THE DIMENSION OF A QUASI-HEREDITARY ALGEBRA

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Quasi-hereditary algebras have been introduced by L. Scott [S] in order to study highest weight categories as they arise in the representation theory of complex Lie algebras and algebraic groups. They have been studied by Cline, Parshall and Scott [CPS], [PS], and in [DR1], [DR2]. Here, we are going to give lower and upper bounds for the dimension of a quasi-hereditary algebra in terms of its species, and we characterize those algebras where one of these bounds is attained: we call them the shallow and the deep quasi-hereditary algebras, respectively.

1. Definitions and results

Let $A$ be a basic semiprimary ring with radical $N$, let $e_1, \ldots , e_n$ be a complete set of orthogonal primitive idempotents. The simple right $A$-module which is not annihilated by $e_i$ will be denoted by $E(i)$, its projective cover by $P(i) = P_A(i)$. The simple left $A$-module not annihilated by $e_i$ is denoted by $E^*(i)$. The species of $A$ is, by definition, $\mathcal{S} = \mathcal{S}(A) = (F_i, \{M_i\})_{1 \leq i, j \leq n}$, where $F_i = e_iAe_i/e_iNe_i$, and $M_j = e_iNe_j/e_iN^2e_j$. In our considerations, the total ordering of the index set $\{1, \ldots , n\}$ of the species will usually be of importance, and in order to stress this, we will speak of a labelled species.

We recall that an ideal $J$ of $A$ is called a heredity ideal provided $J^2 = J$, $JNJ = 0$, and the right module $J_A$ (or, equivalently, the left module $_AJ$) is

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projective. And \( A \) is said to be quasi-hereditary provided there exists a chain \( \mathcal{J} = (J_j)_j \) of ideals

\[
0 = J_0 \subset J_1 \subset \ldots \subset J_m = A
\]
such that \( J_{i}/J_{i-1} \) is a heredity ideal of \( A/J_{i-1} \), such a chain will be called a heredity chain of \( A \). Observe that any heredity ideal \( J \) is generated (as an ideal) by an idempotent, and if \( e \) is any idempotent in \( J \), then the ideal \( \langle e \rangle \) generated by \( e \) is a heredity ideal of \( A \), and \( J/\langle e \rangle \) is a heredity ideal of \( A/\langle e \rangle \). It follows that we can refine any heredity chain of \( A \) to a heredity chain \( \mathcal{J} \) such that, in addition, \( J_{i}/J_{i-1} \) is generated by a primitive idempotent, and we call such a heredity chain a saturated one. So, let \( \mathcal{J} \) be a saturated heredity chain of \( A \), and we always assume that the idempotents \( e_i \) are chosen in such a way that \( J_i = \langle e_{n-i+1} + \ldots + e_n \rangle \), for \( 0 \leq i \leq n \). In this way, the quasi-hereditary algebra \( A \) together with the fixed saturated heredity chain determines uniquely \( \mathcal{S}(A) \) as a labelled species. Note that \( \mathcal{S}(A) \) is a species without loops.

Assume that \( A \) is quasi-hereditary, with heredity chain \( \mathcal{J} = (J_i)_i \), where \( J_i = \langle e_{n-i+1} + \ldots + e_n \rangle \). Let \( A_i = A/J_{i-1} \). Note that \( E(i) \) and \( E^*(i) \) are \( A_i \)-modules, and we denote their \( A_i \)-projective covers by \( \Delta(i) = A_i^*(i) \) and \( \Delta^*(i) = A_i^*(i) \), respectively. Since we deal with a quasi-hereditary algebra, it follows that \( J_{i}/J_{i-1} \), as a right \( A \)-module, is the direct sum of copies of \( A(n-i+1) \) (so the modules \( A(i) \) are just those modules which occur as building blocks in the standard filtrations of the projective right \( A \)-modules: the "Verma modules", or "induced modules"). Similarly, \( J_{i}/J_{i-1} \) is, as left \( A \)-module, the direct sum of copies \( A^*(n-i+1) \).

By definition, both \( A(i) \) and \( A^*(i) \) are local \( A \)-modules. In case all the modules \( A(i) \) and \( A^*(i) \), with \( 1 \leq i \leq n \), have Loewy length at most 2, we call \( A \) shallow. Thus, \( A \) is shallow if and only if all the modules \( \text{rad} A(i) \) and \( \text{rad} A^*(i) \) are semisimple. Observe that these modules are actually \( A_{i-1} \)-modules, and we call \( A \) deep provided \( \text{rad} A(i) \) is a projective right \( A_{i-1} \)-module and \( \text{rad} A^*(i) \) is a projective left \( A_{i-1} \)-module, for all \( 1 \leq i \leq n \).

Now, conversely, let \( \mathcal{S} \) be a labelled species without loops, say \( \mathcal{S} = (F_{i}, M_{i})_{i,j \leq n} \), with \( M_{i} = 0 \) for all \( i \). The tensor algebra \( \mathcal{T}(\mathcal{S}) \) can be decomposed as follows. Let \( T = T(n) \) be the set of all sequences \((t_0, t_1, \ldots, t_m)\) where the \( t_i \) are integers with \( 1 \leq t_i \leq n \), and \( m \geq 1 \), such that, moreover, \( t_{i-1} \neq t_i \) for \( 1 \leq i \leq m \). For \( t = (t_0, t_1, \ldots, t_m) \in T \), let

\[
M(t) = M_{t_0} \otimes_{F_{t_1}} M_{t_1} \otimes_{F_{t_2}} \ldots \otimes_{F_{t_{m-1}}} M_{t_{m}},
\]

and for \( T' \subseteq T \), let

\[
M(T') = \bigoplus_{t \in T'} M(t).
\]

Let \( \mathcal{S}_0(\mathcal{S}) = \prod_{i=1}^{n} F_i \) and \( \mathcal{S}_+(\mathcal{S}) = M(T) \), thus \( \mathcal{S}(\mathcal{S}) = \mathcal{S}_0(\mathcal{S}) \oplus \mathcal{S}_+(\mathcal{S}) \).
We are going to define two factor algebras of $\mathcal{F}(\mathcal{P})$ which will turn out to be quasi-hereditary. Both algebras will be of the form $\mathcal{F}(\mathcal{P})/M(T')$ for suitable choices of $T'$. In order to define the first one, we define complementary subsets $U$, $U^0$ of $T$ as follows: Let

$$U = U(n) = \{(t_0, t_1) \in T\} \cup \{(t_0, t_1, t_2) \in T | t_0 < t_1 > t_2\},$$

thus

$$U^0 = \mathcal{F} \setminus U = \{(t_0, t_1, \ldots, t_m) \in T | \text{there is } 0 < i < m \text{ with } t_i < \max(t_{i-1}, t_{i+1})\}.$$

Obviously, $M(U^0)$ is an ideal of $\mathcal{F}(\mathcal{P})$, and

$$(\mathcal{F}_*(\mathcal{P}))^3 \subseteq M(U^0) \subseteq (\mathcal{F}_*(\mathcal{P}))^2,$$

thus $M(U^0)$ is an admissible ideal. We define $S(\mathcal{P}) = T(\mathcal{P})/M(U^0)$. Note that as abelian groups, we can identify $S(\mathcal{P})$ and $\mathcal{R}_0(\mathcal{P}) \oplus M(U)$.

For the second algebra, we define complementary subsets $V$, $V^0$ of $T$ as follows: Let

$$V = V(n) = \{(t_0, \ldots, t_m) \in T | \text{given } i < j \text{ with } t_i = t_j, \text{ there exists } l \text{ with } i < l < j \text{ and } t_i < t_l\},$$

$$V^0 = T \setminus V = \{(t_0, \ldots, t_m) \in T | \text{there are } i < j \text{ with } t_i = t_j \text{ and } t_i < t_l \text{ for all } i < l < j\}.$$

As usual, we may consider a product on $T$ by using the juxtaposition, thus

$$(t_0, \ldots, t_m) \cdot (t'_0, \ldots, t'_m) = (t_0, \ldots, t_m, t'_0, \ldots, t'_m).$$

Of course, for subsets $T'$, $T''$ of $T$, we define $T' \cdot T'' = \{t' \cdot t'' | t' \in T', t'' \in T'' \text{ and } t' \cdot t'' \in T\}$ and so on. Then, obviously, for $n \geq 2$

$$V(n) = V(n-1) \cup V(n-1) \cdot n \cdot V(n-1) \cup V(n-1) \cdot n \cdot V(n-1).$$

By induction on $n$, we see that $V(n)$ is finite. In particular, the sequences $(t_0, \ldots, t_m) \in V(n)$ are of bounded length, say $m \leq v(n)$ for some $v(n)$. Thus

$$(\mathcal{F}_*(\mathcal{P}))^{v(n)+1} \subseteq M(V^0) \subseteq (\mathcal{F}_*(\mathcal{P}))^2,$$

so that $M(V^0)$ is an admissible ideal. We define $D(\mathcal{P}) = \mathcal{F}(\mathcal{P})/M(V^0)$, and note that $D(\mathcal{P})$ can be identified, as an abelian group, with $\mathcal{R}_0(\mathcal{P}) \oplus M(V)$.

**Theorem 1.** Let $\mathcal{P}$ be a labelled species without loops. The rings $S(\mathcal{P})$ and $D(\mathcal{P})$ are quasi-hereditary, with labelled species $\mathcal{P}$. The ring $S(\mathcal{P})$ is shallow, the ring $D(\mathcal{P})$ is deep.

In particular, we see that the nonexistence of loops is the only condition on a species for being realizable as the species of a quasi-hereditary ring.
Let \( k \) be a (commutative) field. In case \( \mathcal{S} \) is a finite-dimensional \( k \)-species, labelled and without loops, we denote by \( s_k(\mathcal{S}) \) and \( d_k(\mathcal{S}) \) the \( k \)-dimension of \( S(\mathcal{S}) \) and \( D(\mathcal{S}) \), respectively. We are going to formulate an estimate for the Cartan invariants of a quasi-hereditary algebra \( A \) in terms of the Cartan invariants of the corresponding algebras \( S(\mathcal{S}) \) and \( D(\mathcal{S}) \). In this way, we deduce that the dimension of \( A \) is bounded from below by \( s_k(\mathcal{S}) \) and from above by \( d_k(\mathcal{S}) \).

**Theorem 2.** Let \( A \) be a basic, finite-dimensional \( k \)-algebra which is quasi-hereditary with labelled species \( \mathcal{S} \). Then, for any \( i, j \)

\[
\dim_k(e_i S(\mathcal{S}) e_j) \leq \dim_k(e_i A e_j) \leq \dim_k(e_i D(\mathcal{S}) e_j).
\]

In particular,

\[
s_k(\mathcal{S}) \leq \dim_k A \leq d_k(\mathcal{S}).
\]

We have \( s_k(\mathcal{S}) = \dim_k A \) if and only if \( A \) is shallow, and \( d_k(\mathcal{S}) = \dim_k A \) if and only if \( A \) is deep.

The proof of Theorem 1 is given in Section 2, the proof of Theorem 2 in Section 3. We add examples showing that besides the algebras \( S(\mathcal{S}) \) and \( D(\mathcal{S}) \), there are other shallow or deep algebras. A detailed study of the ring-theoretical and homological properties of quasi-hereditary rings which are shallow or deep will be given in a subsequent publication.

2. The rings \( S(\mathcal{S}) \) and \( D(\mathcal{S}) \)

The aim of this section is a proof of Theorem 1. Thus, let \( \mathcal{S} \) be a labelled species without loops, with index set \([1, \ldots, n]\). The proof is by induction on \( n \).

If \( n = 1 \), then \( S(\mathcal{S}) = D(\mathcal{S}) = F_1 \), thus quasi-hereditary (and trivially both shallow and deep). Thus, let \( n \geq 2 \), and let \( \mathcal{S}' \) be the restriction of \( \mathcal{S} \) to \([1, \ldots, n-1]\).

Consider first \( S(\mathcal{S}) \). Given \( m \in \mathbb{N} \), let \([1, m] = \{i \in \mathbb{N} \mid 1 \leq i \leq m\} \). Then

\[
S(\mathcal{S}) e_n = F_n \oplus M([1, n-1] \cdot n),
\]

\[
e_n S(\mathcal{S}) = F_n \oplus M(n \cdot [1, n-1]),
\]

\[
\langle e_n \rangle = F_n \oplus M(\{ t \in U \mid t_i = n \text{ for some } i \})
= F_n \oplus M([1, n-1] \cdot n \cap [1, n-1] \cup [1, n-1] \cdot n \cdot [1, n-1])
= (F_n \oplus M([1, n-1] \cdot n)) \otimes_{F_n} F_n(\mathcal{S} \cdot M([1, n-1]))
= S(\mathcal{S}) e_n \otimes_{F_n} e_n S(\mathcal{S}).
\]

In particular, \( e_n S(\mathcal{S}) e_n = F_n \), and the equalities above show that \( \langle e_n \rangle \) is a heredity ideal. Of course, \( \text{rad} \Delta(n) = M(n \cdot [1, n-1]) \) is a semisimple right
module, \( \text{rad}\ A^*(n) = M([1, n-1] \cdot n) \) is a semisimple left module. Since \( S(\mathcal{S})/\langle e_n \rangle = S(\mathcal{S}') \), we use induction and conclude that \( S(\mathcal{S}) \) is a shallow quasi-hereditary ring.

Next, we consider \( D(\mathcal{S}) \). We have
\[
D(\mathcal{S})e_n = F_n \oplus M(V(n-1) \cdot n),
\]
\[
e_n D(\mathcal{S}) = F_n \oplus M(n \cdot V(n-1)),
\]
\[
\langle e_n \rangle = F_n \oplus M(V(n-1) \cdot n \cup n \cdot V(n-1) \cup V(n-1) \cdot n \cdot V(n-1))
\]
\[
= (F_n \oplus M(V(n-1) \cdot n)) \otimes_{F_n} (F_n \oplus M(n \cdot V(n-1)))
\]
\[
= D(\mathcal{S})e_n \otimes_{F_n} e_n D(\mathcal{S}),
\]
so that \( e_n D(\mathcal{S})e_n = F_n \), and \( \langle e_n \rangle \) is a heredity ideal. Since \( D(\mathcal{S})/\langle e_n \rangle = D(\mathcal{S}') \), it follows by induction that \( D(\mathcal{S}) \) is quasi-hereditary. Now
\[
\text{rad}\ A(n) = M(n \cdot V(n-1)) = \bigoplus_{i=1}^{n-1} M_i \otimes_{F_i} P_{D(\mathcal{S})}(i),
\]
thus \( A(n) \) is a projective right \( D(\mathcal{S}') \)-module. Similarly, \( \text{rad}\ A^*(n) \) is a projective left \( D(\mathcal{S}') \)-module. By induction, it follows that \( D(\mathcal{S}) \) is deep.

3. Quasi-hereditary \( k \)-algebras

Let \( k \) be a field, and \( A \) a basic finite-dimensional quasi-hereditary \( k \)-algebra with labelled species \( \mathcal{S} \). Let \( \{1, \ldots, n\} \) be the index set of \( \mathcal{S} \). Note that \( e_n A e_n = F_n \), and, in the same way, \( e_n S(\mathcal{S}) e_n = e_n D(\mathcal{S}) e_n = F_n \). In particular, for the proof of the dimension inequalities, we may assume \( n \geq 2 \). Let \( \mathcal{S}' \) be the restriction of \( \mathcal{S} \) to \( \{1, \ldots, n-1\} \); clearly, this is the labelled species for \( B = A/\langle e_n \rangle \). By induction, we know that
\[
\dim_k(e_n S(\mathcal{S}) e_j) \leq \dim_k(e_n B e_j) \leq \dim_k(e_n D(\mathcal{S}) e_j),
\]
for all \( i, j \leq n-1 \).

First, consider \( e_n A e_j \), with \( 1 \leq j \leq n-1 \). Let \( X = \bigoplus_{i=1}^{n-1} e_n A e_j \), thus \( X \) is the radical of the right \( A \)-module \( e_n A \); this is a \( B \)-module with top \( X = \bigoplus_{i=1}^{n-1} M_i \). Let \( d_i = \dim_k(M_i) \). We denote by \( P \) the \( B \)-projective cover of \( X \), thus \( P \) is the direct sum of \( d_i \) copies of \( e_i B \), for \( 1 \leq i \leq n-1 \). The epimorphisms \( P \to X \to X e_j \) yield epimorphisms \( P e_j \to X e_j \to X e_j \). Now, \( X e_j = _n M_j \), \( X e_j = e_n A e_j \), and \( P e_j = \bigoplus_{i=1}^{n-1} (e_i B)e_j \), thus
\[
\dim_k(_n M_j) \leq \dim_k(e_n A e_j) \leq \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i B e_j).
\]
However, \( e_n S(\mathcal{S}) e_j = _n M_j \), so the left-hand term is the desired one. Now,
\[ \text{rad}(e_n D(S)_{D(S)}) = \bigoplus_{i=1}^{n-1} (e_i D(S'))_{D(S)}^d. \]

It follows that \( e_n D(S)e_j = \bigoplus_{i=1}^{n-1} (e_i D(S')e_j)^d \), and therefore
\[ \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i Be_j) \leq \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i D(S')e_j) = \dim_k(e_n D(S)e_j). \]

This finishes the proof for \( e_i Ae_j \). The dual proof yields the similar inequality for \( e_j Ae_n \), where \( 1 \leq j \leq n-1 \).

It remains to consider \( e_i Ae_j \), where \( 1 \leq i, j \leq n-1 \). Since \( \langle e_n \rangle = A e_n \otimes_{F_n} e_n A \), there is the exact sequence
\[ 0 \to e_i Ae_n \otimes_{F_n} e_n Ae_j \to e_i Ae_j \to e_i Be_j \to 0, \]
and similar ones for \( S(S) \) and \( D(S) \), namely
\[ 0 \to e_i S(S)e_j \to e_i S(S)e_j \to e_i S(S)e_j \to 0, \]
\[ 0 \to e_i D(S)e_j \to e_i D(S)e_j \to e_i D(S)e_j \to 0. \]

The desired inequalities follow from the inequalities for \( e_i Ae_n \), \( e_n Ae_j \), and \( e_i Be_j \), by taking into account that for a right \( F_n \)-space \( X \) and a left \( F_n \)-space \( Y \), we have
\[ \dim_k X \otimes_{F_n} Y = \frac{1}{\dim_k F_n} \dim_k X \cdot \dim_k Y. \]

This finishes the proof of the first part of Theorem 2.

Now assume that \( A \) is shallow. By induction, we know that \( \dim_k(e_i S(S')e_j) = \dim_k(e_i Be_j) \), for \( i, j \leq n-1 \). Since \( X = X' \), we have \( e_n S(S)e_j = M_j = e_n Ae_j \), for \( j \leq n-1 \), and similarly \( e_j S(S)e_n = e_j Ae_n \) for \( j \leq n-1 \). It follows that \( \dim_k(e_i S(S)e_j) = \dim_k(e_i Ae_j) \), for all \( i, j \).

Similarly, if we assume that \( A \) is deep, then, by induction, \( \dim_k(e_i Be_j) = \dim_k(e_i D(S')e_j) \), for \( i, j \leq n-1 \). On the other hand, we have in this case \( X = P \), thus \( e_n Ae_j = \bigoplus_{i=1}^{n-1} (e_i Be_j)^d \), and therefore
\[ \dim_k(e_n Ae_j) = \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i Be_j) = \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i D(S')e_j) = \dim_k(e_n D(S)e_j). \]

It follows that \( \dim_k(e_i Ae_j) = \dim_k(e_i D(S)e_j) \).

Note that \( \dim_k A = \sum_{i, j} \dim_k(e_i Ae_j) \), thus always \( s_k(S) \leq \dim_k A \leq d_k(S) \).

Let us first assume \( s_k(S) = \dim_k A \), thus \( \dim_k(e_i Ae_j) = \dim_k(e_i S(S)e_j) \), for all \( i, j \). If \( i, j \leq n-1 \), a proper inequality \( \dim_k(e_i S(S)e_j) < \dim_k(e_i Be_j) \) would yield that \( \dim_k(e_i S(S)e_j) < \dim_k(e_i Ae_j) \) for the same pair \( i, j \) of indices, since
\[ \dim_k(e_i Ae_j) - \dim_k(e_i S(S)e_j) = \dim_k(e_i Be_j) - \dim_k(e_i S(S')e_j) = a, \]
with
\[ a = \dim_k (e_i A e_n \otimes_{F_n} e_n A e_j) - \dim_k (e_i S(\mathcal{S}) e_n \otimes_{F_n} e_n S(\mathcal{S}) e_j) \geq 0. \]
Thus \( s_k(\mathcal{S}') = \dim_k B \), and \( B \) is shallow by induction. On the other hand, \( \dim_k (e_n S(\mathcal{S}) e_j) = \dim_k (e_n A e_j) \) implies that \( X e_j = \bar{X} e_j \), for all \( 1 \leq j < n \), and therefore \( X = \bar{X} \) is semisimple. This shows that the right \( A \)-module \( e_n A \) has Loewy length at most 2. Similarly, the left \( A \)-module \( A e_n \) has Loewy length at most 2. As a consequence, \( A \) is shallow.

In the same way, we proceed in case \( \dim_k A = d_k(\mathcal{S}) \). We see immediately that \( \dim_k (e_i A e_j) = \dim_k (e_i D(\mathcal{S}) e_j) \), for all \( i, j \), and conclude that \( \dim_k B = d_k(\mathcal{S}') \). Thus \( B \) is deep by induction. On the other hand, \( \dim_k (e_n A e_j) = \dim_k (e_n D(\mathcal{S}) e_j) \) implies that \( P e_j = X e_j \), for all \( 1 \leq j \leq n-1 \), and therefore \( X = P \) is a projective right \( B \)-module. Similarly, the radical of the left \( A \)-module \( A e_n \) is projective as a left \( B \)-module. Thus \( A \) is deep.

4. Examples

The bounds \( s_k(\mathcal{S}) \leq \dim_k A \leq d_k(\mathcal{S}) \) are optimal, but we should remark that usually \( d_k(\mathcal{S}) - s_k(\mathcal{S}) \) may be rather large. As an example, consider the \( k \)-species \( \mathcal{S}_n = (F, i M_j)_{1 \leq i, j \leq n} \) with \( F_i = k \) and \( i M_j = 0 \) for all \( i \), whereas \( j M_j = k \) for all \( i \neq j \); thus \( T(\mathcal{S}_n) \) is the path algebra for the quiver with \( n \) vertices, a unique arrow \( i \to j \) for \( i \neq j \), and no loops. We are going to exhibit \( s(n) := s_k(\mathcal{S}_n) \) and \( d(n) := d_k(\mathcal{S}_n) \). It suffices to calculate the cardinalities of the index sets \( U(n) \) and \( V(n) \), since
\[ s(n) = n + U(n), \quad d(n) = n + V(n). \]
Clearly, \( |U(1)| = 0 = |V(1)| \). For \( n \geq 2 \), we have
\[ U(n) = U(n-1) \cup [1, n-1] \cdot n \cup n \cdot [1, n-1] \cup [1, n-1] \cdot n \cdot [1, n-1], \]
thus
\[ |U(n)| = |U(n-1)| + 2(n-1) + (n-1)^2 = |U(n-1)| + n^2 - 1, \]
and consequently,
\[ |U(n)| = -n + \sum_{i=1}^n t^2 = -n + \frac{1}{6} n(n+1)(2n+1). \]
Similarly, from
\[ V(n) = V(n-1) \cup V(n-1) \cdot n \cup n \cdot V(n-1) \cup V(n-1) \cdot n \cdot V(n-1) \]
for \( n \geq 2 \), we obtain
\[ |V(n)| = 3|V(n-1)| + |V(n-1)|^2. \]
It follows that \( s(n) = \frac{1}{6} (n+1)(2n+1) \), and that \( d(n) \) is given recursively by
\( d(1) = 1 \), and \( d(n) = d(n-1) + (d(n-1) + 1)^2 \) for \( n \geq 2 \). The first values for \( s(n) \) and \( d(n) \) are the following:

\[
\begin{align*}
  s(1) &= 1, & d(1) &= 1, \\
  s(2) &= 5, & d(2) &= 5, \\
  s(3) &= 14, & d(3) &= 41, \\
  s(4) &= 30, & d(4) &= 1805, \\
  s(5) &= 55, & d(5) &= 3263441.
\end{align*}
\]

Let \( \mathcal{S} \) be a labelled species without loops. Let us assume that there are even no oriented cycles. Then \( D(\mathcal{S}) \) is the tensor algebra of \( \mathcal{S} \). In particular, if \( \mathcal{S} \) is, in addition, a finite-dimensional \( k \)-algebra where \( k \) is a perfect field, then \( D(\mathcal{S}) \) is the only deep quasi-hereditary algebra with species \( \mathcal{S} \). If the labelling is chosen in such a way that \( M_j = 0 \) for \( i > j \), then \( S(\mathcal{S}) = T(\mathcal{S}) / T_1(\mathcal{S})^2 \), so again \( S(\mathcal{S}) \) is the only shallow quasi-hereditary algebra with labelled species \( \mathcal{S} \). Of course, in general there may be shallow rings which are not of the form \( S(\mathcal{S}) \), the first example is the path algebra of the quiver of Fig. 1 with the commutativity relation.

![Fig. 1](image1)

For a labelled species \( \mathcal{S} \) without loops but with oriented cycles there usually also will exist deep rings which are not of the form \( D(\mathcal{S}) \). For example, consider the algebra \( A \) given by the quiver of Fig. 2 with relations \( \beta \alpha - \gamma \delta = 0 \)

![Fig. 2](image2)

For a labelled species \( \mathcal{S} \) without loops but with oriented cycles there usually also will exist deep rings which are not of the form \( D(\mathcal{S}) \). For example, consider the algebra \( A \) given by the quiver of Fig. 2 with relations \( \beta \alpha - \gamma \delta = 0 \)

![Fig. 3](image3)
and $\delta \gamma = 0$. The labelled species corresponding to this quiver will be denoted by $\mathcal{S}$. Then $A$ is deep with labelled species $\mathcal{S}$, but not isomorphic to $D(\mathcal{S})$.

Also, we should remark that there are quasi-hereditary algebras $A$ with radical $N$ such that no ideal $I \subseteq N^2$ yields a shallow algebra $A/I$. A typical example is the algebra $A$ given by the quiver of Fig. 3 with the commutativity relation. Note that $A$ has a unique minimal nonzero ideal $J$. An ideal $I$ with $A/I$ shallow must contain $J$, but there is no ideal $I$ with $J \subseteq I \subseteq N^2$ such that $A/I$ is quasi-hereditary with respect to the given ordering of the vertices.

References