PROJECTIVE REPRESENTATIONS
OF THE GENERALIZED SYMMETRIC GROUPS

J. F. HUMPHREYS

Department of Pure Mathematics, University of Liverpool
Liverpool, England

1. Definitions

The generalized symmetric groups are the wreath products $C_m \wr S_n$, where $C_m$
denotes a cyclic group of order $m$ and $S_n$ is the symmetric group on \{1, \ldots, n\}.
The case $m = 2$ is of special interest since the groups arising in that case are the
finite reflection groups of type $B_n$.

A (complex) projective representation of degree $d$ of a group $G$ is a map
$P : G \to \text{GL}(d, C)$ such that

(i) $P(1_G) = I_d$; and

(ii) for all $x, y$ in $G$, there is an element $\alpha(x, y)$ in $C \setminus \{0\}$ such that

\[ P(x)P(y) = \alpha(x, y)P(xy). \]

If $\alpha(x, y) = 1$ for all $x, y$, we say that $P$ is a linear representation of $G$.

The map $\alpha : G \times G \to C \setminus \{0\}$ satisfies the conditions

(i) $\alpha(1, g) = 1 = \alpha(g, 1)$ for all $g$ in $G$, and

(ii) $\alpha(x, yz)\alpha(y, z) = \alpha(x, y)\alpha(xy, z)$ for all $x, y, z$ in $G$, so that $\alpha$ is

a 2-cocycle. There is an equivalence relation on 2-cocycles: $\alpha \sim \beta$ if and only if

there exists a map $\delta : G \to C \setminus \{0\}$ such that for all $x, y$ in $G$

\[ \alpha(x, y) = \delta(x)\delta(y)\delta(xy)^{-1}\delta(x, y). \]

The equivalence classes of 2-cocycles form a group $M(G)$ under multiplication
of values. This group is the Schur multiplier and is finite if $G$ is finite.

In 1904, Schur [4] established the existence of a representation group $H$ for any finite group $G$. This is a group with the properties:

(i) $H$ has a subgroup $A$ with $A \leq Z(H) \cap [H, H]$;

(ii) $H/A \cong G$; and

(iii) $|A| = |M(G)|$.

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Schur also showed that any projective representation of $G$ can be lifted to a linear representation of $H$.

The Schur multiplier of $C_m \hat{\times} S_n$ has been calculated by Davies and Morris [2]. It is an elementary abelian 2-group of rank $k(m, n)$ where

$$k(m, n) = \begin{cases} 
3 & \text{if } m \text{ is even and } n \geq 4, \\
 n - 1 & \text{if } m \text{ is even and } n \leq 3, \\
1 & \text{if } m \text{ is odd and } n \geq 4, \\
0 & \text{if } m \text{ is odd and } n \leq 3.
\end{cases}$$

Thus the projective representations of $C_m \hat{\times} S_n$ are determined by the linear representations of various “double covers” $G$ of $C_m \hat{\times} S_n$

$$1 \to C_2 \to G \to C_m \hat{\times} S_n \to 1.$$ 

In an irreducible representation of degree $d$ of such a group $G$, by Schur’s lemma, the generator of $C_2$ is represented either as $I_d$ or $-I_d$. In the latter case, the representation is said to be negative. The irreducible nonlinear projective representations of $C_m \hat{\times} S_n$ therefore correspond to irreducible negative representations of various double covers of $C_m \hat{\times} S_n$.

2. A general construction for double covers

Let $\mathcal{G}$ be the category whose objects are triples $(G, z, \sigma)$ where $G$ is a finite group, $z$ is a central involution in $G$ and $\sigma: G \to Z/2$ is a homomorphism with $\sigma(z) = 0$. A morphism between $(G_1, z_1, \sigma_1)$ and $(G_2, z_2, \sigma_2)$ is a group homomorphism $\theta: G_1 \to G_2$ such that $\theta(z_1) = z_2$ and with $\sigma_2(\theta(g_1)) = \sigma_1(g_1)$ for all $g_1$ in $G_1$. Thus $G$ is a double cover for $\hat{G} = G/[1, z]$.

Given $N, H$ in $\mathcal{G}$ and a homomorphism $\theta: H \to \text{Aut}_{\hat{G}} N$, such that $\theta(z)$ is the identity. Let $N \hat{\times}_{\theta} H$ be the Cartesian product $N \times H$ with multiplication

$$(n, h)(n_1, h_1) = (n\theta(h)(n_1), z^{\sigma(h)\sigma(n_1)} hh_1).$$

It may be checked that $N \hat{\times}_{\theta} H$ is then a group with $\{(1, 1), (1, z), (z, 1), (z, z)\}$ a central subgroup. Let $Z$ be the central subgroup $\{(1, 1), (z, z)\}$ and put

$$N\hat{\mathcal{V}}_{\theta} H = N \hat{\times}_{\theta} H/Z.$$ 

This is given the structure of an object in $\mathcal{G}$ by setting $z = (1, z)Z$ and

$$\sigma(n, h) = \sigma(n) + \sigma(h).$$

If $\theta$ is the homomorphism $H \to \text{Aut}(\hat{N})$ produced in the obvious way from $\theta$, then $N\hat{\mathcal{V}}_{\theta} H$ is a double cover of the semi-direct product $N \times_{\theta} H$. 
3. An application of the construction

From now on we take $m$ to be even. Let $A$ be the direct product $C_m \times \{1, z\}$. Regard $A$ as an object in $\mathcal{G}$ by choosing a homomorphism $\sigma: A \to \mathbb{Z}/2$ with $\sigma(z) = 0$. Now iterate the construction of Section 2 on $A$ $n$ times to produce a group $N$ which is a double cover of $C_m^n$. (The maps $\theta$ are all taken to be the trivial map $h \mapsto \text{id}_N$).

Next let $H$ be the group $\tilde{S}_n(\varepsilon)$ generated by $z, t_1, \ldots, t_{n-1}$ subject to the relations

\begin{align*}
  z^2 &= 1, & zt_i &= t_i z, & 1 \leq i \leq n-1, \\
  t_i^2 &= 1, & 1 \leq i \leq n-1, \\
  (t_it_{i+1})^3 &= 1, & 1 \leq i \leq n-2, \\
  t_it_j &= z^{|i-j|}t_j t_i, & |i-j| \geq 2 & \text{and} & 1 \leq i, j \leq n-1.
\end{align*}

Thus $S_n(0)$ is the direct product $S_n \times \{1, z\}$ while $\tilde{S}_n(1)$ is one of the representation groups of $S_n$ constructed by Schur if $n \geq 4$. In either case, $H$ is regarded as an object in $\mathcal{G}$ by taking $\sigma(z)$ to be 0 and $\sigma(t_i) = \varepsilon$.

One further homomorphism $\alpha: A \to \mathbb{Z}/2$ is used to specify the map $\theta: H \to \text{Aut}_\mathcal{G}(N)$. Thus $\theta$ maps $t_i$ to $\tau_i$ where

$$
\tau_i (g_1, \ldots, g_n) = z^{n_i}(g_1, \ldots, g_{i-1}, g_{i+1}, g_i, \ldots, g_n)
$$

with

$$
n_i = \sigma(g_i)\sigma(g_{i+1}) + \sum_{k=i+1}^{n} \alpha(g_k).
$$

The construction of Section 2 is then applied to yield a group $Y_n(\alpha, \sigma, \varepsilon)$. There are therefore eight groups arising from the possible choices of $\alpha, \sigma$ and $\varepsilon$ and these are precisely the eight double covers of $C_m^2 S_n$.

When $\alpha = 0$, we have the important "Young" property that $Y_k(0, \sigma, \varepsilon) \tilde{\gamma}_{\theta=1} Y_l(0, \sigma, \varepsilon)$ is a subgroup of $Y_{k+l}(0, \sigma, \varepsilon)$.

4. Representation theory

For $G$ in $\mathcal{G}$, let $T^1(G)$ be the Grothendieck group generated by the finite dimensional negative representations. We also consider $\mathbb{Z}/2$-graded negative representations of $G$. These are pairs $\{V_0, V_1\}$ of finite-dimensional vector spaces such that $V_0 \oplus V_1$ is a negative representation of $G$ and also that $gV_i \subseteq V_{i+\sigma(g)}$. Let $T^*(G)$ be the $\mathbb{Z}/2$-graded group $T^0(G) \otimes T^1(G)$.

Now define $L$ to be the ring $\mathbb{Z}[\lambda]/(\lambda^3 - 2\lambda)$. A $\mathbb{Z}/2$-grading is determined on $L$ by requiring that $\lambda \in L^1$. As abelian groups, $L^1$ has basis $\lambda$ and $L^0$ has basis $\{1, \varrho\}$ where $\varrho = \lambda^2 - 1$. In fact $T^*(G)$ is a $\mathbb{Z}/2$-graded $L$-module. The following result was proved in [3].
**Theorem.** Given $A, B$ in $\mathcal{G}$, there is an isomorphism

$$T^*(A) \otimes_L T^*(B) \rightarrow T^*(A \hat{\otimes} B).$$

As a consequence of this, the graded ring $\bigoplus_{n \geq 1} T^*(Y_n(0, \sigma, \epsilon))$ has a multiplication (induction product). It also has a comultiplication (arising from restrictions) together with a positivity property (elements corresponding to irreducible representations) and a self-adjointness property (from inner products). We therefore have the following result.

**Theorem.** The algebra $\bigoplus_{n \geq 1} T^*(Y_n(0, \sigma, \epsilon))$ is an $L$-PSH algebra.

The notion of an $L$-PSH algebra was introduced in [1]. It is a natural generalization of the concept of $Z$-PSH algebra as introduced by Zelevinsky [5]. The standard example of a $Z$-PSH algebra is the graded algebra of Grothendieck groups of linear representations of $S_n$.

In [1], Bean and Hoffman have given a classification of $L$-PSH algebras analogous to Zelevinsky's classification of $Z$-PSH algebras. In the case of $L$-PSH algebras, each is a tensor product of "atoms". There are four types of atom, each of which is realized in one of our four algebras. Two of our four algebras are atomic PSH-algebras, and the other two are both tensor products of two atoms. In each case the algebra structure is known and information such as the rank of each algebra in dimension $n$ may easily be calculated.

**References**


