HYPERSURFACES WITH SINGULAR LOCUS A PLANE CURVE AND TRANSVERSAL TYPE $A_1$

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1. Introduction

In [Si-1] we studied functions $f: (C^{n+1}, 0) \rightarrow (C, 0)$ where the singular locus $\Sigma$ was a smooth line and with transversal singularities on $\Sigma - \{0\}$ of type $A_1$. We called those singularities isolated line singularities. In this paper we generalize this to the case, where $\Sigma$ is a plane curve in $C^{n+1}$.

We study the topology of the Milnor fibre with the help of a generic approximation, which has $\Sigma$ as part of the critical locus, and where only special types of singularities are allowed:

(a) $A_1$-points; local formula $w_0^2 + \ldots + w_n^2$;
(b) $A_\infty$-points; local formula $w_1^2 + \ldots + w_n^2$;
(c) $D_x$-points; local formula $w_0 w_1^2 + w_2^2 + \ldots + w_n^2$;
(d) central type; local formula $u \cdot g^2 + w_2^2 + \ldots + w_n^2$ where $g(x, y) = 0$ is a reduced equation of the plane curve $\Sigma$ and $u$ is a unit.

The existence of the deformations follows from work of Pellikaan [Pe].

The homotopy type of the local Milnor fibres of the above elementary types are as follows:

(a) $A_1$-points: $S^n$;
(b) $A_\infty$-points: $S^{n-1}$;
(c) $D_\infty$-points: $S^n$;
(d) central type: $S^{n-1} \vee S^n \vee \ldots \vee S^n$.

The $A_\infty$-points occur in 1-dimensional bundles along the critical set $\Sigma$.

By methods similar to Lê (cf. [Br]) in the isolated singularity case, we construct the Milnor fibre of $f$ by gluing together the local contributions. Our main result is:

Theorem 3.11. Let $\Sigma$ be a plane curve and $f: (C^{n+1}, 0) \rightarrow (C, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ and transversal type $A_1$ on
\[ \Sigma - \{0\} \text{ and let } *D_{\infty} > 0, \text{ then the homotopy type of the Milnor fibre } F \text{ of } f \text{ is a bouquet of } \mu_n(f) \text{ n-spheres, where} \]
\[ \mu_n(f) = 2\mu(\Sigma) + *A_{1} + 2* D_{\infty} - 1; \]

\[ \mu(\Sigma) = \text{Milnor number of } \Sigma, \quad *D_{\infty} = \text{number of } D_{\infty}\text{-points in the generic approximation with } \Sigma \text{ fixed}, \quad *A_{1} = \text{number of } A_{1}\text{-points in the generic approximation with } \Sigma \text{ fixed}. \]

The proof is similar to [Si-1].

The method of the proof gives no result in the case \( *D_{\infty} = 0 \). For this case and the more general case that \( \Sigma \) is a 1-dimensional complete intersection singularity (cis) and transversal type \( A_{1} \) on \( \Sigma - \{0\} \) we refer to [Si-2]. In that paper we first compute the homology of the Milnor fibre in terms of \( \mu(\Sigma), *D_{\infty} \) and \( *A_{1} \). From the homology and additional information about the fundamental group from [LeSa] we can determine the homotopy type of the Milnor fibre, which is as follows:

\[ S^n \vee \ldots \vee S^n \quad \text{if } *D_{\infty} > 0, \]
\[ S^{n-1} \vee S^n \vee \ldots \vee S^n \quad \text{if } *D_{\infty} = 0. \]

As general references for singularities of functions \( C^{n+1} \rightarrow C \) we mention the book of Arnol’d–Gusein Zade–Varchenko [Ar]. For the topology of singularities we refer to [Mi] and [Lo]. For non-isolated singularities, see also [Le] and [Yo].

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2. Generic approximations of the function

2.1. Let \( \Sigma \) be a 1-dimensional complete intersection with isolated singularity at \( 0 \in C^{n+1} \). We consider \( f: (C^{n+1}, 0) \rightarrow (C, 0) \) a holomorphic function germ with critical locus \( \Sigma(f) = \Sigma \). This situation is treated (in more generality) in the thesis of Pellikaan [Pe]. On every branch of \( \Sigma(f) \) there is if \( z \neq 0 \) a well-defined transversal singularity type.

Let \( g_1, \ldots, g_n \) define the complete intersection \( \Sigma \) as a reduced algebraic set and let \( I = (g_1, \ldots, g_n) \). Then we have:

\[ f \text{ is singular on } \Sigma \iff f \in I^2 \quad ([Pe] 11.6). \]

In this case we can write \( f = \sum h_{ij} g_i g_j \) with \( h_{ij} \in C_{n+1} \) and \( h_{ij} h_{ji} = h_{ji} \). On \( I \) and \( I^2 \) acts the subgroup \( \mathcal{G} \) of \( \mathcal{G} \) defined by \( \mathcal{G} = \{ \varphi \in \mathcal{G} | \varphi \ast (I) \subset I \} \). Let \( \tau_\Sigma(f) \)
be the tangent space to the \( \mathcal{O}_X \)-orbit and \( J_f = \left( \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right) \) the Jacobian ideal of \( f \).

Define
\[
j(f) = \dim \mathcal{O}_X \frac{I}{J_f} \quad \text{and} \quad c(f) = \dim \mathcal{O}_X \frac{I^2}{\tau_2(f)},
\]
the Jacobi number and the codimension.

2.2. **Proposition ([Pe]).** Equivalent are:
(a) \( c(f) < \infty \);
(b) \( j(f) < \infty \);
(c) the transversal type of \( f \) along \( \Sigma - \{0\} \) is \( A_1 \).

Moreover if \( c(f) < \infty \) then \( f \) is finitely determined inside \( I^2 \).

2.3. We specialize now to the case that \( \Sigma \) is a plane curve. We can choose coordinates
\[(z_0, z_1, \ldots, z_n) = (x, y, z_2, \ldots, z_n)\]
such that \( \Sigma \) is given by
\[
g_1 = g(x, y) = 0,
\[
g_2 = z_2 = 0, \ldots, g_n = z_n = 0.
\]
So \( I = (g, z_2, \ldots, z_n) \).

If \( f \) is \( \mathcal{O}_X \)-equivalent with \( u \cdot g^2 + z_2^2 + \ldots + z_n^2 \) then \( f \) is called of central singularity type (\( u \) unit in \( \mathcal{O}_{(x,y)} \)). According to the splitting lemma [Gr–Me] we can suppose in general:
\[
f(x, y, z) = f'(x, y, z_2, \ldots, z_n) + z_2^2 + \ldots + z_n^2
\]
with \( f' \in (x, y, z_1, \ldots, z_p)^3 \).

2.4. Consider the following deformation of \( f \) with fixed critical locus (cf. [Pe] (7.18))
\[
f_s(z) = f(x, y, z) + \sum_{k,l} a_{k,l} y_k g_l + \sum_{i,k} b_{ik} z_i g_k,
\]
where we choose the matrix \( (a_{k,l}) \) in diagonal form with diagonal elements \( (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0) \) and \( s \in S = \{(\lambda_1, \ldots, \lambda_p), (b_{ik})\} \).

**Proposition.** There exists a dense open subset \( V \) of \( S \) and an open neighbourhood \( U \) of \( 0 \) in \( \mathbb{C}^{n+1} \) such that for all \( s \in V \) sufficiently small
(i) \( f_s \) has only \( A_1 \)-singularities in \( U \setminus \Sigma \);
(ii) \( f_s \) has only \( A_\infty \)- and \( D_\infty \)-singularities on \( U \cap \Sigma - \{0\} \);
(iii) \( f \) has the central singularity type in 0;
(iv) \( j_f = \# \{A_1\text{-points of } f \text{ on } U \setminus \Sigma \} + \# \{D_\infty\text{-points of } f \text{ on } U \cap \Sigma \} + j_{f_{\nu,0}} \)
where \( j_{f_{\nu,0}} \) is the Jacobi number of the central singularity.

Proof. Almost all the assertions are shown by Pellikaan ([Pe], I (7.18)). The special choice of the matrix \((a_{ij})\) doesn’t influence his proof. A computation shows that the 2-jet-extension

\[ j^2 F : C^{n+1} \times S \to J^2_{(0)}(C^{n+1}, C) \]

is transversal to the \( A_1 \)-stratum outside \( \Sigma \) and to the \( D_\infty \)-stratum on \( \Sigma - \{0\} \).
(For details cf. [Pe]). The assertions (i) and (ii) follow as an application of Sard’s theorem.

For (iii) a more careful analysis of Pellikaan’s proof is necessary. Write \( f = h_{ij} g_i g_j \) where

\[ (h_{ij}) = \begin{pmatrix} * & 0 \\ 0 & I_{n-r} \end{pmatrix}. \]

Since the 2-jet of \( f \) is equal to \( z_{r+1}^2 + \ldots + z_n^2 \) this implies that \( h_{ij} \in m \) if \( 2 \leq i \leq r \) and \( 2 \leq j \leq r \) (\( m \) is the maximal ideal).

Let \( f = \sum H_{ij} g_i g_j \) where

\[ H_{ij} = h_{ij} + a_{ij} + \sum z_i b_i \delta_{ij} = h_{ij} + (\lambda_i + \sum z_i b_i) \delta_{ij}. \]

Remark that:

\[ H_{ij} = h_{ij} \quad \text{for } i \neq j \]

and

\[ H_{ii} = h_i + \lambda_i + \sum z_i b_i \]

so

\[ H_{ii} \equiv \lambda_i \pmod{m} \quad \text{if } 2 \leq i \leq r, \]

\[ H_{ii} = 0 \quad \text{if } i \geq r + 1, \]

\[ H_{ij} \equiv 0 \pmod{m} \quad \text{if } i \geq 2 \text{ and } j \geq 2. \]

For \( \lambda_i \) sufficiently general we can suppose that \( H_{ii} \) are units. Moreover we can suppose: \( \det H_{ij} \) is invertible (this corresponds to Pellikaan’s statement \( \delta_{f_{\nu,0}} = 0 \)).

We next transform the matrix \((H_{ij})\) into a normal form with standard technics from quadratic forms. We first treat \( z_2 \). Set

\[ \bar{z}_2 := z_2 + \frac{H_{23}}{H_{22}} z_3 + \ldots + \frac{H_{2n}}{H_{22}} z_n + \frac{H_{1n}}{H_{22}} g. \]
which defines a coordinate transformation on \((\mathbb{C}^{n+1}, 0)\). Remark \(\tilde{z}_2 \equiv z_2 \pmod{m}\). Also \(\text{mod } m\) we have:

\[
(H_{ij}) \equiv \begin{pmatrix}
\ast & \ast & \ast \\
\ast & \lambda_2 & 0 \\
\ast & 0 & \lambda_n
\end{pmatrix}
\]

The form of this matrix and the \(\lambda_2, \ldots, \lambda_n\) are not changed by this coordinate transformation (nb. \(\lambda_{r+1} = \ldots = \lambda_n = 1\)).

We treat \(z_3, \ldots, z_n\) in the same way and get

\[
f_s = H_{11}^* g^2 + \lambda_2 \tilde{z}_2^2 + \ldots + \lambda_n \tilde{z}_n^2.
\]

Since \(\tilde{z}_k \equiv z_k \pmod{m}\) we have \((\tilde{z}_2, \ldots, \tilde{z}_n, g) = (z_2, \ldots, z_n, g)\). Moreover \(\det(H_{ij})\) remains invertible. So \(H_{11}^*\) is a unit in \(\mathcal{O}_{n+1}\). Since \(f_s\) is finitely determined we can change coordinates again (by completing squares) such that \(H_{11}^*\) is a function of \(x\) and \(y\) only. So \(f_s\) is right-equivalent to

\[
u \cdot g^2 + \tilde{z}_2^2 + \ldots + \tilde{z}_n^2 \quad (\text{u unit in } \mathcal{O}_{(x,y)}).
\]

2.5. Remark. It can happen that \(\# D_\infty = 0\). In the case of isolated line singularities, this only happens for type \(A_\infty\).

C. Cox showed me the examples

\[
f = xyz + z^p \quad (p \geq 2).
\]

The critical locus is the union of the \(x\)-axis and the \(y\)-axis, the transversal type is \(A_1\). The deformation

\[
f_s = xyz + z^p + sz^2
\]

has the properties:

\[
\# D_\infty = 0, \quad \# A_1 = p - 2,
\]

and has central type for \(s \neq 0\).

Also \(f = z \cdot g + z^p \ (p \geq 2)\), where \(g = 0\) is a plane curve, has the property \(\# D_\infty = 0\).

Pellikaan [Pe] showed in Lemma 1.7.17, that if \(\Sigma\) is a reduced 1-dimensional complete intersection, defined by the ideal \(I\) and \(\# D_\infty = 0\), then there exist generators \(g_1, \ldots, g_n\) of \(I\) such that \(f\) is equivalent to \(g_1^2 + \ldots + g_n^2\). This shows that in general there are plenty possible \(f\) with \(\# D_\infty = 0\). At the other hand it is not difficult to degenerate such \(f\) to \(\xi \cdot g_1^2 + \ldots + g_n^2 \ (\xi \in m)\) with \(\# D_\infty > 0\).
2.6. Remark. It is in principle possible to classify the non-isolated singularities of this paper in the same way as isolated singularities. Proposition 2.2 makes this possible.

For the case that \( \Sigma \) is a smooth line, we refer to [Si-1].

In the case that \( \Sigma \) is of type \( A_1 \):

\[
g(x, y) = xy = 0, \quad z = 0
\]

the beginning of a list of singularities is as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>( f )</th>
<th>corank</th>
<th>( j_f )</th>
<th>( c_f )</th>
<th>( *A )</th>
<th>( *D_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{x,y, z} )</td>
<td>( x^2 y^2 )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( T_{x,y, z} )</td>
<td>( x^2 y^2 (y + x^n) ) (( n \geq 1 ))</td>
<td>2</td>
<td>2n + 2</td>
<td>n</td>
<td>n</td>
<td>n + 1</td>
</tr>
<tr>
<td>( T_{x, x, y} )</td>
<td>( xyz + z' ) (( r \geq 3 ))</td>
<td>3</td>
<td>( r - 1 )</td>
<td>( r - 2 )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( T_{x, x, y} )</td>
<td>( x^2 z + y^2 + x^2 y^2 )</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The list contains all simple singularities and all singularities with \( *D_\infty = 0 \).

2.7. We next are interested in the Milnor fibres. Let \( f : (C^{n+1}, 0) \to (C, 0) \) and let \( \varepsilon_0 \) be an admissible radius for the Milnor fibration, that is \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \) holds

\[
f^{-1}(0) \nsubseteq \partial B_\varepsilon \quad \text{(as a stratified set)}.
\]

For each admissible \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that

\[
f^{-1}(t) \nsubseteq \partial B_\varepsilon \quad \text{for all} \quad 0 < t \leq \delta(\varepsilon).
\]

We fix now \( \varepsilon \leq \varepsilon_0 \) and consider \( 0 < \delta \leq \delta(\varepsilon) \) and take the representative

\[
f : X_\Delta = f^{-1}(\Delta) \cap B_\varepsilon \to \Delta
\]

where \( \Delta \) is a disc of radius \( \delta \).

**Lemma.** Let \( f_s \) be as above. Consider the restriction

\[
f_s : X_{\Delta, s} = f_s^{-1}(\Delta) \cap B_\varepsilon \to \Delta.
\]

For \( s \in S \) and \( \delta > 0 \) sufficiently small we have:

1. \( f_s^{-1}(t) \nsubseteq \partial B_\varepsilon \) for all \( t \in \Delta \);
2. above the boundary circles \( \partial \Delta \) the fibrations induced by \( f \) and \( f_s \) are equivalent;
3. \( X_\Delta \) and \( X_{\Delta, s} \) are homeomorphic.

Proof, cf. [Si-2]. \( \square \)

3. The homotopy type of the Milnor fibre

3.1. From now on we choose \( s \) such that \( f_s : X_{\Delta, s} \to \Delta \) satisfies the conditions of Proposition 2.4 and Lemma 2.7.
We omit the suffix $s$ and write again

$$f: X_\Delta \to \Delta.$$ 

The critical set of $f$ consists of

(a) The 1-dimensional cis $\Sigma$, where local singularities are $A_\infty$, $D_\infty$ or the central type.

(b) isolated points $\{c_1, \ldots, c_\sigma\}$ where the local singularity is of type $A_1$.

We can suppose that all critical values of $f$ are different (this is mostly for notational convenience). The critical value $0$ corresponds to the non-isolated singularities on $\Sigma$. We follow now the construction in [Si-1].

3.2. Define $B_0, B_1, \ldots, B_\sigma$ disjoint $(2n+2)$-balls around $c_0 = 0$, $c_0 = 0, c_1, \ldots, c_\sigma$ and inside $B = B_\varepsilon$. Let $D_0, \ldots, D_\sigma$ be disjoint 2-discs around $f(c_0), \ldots, f(c_\sigma)$ and inside $D = D_\varepsilon$ chosen in such a way that we get locally

$$f: B_i \cap f^{-1}(D_i) \to D_i$$

which are Milnor fibrations above $D_i - \{f(c_i)\}$.

Let $\Sigma^* = \Sigma - B_0$. The number of topological components of $\Sigma^*$ is equal to the number of irreducible branches of $\Sigma$. Each branch $\Sigma^*_k (k = 1, \ldots, r)$ is a disc with one hole.

3.3. We want to construct a nice tube neighbourhood of $\Sigma^*$. To do this we consider the map $w: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ defined by

$$w_1 = g,$$

$$w_2 = z_2,$$

$$\ldots \ldots \ldots$$

$$w_n = z_n.$$

Let

$$r(z) = |w_1(z)|^2 + \ldots + |w_n(z)|^2.$$
Remark that
\[ r^{-1}(0) = w^{-1}(0) = \Sigma. \]

For \( \epsilon^* > 0 \) sufficiently small, define:
\[ T = \{ z \in B \setminus B_0 \mid r(z) \leq \epsilon^* \}, \]
\[ \partial T = \{ z \in B \setminus B_0 \mid r(z) = \epsilon^* \}. \]

\( T \) and \( \partial T \) have topological components, which we denote by \( T_1, \ldots, T_r \) and \( \partial T_1, \ldots, \partial T_r \), where the numbering corresponds to the branches \( \Sigma^*_1, \ldots, \Sigma^*_r \).

3.4. Lemma. \( T = \Sigma^* \times Q^n \) where \( Q^n \) is a closed \( n \)-ball in \( C^n \).

Proof. The lemma follows from the Ehresmann fibration theorem since \( f \) is submersive on \( T \) and so are its restrictions to \( T \cap \partial B_0 \) and \( T \cap \partial B_0 \).

3.5. Lemma. (a) There exist \( \epsilon^* \) such that for all \( 0 < \epsilon \leq \epsilon^* \)
\[ f^{-1}(0) \include \partial B_0 \quad \text{and} \quad f^{-1}(0) \include \partial T. \]
(b) For every \( 0 < \epsilon \leq \epsilon^* \) there exist a \( \tau = \tau(\epsilon) \) such that for all \( 0 < |t| \leq \tau \)
\[ f^{-1}(t) \include \partial B_0 \quad \text{and} \quad f^{-1}(t) \include \partial T. \]

Proof. Application of the curve selection lemma and the openness of the transversality conditions.

3.6. Along \( \Sigma \) we have 3 types of singularities: \( A_\infty, D_\infty \) and central type. In each case we consider the pair, consisting of the Milnor fibre of \( f \) and the Milnor fibre of the restriction of \( f \) to a nearby slice transversal to \( \Sigma \). The topology of these pairs can be described as follows:
\( A_\infty \): Milnor pair \( h \simeq (S^{n-1}, S^{n-1}) \),
\( D_\infty \): Milnor pair \( h \simeq (S^n, S^n) \),
central: Milnor pair \( h \simeq (S^{n-1} \vee S^n \vee \ldots \vee S^n, S^{n-1}) \).

The first two cases are treated in [Si-1], for the central type we refer to the next proposition.

3.7. Proposition. Let \( f \) be of central type
\[ f = u \cdot g^2 + z_2^2 + \ldots + z_n^2, \]
and let \( f' \) be the restriction of \( f \) to a transversal slice \( w_0 = \text{const} \) at a point of \( \partial B_0 \cap \Sigma^*_r \). The pair consisting of the Milnor fibres of \( f \) and \( f' \) is homotopy
equivalent to the pair \((S^{n-1} \vee S^n \vee \ldots \vee S^n, S^{n-1})\) where the wedge contains \(2\mu(\Sigma)\) copies of \(S^n\) and one \(S^{n-1}\).

**Proof.** We first consider \(n = 1\). It is sufficient to consider

\[
f(x, y) = g(x, y)^2 = 1.
\]

We get two components: \(g(x, y) = 1\) and \(g(x, y) = -1\), each corresponding to a Milnor fibre of the curve \(g = 0\). This Milnor fibre is a bouquet of \(\mu(\Sigma)\) \(n\)-spheres.

At points of \(\partial B_0 \cap \Sigma^*_k\) we can use \(w_0\) and \(g\) as local coordinates from 3.4. So the transversal Milnor fibre is given by

\[
f = g^2 = 1, \quad w_0 \text{ const}
\]

and consists of two points, one in each of the two components of the Milnor fibre of \(f\).

So the pair is homotopy equivalent to \(((S^1 \vee \ldots \vee S^1) \cup (S^1 \vee \ldots \vee S^1), S^0)\). If \(n = 2\) we have to take the double suspension of the spaces and we get \(((S^2 \vee \ldots \vee S^2) \cup (S^2 \vee \ldots \vee S^2), S^1)\) where the copies of \(S^2 \vee \ldots \vee S^2\) are connected in 2 points of \(S^1\).

This pair is homotopy equivalent to \((S^1 \vee S^2 \vee \ldots \vee S^2, S^1)\). For \(n \geq 3\) further double suspension gives the result. \(\square\)

**3.8.** For \(B_0, T\) and \(D_0\) small enough we define for \(t \in \partial D_0\)

\[
F^* = f^{-1}(t) \cap T,
\]

\[
F^*_k = f^{-1}(t) \cap T_k,
\]

\[
F^c = f^{-1}(t) \cap B_0,
\]

\[
F^0 = F^* \cup F^c.
\]

We use coordinates \((w_0, w_1, \ldots, w_n)\) in \(T\) with \(w_0 \in \Sigma^*\) and \((w_1, \ldots, w_n) \in Q^*\). Consider the projection, which we can suppose to be holomorphic

\[
w_0: T_k \to \Sigma^*_k
\]
and its restriction to \( F^\bullet_k \). This projection is singular at point \( s \) of \( \Gamma \cap F^\bullet_k \) where \( \Gamma \) is the polar curve of \( f \) with respect to \( w_0 \) and is given by

\[
\frac{\partial f}{\partial w_1} = \ldots = \frac{\partial f}{\partial w_n} = 0.
\]

Since \( \Gamma \) cuts \( \Sigma^\bullet \) only in the \( D_\infty \)-points of \( f \) ([Si-1]) it follows that

\[
w_0: F^\bullet_k \to \Sigma^\bullet_k
\]

can only be singular in the neighbourhoods of \( D_\infty \)-points of \( f \).

Let \( S_{k,1}, \ldots, S_{k,r_k} \) be small disjoint discs around the \( D_\infty \)-points in \( \Sigma^\bullet_k \). Set

\[
S_k = \bigcup_i S_{k,i}, \quad M_k = \overline{\Sigma^\bullet_k \setminus S_k}.
\]

We also suppose that

\[
f: w_0^{-1}(S_{k,i}) \cap f^{-1}(D_0) \to D_0
\]

satisfies the Milnor conditions with respect to the polyball \( S_{k,i} \times Q^n \).

3.9. **Lemma.** For the diameter of \( T \) sufficiently small the projection

\[
w_0: F^\bullet_k \to \Sigma^\bullet_k
\]

is locally trivial above \( M_k \) with fibre equivalent to the Milnor fibre of the quadratic singularity: \( w_1^2 + \ldots + w_n^2 \).

**Proof.** For families of quasi-homogeneous singularities there is a stable radius for the Milnor construction (cf. [Ok] or [Os]). This implies that the various transversality conditions are satisfied and the lemma follows from Ehresmann’s fibration theorem. \( \square \)

3.10. **Proposition.** Let \( *D_\infty > 0 \), then \( F^0 \) is homotopy equivalent to the union of the Milnor fibre of the central singularity and the Milnor fibres of the \( D_\infty \)-singularities, glued together along a common \( S^{n-1} \). So

\[
F^0 \simeq S^n \vee \ldots \vee S^n; \quad b_n(F^0) = 2\mu(\Sigma) + 2 * D_\infty - 1.
\]

If \( *D_\infty = 0 \) then

\[
F^0 \simeq F^\varepsilon \simeq S^n \vee \ldots \vee S^n \vee S^{n-1}; \quad b_n(F^0) = 2\mu(\Sigma).
\]

**Proof.** Let \( \gamma_{k,0} = \Sigma^\bullet_k \cap \partial B_0 \) and \( s_{k,0} \in \gamma_{k,0} \). Choose a system of paths \( \gamma_{k,1}, \ldots, \gamma_{k,r_k} \) from \( s_{k,0} \) to \( S_{k,1}, \ldots, S_{k,r_k} \) (in the usual way; see the diagram). Set

\[
\gamma_k = \bigcup_i \gamma_{k,i}.
\]
$S_k \cup \gamma_k$ is a deformation retract of $\Sigma_k^*$ and $\gamma_{k,0}$ is a deformation retract of $\gamma_k$. Since we can suppose that $w_0$ is locally trivial above $M_k$ it follows from the homotopy lifting property that

$$(F_k^*, \, w_0^{-1}(\gamma_{k,0})) \cong (w_0^{-1}(S_k \cup \gamma_k), \, w_0^{-1}(\gamma_k)).$$

If $^\ast D_\infty > 0$ on $\Sigma_k^*$ this is homotopy equivalent to $(w_0^{-1}(\gamma_k) \cup E_k, \, w_0^{-1}(\gamma_k))$ where $E_k$ is the disjoint union of $2^\ast D_\infty$-cells, which are attached to the vanishing cycle $S^{n-1}$ in the standard way. The attachment takes place in $w_0^{-1}(s_{k,0})$.

If $^\ast D_\infty = 0$ on $\Sigma_k^*$ then $F_k^* \cup F^c \cong F^c$.

In both cases

$$F^0 = F^c \cup F_1^* \cup \ldots \cup F_r^*.$$ 

From 3.7 we know that

$$F_c \cong S^{n-1} \vee S^n \vee \ldots \vee S^n; \quad b_n(F^c) = 2\mu(\Sigma),$$

and that each $w_0^{-1}(s_{k,0})$ can up to homotopy be identified with the $S^{n-1}$ of the wedge.

If $^\ast D_\infty > 0$ on a $\Sigma_k^*$ then this $S^{n-1}$ is killed and we have

$$F^0 \cong S^n \vee \ldots \vee S^n; \quad b_n(F^0) = 2\mu(\Sigma) + 2^\ast D_\infty - 1.$$ 

If $^\ast D_\infty = 0$ then $F^0 \cong F^c$. □

**3.11. Theorem.** Let $\Sigma$ be a plane curve and $f: (C^{n+1}, 0) \rightarrow (C, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ and transversal singularity type $A_1$ on $\Sigma - \{0\}$ and let $^\ast D_\infty > 0$, then the homotopy type of the Milnor
fibre $F$ of $f$ is a bouquet of $\mu_n(f)$ spheres $S^n$ where

$$\mu_n(f) = 2\mu(\Sigma) + ^*A_1 + 2^*D_\infty - 1;$$

$\mu(\Sigma)$ = Milnor number of $\Sigma$, $^*D_\infty$ = number of $D_\infty$ points in the generic approximation with $\Sigma$ fixed, $^*A_1$ = number of $A_1$ points in the generic approximation with $\Sigma$ fixed.

Proof. Take $\Delta$, $D_0$, $D_1$, ..., $D_\sigma$ and $B$, $B_0$, $B_1$, ..., $B_\sigma$, $T$ as before. Let $t \in \partial D_0$. Choose a system of paths $\psi_1$, ..., $\psi_\sigma$ from $t$ to $D_1$, ..., $D_\sigma$. For $T \subset D$ we set $X_T = f^{-1}(T) \cap B$. As in the preceding proposition there is a homotopy equivalence

$$(X_\Delta, X_t) \overset{h}{\simeq} (X_{D_0} \cup_{\psi_1} e_1^{n+1} \cup \ldots \cup_{\psi_\sigma} e_\sigma^{n+1}, X_t).$$

Moreover

$$(X_{D_0}, X_t) \overset{h}{\simeq} (X_{D_0} \cap (B_0 \cup T) \cap X_t, X_t).$$

Let $\phi_i: S^n \to F^0 = X_t \cap (B_0 \cup T)$ represent the $2\mu(\Sigma) + 2^*D_\infty - 1$ generators of $\pi_n(F^0)$. Use $\{\phi_i\}$ to attach $(n+1)$-cells $f_i^{n+1}$ to $F^0$. The inclusion mapping

$$F^0 = X_t \cap (B_0 \cup T) \subset X_{D_0} \cap (B_0 \cup T)$$

extends to a homotopy equivalence

$$F^0 \cup f_1^{n+1} \cup \ldots \cup f_\sigma^{n+1} \to X_{D_0} \cap (B_0 \cup T)$$

since both spaces are contractible. So we get a homotopy equivalence:

$$(X_{D_0}, X_t) \overset{h}{\simeq} (X_t \cup f_1^{n+1} \cup \ldots \cup f_\sigma^{n+1}, X_t);$$

$X_\Delta$ is obtained from $X_t$ by attaching $2\mu(\Sigma) + ^*A_1 + 2^*D_\infty - 1$ $(n+1)$-cells. So $X_\Delta$ is $(n-1)$-connected, since $X_\Delta$ is contractible. Since $X_t$ has the homotopy type of a $n$-dimensional finite CW-complex, it follows that $X_t$ has the homotopy type of a bouquet of $\mu_n(f) = 2\mu(\Sigma) + ^*A_1 + 2^*D_\infty - 1$ $n$-spheres.

3.12. Remark. As we already mentioned in the introduction we showed in [Si-2] in a slightly different way that in case of $\Sigma$ a 1 dimensional isolated complete intersection singularity (cis) the homotopy type of the Milnor fibre $F$ is as follows:

$$^*D_\infty > 0: S^n \vee \ldots \vee S^n, \quad b_n(F) = \mu(\Sigma) + ^*A_1 + 2^*D_\infty - 1;$$

$$^*D_\infty = 0: S^{n-1} \vee S^n \vee \ldots \vee S^n, \quad b_n(F) = \mu(\Sigma) + ^*A_1,$$

where $^*A_1$ and $^*D_\infty$ denote the number of $A_1$-points, respectively $D_\infty$-points in an approximation $f_\epsilon$, which deforms $\Sigma$ into a smooth singular locus. So
the notation \( A_1 \) is used in [Si-2] in an other way and differs \( \mu(\Sigma) \) with the notation in this paper.

3.13. Example. \( f = x^2 y^2 + y^2 z^2 + z^2 x^2 \).

Now the critical locus is not a plane curve, even not a complete intersection. It consist of the three coordinate axis in \( C^3 \). The transversal type is \( A_1 \).

Consider the deformation

\[ f_s = x^2 y^2 + y^2 z^2 + z^2 x^2 + sxyz. \]

The singular locus of \( f_s \) consists of

(a) four isolated \( A_1 \)-points;

(b) the three coordinate axis with each two \( D_\infty \)-points and a central singularity at the origin, which is equivalent to \( xyz \).

The Milnor fibre of the central singularity is homotopy equivalent to a 2-torus. In fact in polar coordinates this fibre is given by:

\[ |x| \cdot |y| \cdot |z| = 1, \]

\[ \arg x + \arg y + \arg z \equiv 0 \mod 2\pi \text{ (on 3-torus)}. \]

The transversal Milnor fibres (corresponding to the 3 axis) are three independent circles (up to homotopy) and indicated by \( \rightarrow, \longrightarrow, \text{ and } \longleftrightarrow \). Every two of them form a basis of \( \pi_1 = H_1 \).

The constructions of this paper, apply also to this example. For the definitions of the tubes along \( \Sigma^* \) one can use here the ordinary distance function.

The part \( F^0 \) of the Milnor fibre is homotopy equivalent to the union of the torus and the Milnor fibres of the \( D_\infty \)-points, which are glued together along the torus above three 1-spheres on the torus. Since on every branch of \( \Sigma \) we have \( ^*D_\infty > 0 \), the generators of the fundamental group of the central singularity are killed and so

\[ F^0 \cong S^2 \vee \ldots \vee S^2, \quad b_2(F^0) = 11. \]
For the full Milnor fibre $F$ we must also consider the contributions from the $A_4$-points and we get

$$F \cong S^2 \vee \ldots \vee S^2, \quad b_2(F) = 15.$$ 

In this example

$$2\mu(\Sigma) + \#A_4 + 2 \#D_\infty - 1 = 2 \cdot 2 + 4 + 2 \cdot 6 - 1 = 19$$

which is different from 15.

References


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