq. (3.4) in the domain $z \in \boldsymbol{R} : z > 1$, to obtain formulas for the Stoneley wave slowness we need to find expressions of real roots bigger than 1 of Eq. (3.5).

3.1. Case 1: $1 \geq B \geq E \geq F > 0$

Case 1.1: $1 > B > E > F > 0$.

THEOREM 1. *Let $1 > B > E > F$. If a Stoneley wave exists, then it is unique, and its slowness $y_s = c_T^2/c^2$ is defined by:*

$$y_s = -\frac{A_1}{A_2} - \frac{1}{2}(F+E) - \hat{I}_0 \tag{3.6}$$

where A_1, A_2 given by:

$$\begin{aligned} A_1 &= \frac{2B(1+E-F)+F(F-2E-2)+B^2[3+D(5+E^2+2F-2E(2+F))]}{2DB^2}, \\ A_2 &= \frac{2[DB^2(E-1)+F-B]}{DB^2}, \end{aligned} \tag{3.7}$$

and

$$\hat{I}_0 = \frac{1}{\pi}\Big(-\int_F^E \theta_1(t)\,dt + \int_E^B \theta_2(t)\,dt + \int_B^1 \theta_3(t)\,dt\Big), \tag{3.8}$$

where

$$\theta_k(t) = atan\,\varphi_k(t), \quad k = 1,2,3, \tag{3.9}$$

in which:

$$\begin{aligned} \varphi_1(t) = &\Big[-\frac{1}{DB^2}(2t-B)^2\sqrt{E-t}\Big]\Big/\Big\{(2t-1)^2\sqrt{t-F} \\ &+\frac{4t}{DB^2}\sqrt{t-F}\sqrt{E-t}\sqrt{B-t}+4t\sqrt{1-t}\sqrt{E-t}\sqrt{t-F}\Big\}, \end{aligned} \tag{3.10}$$

$$\varphi_2(t) = \frac{(2t-1)^2\sqrt{t-F}+\dfrac{(2t-B)^2}{DB^2}\sqrt{t-E}}{\dfrac{4t}{DB^2}\sqrt{t-F}\sqrt{t-E}\sqrt{B-t}+4t\sqrt{1-t}\sqrt{t-E}\sqrt{t-F}}, \tag{3.11}$$

$$\varphi_3(t) = \frac{(2t-1)^2\sqrt{t-F}+\dfrac{(2t-B)^2}{DB^2}\sqrt{t-E}-\dfrac{4t}{DB^2}\sqrt{t-F}\sqrt{t-E}\sqrt{t-B}}{4t\sqrt{1-t}\sqrt{t-E}\sqrt{t-F}}. \tag{3.12}$$

Proof: Denote $L = L_1 \cup L_2 \cup L_3$ with $L_1 = [F,E]$, $L_2 = [E,B]$, $L_3 = [B,1]$, $S = \{z \in \mathbf{C},\, z \notin L\}$, $N(z_0) = \{z \in S : 0 < |z - z_0| < \varepsilon\}$, ε is a sufficiently small positive number, z_0 is some point of the complex plane $\mathbf{C}$. If a function $\phi(z)$ is holomorphic in $\Omega \subset \mathbf{C}$ we write $\phi(z) \in H(\Omega)$. From (3.5) it is not difficult to show that the function $f(z)$ has the properties:

(f_1) $f(z) \in H(S)$,

(f_2) $f(z)$ is bounded in $N(F)$ and $N(1)$,

(f_3) $f(z) = O(z^2)$ as $|z| \to \infty$,

(f_4) $f(z)$ is continuous on L from the left and from the right with the boundary values $f^+(t)$ (the right boundary value of $f(z)$), $f^-(t)$ (the left boundary value of $f(z)$) defined as follows:

$$f^{\pm}(t) = \begin{cases} f_1^{\pm}(t), & t \in L_1, \\ f_2^{\pm}(t), & t \in L_2, \\ f_3^{\pm}(t), & t \in L_3 \end{cases} \tag{3.13}$$

where:

$$f_k^-(t) = \overline{f_k^+(t)}, \quad k = 1, 2, 3, \tag{3.14}$$

$$f_1^+(t) = i\Big[(2t-1)^2\sqrt{1-t}\sqrt{t-F} + \frac{4t}{DB^2}\sqrt{1-t}\sqrt{t-F}\sqrt{E-t}\sqrt{B-t} + 4t(1-t)\sqrt{E-t}\sqrt{t-F}\Big] - \frac{(2t-B)^2}{DB^2}\sqrt{1-t}\sqrt{E-t}, \tag{3.15}$$

$$f_2^+(t) = i\Big[(2t-1)^2\sqrt{1-t}\sqrt{t-F} + \frac{(2t-B)^2}{DB^2}\sqrt{1-t}\sqrt{t-E}\Big] + \frac{4t}{DB^2}\sqrt{1-t}\sqrt{t-F}\sqrt{t-E}\sqrt{B-t} + 4t(1-t)\sqrt{t-E}\sqrt{t-F}, \tag{3.16}$$

$$f_3^+(t) = i\Big[(2t-1)^2\sqrt{1-t}\sqrt{t-F} + \frac{(2t-B)^2}{DB^2}\sqrt{1-t}\sqrt{t-E} - \frac{4t}{DB^2}\sqrt{1-t}\sqrt{t-F}\sqrt{t-E}\sqrt{t-B}\Big] + 4t(1-t)\sqrt{t-E}\sqrt{t-F}. \tag{3.17}$$

Note that $f_k^+(t)$ ($f_k^-(t)$) is the right (left) boundary value of $f(z)$ on L_k and $i = \sqrt{-1}$. Now we introduce function $g(t)$ ($t \in L$):

$$g(t) = \begin{cases} \dfrac{f_1^+(t)}{f_1^-(t)}, & t \in L_1, \\ \dfrac{f_2^+(t)}{f_2^-(t)}, & t \in L_2, \\ \dfrac{f_3^+(t)}{f_3^-(t)}, & t \in L_3. \end{cases} \tag{3.18}$$

From (3.13) and (3.18) it is clear that:

$$f^+(t) = g(t) f^-(t), \quad t \in L. \tag{3.19}$$

Consider the function $\Gamma(z)$ defined as:

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\log g(t)}{t-z}\, dt. \tag{3.20}$$

It is not difficult to verify that

(γ_1) $\Gamma(z) \in H(S)$,

(γ_2) $\Gamma(\infty) = 0$,

(γ_3) $\Gamma(z) = -(1/2)\log(z-1) + \Omega_0(z)$ $z \in N(1)$, $\Gamma(z) = \Omega_1(z)$, $z \in N(F)$ where $\Omega_0(z)$ ($\Omega_1(z)$) bounded in $N(1)$ ($N(F)$) and takes a defined value at $z = 1$ ($z = F$).

It is noted that (γ_3) comes from the fact (see [22, Chap. 4, Section 29]):

$$\log g(F) = 0, \quad \log g(1) = -i\pi. \tag{3.21}$$

Introduce a new function $\Phi(z)$ defined by:

$$\Phi(z) = \exp \Gamma(z) \tag{3.22}$$

it is implied from (γ_1)–(γ_3) that:

(ϕ_1) $\Phi(z) \in H(S)$,

(ϕ_2) $\Phi(z) \neq 0$ $\forall z \in S$,

(ϕ_3) $\Phi(z) = O(1)$ as $|z| \to \infty$,

(ϕ_4) $\Phi(z) = (z-1)^{-1/2} \exp \Omega_0(z)$ for $z \in N(1)$, $\Phi(z) = \exp \Omega_1(z)$, $z \in N(F)$.

From the Plemelj formula [22], the function $\Phi(z)$ is seen directly to satisfy the boundary condition:

$$\Phi^+(t) = g(t)\Phi^-(t), \quad t \in L. \tag{3.23}$$

We now consider the function $Y(z)$ defined by:

$$Y(z) = f(z)/\Phi(z). \tag{3.24}$$

From (f_1)–(f_3), (3.19), (ϕ_1)–(ϕ_4) and (3.24), it follows that:

(y_1) $Y(z) \in H(S)$,

(y_2) $Y(z) = O(z^2)$ as $|z| \to \infty$,

(y_3) $Y(z)$ is bounded in $N(1)$ and $N(F)$,

(y_4) $Y^+(t) = Y^-(t)$, $t \in L$.

Properties (y_1) and (y_4) of the function $Y(z)$ show that $Y(z)$ is holomorphic in entire complex plane $\mathbf{C}$, with the possible exception of points: $z = 1$ and $z = F$. By (y_3) these points are removable singularity points and it may be assumed that the function $Y(z)$ is holomorphic in the entire complex plane $\mathbf{C}$ (see [23]). Thus, by the generalized Liouville theorem [23] and taking into account (y_2) we have:

$$Y(z) = P(z) \tag{3.25}$$

where $P(z)$ is a polynomial of order 2.

From (3.24) and (3.25) we have:

$$f(z) = \Phi(z)P(z). \tag{3.26}$$

Since $\Phi(z) \neq 0\ \forall z \in S$ (by (ϕ_2)), and $\Phi(F) \neq 0$ (by (ϕ_4)), from (3.26) we deduce:

$$f(z) = 0 \leftrightarrow P(z) = 0 \text{ in } S \cup \{F\}.$$

As $\Phi(z) \to \infty$ as $z \to 1$ (by (ϕ_4)), from (3.26): if $f(1) = 0 \Rightarrow P(1) = 0$. Suppose $P(1) = 0 \Rightarrow P(z) = z_0(z-1)(z-z_1)$ because $P(z)$ is a second-order polynomial ($z_0 \neq 0$ and z_1 are some complex constants). From this fact, (ϕ_4) and Eq. (3.26) it follows $f(1) = 0$. Thus, we have:

$$f(z) = 0 \leftrightarrow P(z) = 0 \text{ in } S \cup \{F\} \cup \{1\}. \tag{3.27}$$

In view of (3.27), instead of finding zeros of $f(z)$ we now looking for two zeros of the second-order polynomial $P(z)$ that may be an easier task. In order to do that, first we have to determine $P(z)$. It should be noted that equation $f(z) = 0$ has no solutions in the interval $(F, 1)$ because of its discontinuity on this interval. From (3.22) and (3.26) we have:

$$P(z) = f(z)\mathrm{e}^{-\Gamma(z)}. \tag{3.28}$$

From (3.14)–(3.18) it follows:

$$\log g(t) = i\phi(t) - 2\pi i, \qquad \phi(t) =: \operatorname{Arg} g(t) \tag{3.29}$$

where:

$$\phi(t) = \begin{cases} \phi_1(t), & t \in L_1, \\ \phi_2(t), & t \in L_2, \\ \phi_3(t), & t \in L_3, \end{cases} \tag{3.30}$$

and

$$\phi_1(t) = \pi - 2\theta_1(t), \quad \phi_2(t) = 2\theta_2(t), \quad \phi_3(t) = 2\theta_3(t) \tag{3.31}$$

where $\theta_k(t)$ $(k = 1, 2, 3)$ are given by (3.9)–(3.12). From (3.20) and (3.29) it follows (see also [24]):

$$-\Gamma(z) = \sum_{n=0}^{\infty} \frac{I_n}{z^{n+1}} \tag{3.32}$$

in which:

$$I_n = \frac{1}{2\pi} \int_F^1 t^n \hat{\phi}(t)\, dt, \quad n = 0, 1, \dots, \ \hat{\phi}(t) = \phi(t) - 2\pi. \tag{3.33}$$

On use of (3.32) we can express $e^{-\Gamma(z)}$ as follow:

$$e^{-\Gamma(z)} = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + O(z^{-4}) \tag{3.34}$$

where a_1, a_2, a_3 are constants to be determined. Employing the identify

$$(\mathrm{e}^{-\Gamma(z)})' = (-\Gamma(z))'\mathrm{e}^{-\Gamma(z)} \tag{3.35}$$

and substituting (3.32), (3.34) into (3.35) provide:

$$a_1 = I_0, \quad a_2 = \frac{I_0^2}{2} + I_1, \quad a_3 = \frac{I_0^3}{6} + I_1 I_0 + I_2. \tag{3.36}$$

By expanding $\sqrt{z-1}$, $\sqrt{z-B}$, $\sqrt{z-E}$, $\sqrt{z-F}$ into Laurent series at infinity, it is not difficult to verify that:

$$f(z) = A_2 z^2 + A_1 z + A_0 + O(z^{-1}) \tag{3.37}$$

where A_1 and A_2 are calculated by (3.7). Substituting (3.34) and (3.37) into (3.28) yields:

$$P(z) = A_2 z^2 + (a_1 A_2 + A_1) z + a_2 A_2 + a_1 A_1 + A_0. \tag{3.38}$$

It is clear from (3.5) that $f(1) = 0$, hence by (3.27):

$$P(1) = 0. \tag{3.39}$$

On use of (3.38), (3.39) and taking into account the first of (3.36) and (3.33) with $n = 0$, it is easy to see that the second root of the equation $P(z) = 0$, denoted by y_s, is given by (3.6).

Now we suppose there exists a Stoneley wave. Then the equation $f(z) = 0$ has a real root bigger than 1, denoted by y_s. Due to (3.27): $P(y_s) = 0$. Since $y_s > 1$ and (3.39), y_s is the second root of the equation $P(z) = 0$, given by (3.6). That means the slowness of the Stoneley wave, say y_s, is calculated by (3.6).

If there exist two Stoneley waves, then the equation $f(z) = 0$ has two distinct real roots $y_s^{(1)}$ and $y_s^{(2)}$ which are both bigger than 1. It follows from (3.27) and (3.39) that the second-order polynomial $P(z)$ has three different zeros. But this is impossible. Therefore, if a Stoneley wave exists, then it is unique. The proof of Theorem 1 is completed.

Now we consider the special cases when at least one of the inequalities $1 > B > E > F$ is replaced by an equality. There exist only three following possibilities (noting that $B > F$, $E < 1$):

Case 1.2: $1 = B > E > F > 0$.
Case 1.3: $1 > B = E > F > 0$.
Case 1.4: $1 = B > E = F > 0$.

The results of these special cases are deduced directly from the ones of the case 1.1 by taking $B = 1$ for the case 1.2, $B = E$ for the case 1.3 and $B = 1$ and $E = F$ for the case 1.4.

For checking the obtained formula, a number of numerical values of the Stoneley wave slowness are calculated by using the formula (3.6) (denoted by y_s) and by solving directly the secular equation (3.2) in the domain $y > 1$ (denoted by y_*). It is seen from Table 1 that they are the same. They are also identical to the corresponding values of $1/x_s$, where x_s is the Stoneley wave velocity calculated by Eq. (24) in [17].

Table 1. Some values of the Stoneley wave slowness computed by using formula (3.6) (denoted by y_s) and by directly solving Eq. (3.2) (denoted by y_*).

$F; E; B; D$	$1/6; 1/3; 1/2; 3.4$	$0.6; 0.7; 0.8; 0.5$	$0.2; 0.4; 0.6; 3.3$	$0.2; 0.35; 0.7; 2.5$
y_*	1.0546	1.7281	1.0959	1.0866
y_s	1.0546	1.7281	1.0959	1.0866
$1/x_s$	1.0546	1.7281	1.0959	1.0866

3.2. Case 2: $1 \geq B > F \geq E > 0$

Case 2.1: $1 > B > F > E > 0$. Following the same procedure carried out for the case 1.1 we have:

THEOREM 2. *Let* $1 > B > F > E$. *If a Stoneley wave exists, then it is unique, and its slowness* $y_s = c_T^2/c^2$ *is defined by* (3.6) *in which* A_1, A_2 *given by*

(3.7) and $\hat{I}_0$ is defined as:

$$\hat{I}_0 = \frac{1}{\pi}\Big(-\int_E^F \theta_1(t)\,dt + \int_F^B \theta_2(t)\,dt + \int_B^1 \theta_3(t)\,dt\Big) \tag{3.40}$$

where $\theta_k(t)$ are computed by (3.9) in which $\varphi_k(t)$ $(k = 2, 3)$ are given by (3.11), (3.12) and

$$\varphi_1(t) = -(2t-1)^2\sqrt{F-t}\Big/\Big\{\Big[\frac{1}{DB^2}(2t-B)^2 + 4t\sqrt{1-t}\sqrt{F-t} + \frac{4t}{DB^2}\sqrt{F-t}\sqrt{B-t}\Big]\sqrt{t-E}\Big\}. \tag{3.41}$$

Case 2.2:. $1 = B > F > E > 0$. The result for this case is also derived directly from the one for the case 2.1 by taking $B = 1$.

Table 2. Some values of the Stoneley wave slowness computed by using formula (3.6) (denoted by y_s) and by solving directly Eq. (3.2) (denoted by y_*).

$E; F; B; D$	$0.25; 0.45; 0.7; 3.0$	$0.25; 0.4; 0.6; 3.0$	$0.2; 0.45; 0.8; 3.2$	$0.4; 0.5; 0.7; 0.6$
y_*	1.1449	1.1306	1.1328	1.2646
y_s	1.1449	1.1306	1.1328	1.2646
$1/x_s$	1.1449	1.1306	1.1328	1.2646

Table 2 shows that the Stoneley wave slownesses computed by using the formula (3.6) are identical to the ones obtained by solving directly the secular equation (3.2), and they coincide with the corresponding values of $1/x_s$, where x_s is the Stoneley wave velocity calculated by Eq. (60) in [17].

3.3. Case 3: $1 > E \geq B > F > 0$

Note that, the case when $B = E$ has been considered already above (case 1.3), thus we need to examine here only the case: $1 > E > B > F > 0$.

Following the same procedure used for the case 1.1 we have:

THEOREM 3. *Suppose that $1 > E > B > F$. If a Stoneley wave exists, then it is unique, and its slowness $y_s = c_T^2/c^2$ is defined by (3.6) in which A_1, A_2 given by (3.7) and:*

$$\hat{I}_0 = \frac{1}{\pi}\Big(-\int_F^B \theta_1(t)\,dt - \int_B^E \theta_2(t)\,dt + \int_E^1 \theta_3(t)\,dt\Big) \tag{3.42}$$

where $\theta_k(t)$ *are given by* (3.9) *in which* $\varphi_k(t)$ $(k = 1, 3)$ *defined by* (3.10), (3.12) *and*

$$\varphi_2(t) = \frac{\frac{1}{DB^2}\sqrt{E-t}\big(4t\sqrt{t-F}\sqrt{t-B}-(2t-B)^2\big)}{(2t-1)^2\sqrt{t-F}+4t\sqrt{1-t}\sqrt{E-t}\sqrt{t-F}}. \tag{3.43}$$

Table 3. Some values of the Stoneley wave slowness calculated by using formula (3.6) (denoted by y_s) and by solving directly Eq. (3.2) (denoted by y_*).

$F; B; E; D$	$0.2; 0.4; 0.6; 3.0$	$0.15; 0.35; 0.5; 3.2$	$0.15; 0.3; 0.7; 3.0$	$0.6; 0.7; 0.9; 0.8$
y_*	1.1240	1.0446	1.1067	3.2322
y_s	1.1240	1.0446	1.1067	3.2322
$1/x_s$	1.1240	1.0446	1.1067	3.2322

It is seen from Table 3 that the formula (3.6) and the secular equation (3.2) give the same Stoneley wave slownesses and they are identical to the corresponding values of $1/x_s$, where x_s is the Stoneley wave velocity calculated by Eq. (65) in [17].

4. Stoneley waves in two half-spaces with the same bulk wave velocities

Suppose two half-spaces have the same bulk wave velocities, that means $c_T = c_T^*$ and $c_L = c_L^*$. With this assumption, it follows from (3.1): $B = 1$ and $F = E\,(= c_T^2/c_L^2)$. *This case is the case 1.4, a special case of the case 1.*

THEOREM 4.

(i) *There always exists a Stoneley wave that travels along the sliding interface between two isotropic half-spaces with the same bulk wave velocities.*

(ii) *Its slowness is calculated by:*

$$y_s = \frac{2+(1-E)^2}{4(1-E)} + \frac{1}{\pi}\int_E^1 \operatorname{atan}\left\{\frac{4t\sqrt{1-t}\sqrt{t-E}}{(2t-1)^2}\right\}dt. \tag{4.1}$$

Proof. (ii) From (3.7) and $B = 1$, $E = F$ we have:

$$A_1 = \frac{(5-2E-E^2)(D+1)}{2D}, \qquad A_2 = \frac{2(E-1)(D+1)}{D}. \tag{4.2}$$

With $B = 1$ and $E = F$, it follows from (3.8), (3.9) and (3.11):

$$\hat{I}_0 = \frac{1}{\pi}\int_E^1 \operatorname{atan}\{\varphi_2(t)\}dt \tag{4.3}$$

where:

$$\varphi_2(t) = \frac{(2t-1)^2}{4t\sqrt{1-t}\sqrt{t-E}}. \tag{4.4}$$

Because $\operatorname{atan}\{\varphi_2(t)\} = \frac{\pi}{2} - \operatorname{atan}\{1/\varphi_2(t)\}$, it follows from (4.3) and (4.4):

$$\hat{I}_0 = \frac{1}{2}(1-E) - \frac{1}{\pi}\int_E^1 \operatorname{atan}\left\{\frac{4t\sqrt{1-t}\sqrt{t-E}}{(2t-1)^2}\right\} dt. \tag{4.5}$$

Substitution of (4.2) and (4.5) into (3.6) (with $E = F$) yields (4.1).

(i) In order to prove the existence of Stoneley waves, according to Proposition 2 we have to show: $y_s > 1\ \forall E \in (0, 1)$, where y_s is given by (4.1). We recall that y_s is a root of the equation $P(z) = 0$, i. e. $P(y_s) = 0$, where $P(z)$ is defined by (3.38) (in which $E = F$ and $B = 1$). If $y_s > 1$, from (3.27) it follows: $f(y_s) = 0$. Therefore, a Stoneley wave is possible, according to Proposition 2.

First, we show that $y_s \notin (E, 1)$. Indeed, from (3.16), $E = F$ and $B = 1$, it is easy to verify that $f_2^{\pm}(t) \neq 0\ \forall t \in (E, 1)$. Suppose $y_s \in (E, 1)$. As $P(y_s) = 0 \Rightarrow P^{\pm}(y_s) = 0$. From (3.28) and the fact $1/\Phi^{\pm}(y_s) \neq 0$ it follows $f_2^{\pm}(y_s) = 0$. But this contradicts $f_2^{\pm}(t) \neq 0\ \forall t \in (E, 1)$. Therefore, we conclude: $y_s \notin (E, 1)$ (*).

It is not difficult to verify that:

$$\frac{2 + (1-E)^2}{4(1-E)} > E \quad \forall E \in (0, 1). \tag{4.6}$$

Therefore, from (4.1): $y_s > E\ \forall E \in (0, 1)$ (**).

Suppose $y_s = 1 \Rightarrow P(z) = z_0(z-1)^2$, $z_0 \neq 0$, because $z = 1$ is double root of $P(z) = 0$ (noting (3.39)) and $P(z)$ is a second-order polynomial. From this fact, (3.26) and (ϕ_4) it follows that:

$$\lim_{z\to 1} \frac{f(z)}{\sqrt{z-1}} = 0. \tag{4.7}$$

However, one can see from Eq. (3.5) with $E = F$, $B = 1$ that:

$$\lim_{z\to 1} \frac{f(z)}{\sqrt{z-1}} = \sqrt{1-E}\left(1 + \frac{1}{D}\right) > 0 \quad \forall E \in (0, 1),\ D > 0. \tag{4.8}$$

Therefore, we have: $y_s \neq 1$ (***). From (*), (**) and (***) it follows: $y_s > 1$. The proof of Theorem 4 is completed.

REMARK 1. (i) Since y_s given by (4.1) is the slowness of a Rayleigh wave propagating in a traction-free isotropic elastic half-space [24, 25], *it is interesting that in this case, the Stoneley travels as a Rayleighh wave in each half-space* *subject to traction-free condition*. Because two isotropic elastic half-spaces have the same Rayleigh wave velocities if they have the same bulk wave velocities, this conclusion agrees with Barnett's statement [18] saying that the sliding interface of two isotropic half-spaces with the same Rayleigh wave velocities always supports a Stoneley wave.

(ii) Since two isotropic elastic half-spaces have the same Rayleigh wave velocities if they satisfy the Wiechert condition [6]:

$$\frac{\lambda}{\lambda^*} = \frac{\mu}{\mu^*} = \frac{\rho}{\rho^*} \tag{4.9}$$

it follows that Theorem 5 holds for two elastic-half-spaces subject the Wiechert condition.

5. Conclusions

In this paper, by using the complex function method we have derived formulas for the slowness of Stoneley waves propagating along the sliding interface of two isotropic elastic half-spaces. The derivation of these formulas shows that if a Stoneley wave exists, then it is unique. The obtained formulas help us to show that a Stoneley wave is always possible for two elastic half-spaces with the same bulk wave velocities that includes the Wiechert condition, a condition that plays an important role in acoustic analyses. This existence result cannot be established if using the formulas for the wave velocity derived in [17]. The obtained formulas are of theoretical as well as practical interest.

Acknowledgments

The work was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant N0 107.02-2019.314.

References

1. R. Stoneley, *Elastic waves at the surface of seperation of two solids*, Proceedings of the Royal Society of London, A, **106**, 416–428, 1924.
2. K. Sezawa, K. Kanai, *The range of possible existence of Stoneley waves, and some related problems*, Bulletin of the Earthquake Research Institute of Tokyo University, **17**, 1–8, 1939.
3. J.G. Scholte, *On the Stoneley wave equation*, Proceedings of the Royal Academy of Science Amsterdam, **45**, 159–164, 1942.
4. J.G. Scholte, *The range of existence of Rayleigh and Stoneley waves*, Monthly Notices of the Royal Astronomical Society Geophysics Supplement, **5**, 120–126, 1947.

5. P.C. Vinh, P.G. Malischewsky, P.T.H. Giang, *Formulas for the speed and slowness of Stoneley waves in bonded isotropic elastic half-spaces with the same bulk wave velocities*, International Journal of Engineering Science, **60**, 53–58, 2012.

6. E. Wiechert, K. Zopp Ritz, *Our present knowledge of the Earth*, [in:] Report Board of Regents Smithsonian Institution, 431–439, 1908.

7. A.V. Ilyashenko, *Stoneley waves in a vicinity of the Wiechert condition*, International Journal of Dynamics and Control, online, https://doi.org/10.1007/s40435-020-00625-y, 2020.

8. A.N. Stroh, *Steady state problems in anisotropic elasticity*, Journal of Mathematical Physics, **41**, 77–103, 1962.

9. D.M. Barnett, J. Lothe, S.D. Gavazza, M.J.P. Musgrave, *Consideration of the existence of interfacial (Stoneley) waves in bonded anisotropic elastic half-spaces*, Proceedings of the Royal Society London, A, **412**, 153–166, 1985.

10. P. Chadwick, D.A. Jarvis, *Interfacial waves in a pre-strain neo-Hookean body I. Biaxial state of strain*, Quaterly Journal of Mechanics and Applied Mathematics, **32**, 387–399, 1979.

11. P. Chadwick, D.A. Jarvis, *Interfacial waves in a pre-strain neo-Hookean body II. Triaxial state of strain*, Journal of Mechanics and Applied Mathematics, **32**, 401–418, 1979.

12. A. Dasgupta, *Effect of high initial stress on the propagation of Stoneley waves at the interface of two isotropic elastic incompressible media*, Indian Journal of Pure and Applied Mathemaics, **12**, 919–926, 1981.

13. J. Dunwoody, *Elastic interfacial standing waves*, [in:] M.F. McCarthy, M.A. Hayes [eds.], *Elastic Waves Propagation*, pp. 107–112, North-Holland, Amsterdam, 1989.

14. M.A. Dowaikh, R.W. Ogden, *Interfacial waves and deformations in pre-stressed elastic media*, Proceedings of the Royal Society of London, A, **433**, 313–328, 1991.

15. P.C. Vinh, P.T.H. Giang, *Uniqueness of Stoneley waves in pre-stressed incompressible elastic media*, International Journal of Non-Linear Mechanics, **47**, 128–134, 2012.

16. G.S. Murty, *A theoretical model for the attenuation and dispersion of Stoneley waves at the loosely bonded interface of elastichalf-spaces*, Physics of the Earth and Planetary Interiors, **11**, 65–79, 1975.

17. P.C. Vinh, P.T.H. Giang, *On formulas for the velocity of Stoneley waves propagating along the loosely bonded interface of two elastic half-spaces*, Wave Motion, **48**, 647–657, 2011.

18. D.M. Barnett, S.D. Gavazza, J. Lothe, *Slip waves along the interface between two anisotropic elastic half-spaces in sliding contact*, Proceedings of the Royal Society of London, A, **415**, 389–419, 1988.

19. H.D. Phan, T.Q. Bui, H.T.L. Nguyen, P.C. Vinh, *Computation of interface wave motions by reciprocity considerations*, Wave Motion, **79**, 10–22, 2018.

20. P.C. Vinh, *Scholte-wave velocity formulae*, Wave Motion, **50**, 180–190, 2013.

21. G.S. Murty, *Wave propagation at an unbounded interface between two elastic half-spaces*, Journal of Acoustical Society of America, **58**, 1094–1095, 1975.

22. N.I. Muskhelishvili, *Singular Intergral Equations*, Noordhoff, Groningen, 1953.

23. N.I. MUSKHELISHVILI, *Some Basic Problems of Mathematical Theory of Elasticity*, Noordhoff, Netherlands 1963.

24. D. NKEMZI, *A new formula for the velocity of Rayleigh waves*, Wave Motion, **26**, 199–205, 1997.

25. M. ROMEO, *Uniqueness of the solution to the secular equation for viscoelastic surface waves*, Applied Mathematics Letters, **15**, 649–653, 2002.

Received July 15, 2020; revised version August 24, 2020.
Published online September 24, 2020.