A MODEL OF HYPERBOLIC STEREOMETRY
BASED ON THE ALGEBRA OF QUATERNIONS

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Let \( R, C \) and \( H \) denote fields of reals, complex numbers and quaternions, respectively. We consider the multiplicative group on \( C \times H \) which acts on \( H^*_3 := H \times H \setminus \{(0, 0)\} \) by the rule

\[
(C \times H) \times H^*_3 \to H^*_3, \quad ((z, h), (x^1, x^2)) \mapsto (zx^1 h, zx^2 h).
\]

Denote by \( N \) the space of orbits of \( H^*_3 \) under this action. \( N \) is a basic space which we shall provide with a hyperbolic metric. It is known that the hyperbolic stereometry can be considered as the Riemannian geometry with the basic manifold \( H^+_3 := \{(x^1, x^2, x^3) \mid x^3 > 0\} \) and with the fundamental metric form

\[
ds^2_{(x^1, x^2, x^3)} = \frac{K^2}{(x^3)^2} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2).
\]

We are going to obtain this metric and some other properties of the hyperbolic space by considering \( N \) as a base of a certain Klein space. An analogical treatment has been performed in a 2-dimensional case and resulted in a brief and consequent system of analytical geometry of the Lobachevski plane ([2], cf. also [4]).

I. THE FUNDAMENTAL GROUP

1. Proposition. The space \( N \) is a compact 3-dimensional manifold with a boundary.

Proof. Orbits of the action

\[H \times H^*_3 \to H^2 / (a, (x^1, x^2)) \mapsto (x^1 a, x^2 a)\]
are points of the projective space over $H$, denoted by $PH$ (cf. [1]). This space can be covered by two charts defined by mappings

$$
\mu_1: \{x^1, x^2\} \mapsto x^1(x^2)^{-1} \quad \text{and} \quad \mu_2: \{x^1, x^3\} \mapsto x^3(x^1)^{-1},
$$

where $\{x^1, x^2\}$ denotes an image of $(x^1, x^3)$ under the canonical projections $H^2_\mathbb{C} \to PH$. The real dimension of $PH$ is 4. Thus $N$ can be viewed as the space of orbits in $PH$ under the action

$$
C \times PH \to PH / (z, \{x^1, x^2\}) \mapsto \{zx^1, zx^2\}.
$$

We have

$$
\mu_i\{zx^1, zx^2\} = z(\mu_i\{x^1, x^2\})z^{-1}, \quad i = 1, 2.
$$

We shall see what are orbits in $H$ under the action $a \mapsto zaz^{-1}$, where $z \in C$ and $a \in H$. For that purpose write $a$ in the form $a = a' + a''j$, where $a'$ and $a''$ are complex numbers and $j$ is the third unity in $H$. Note that this decomposition can be obtained as follows:

$$
(2) \quad a' = \frac{1}{2} \left( a + ia(-i) \right), \quad a'' = \frac{1}{2} \left( a - ia(-i) \right)(-j).
$$

Thus

$$
zas^{-1} = za\bar{z}|z|^{-2} = a' + \exp(i2\arg z)a''j.
$$

Since $a'$ and $|a''|$ are invariant, we can define mappings $m_1$ and $m_2$ as follows: if $(x^1, x^2) \in H$ and $x^2 \neq 0$, then

$$
(3) \quad \mu_1\{x^1, x^2\} = x^1(x^2)^{-1} = h' + h''j, \quad \text{where} \quad h', h'' \in C.
$$

We put

$$
m_1(p(x^1, x^2)) = (\text{re} h', \text{im} h', |h''|),
$$

where $p$ denotes the canonical projection of $H^2_\mathbb{C}$ onto $N$. Thus the values of $m_1$ lie in $\text{cl}R^3_+$. Similarly, we define

$$
m_2: \{p(x^1, x^2)|x^1 \neq 0\} \to \text{cl}R^3_+,
$$

where

$$
p(x^1, x^2) \mapsto \left( \text{re} \left( \mu_2(x^1, x^2)' \right), \text{im} \left( \mu_2(x^1, x^2)' \right), |\mu_2(x^1, x^2)'| \right),
$$

and $\mu_2(-) = \mu_2(-)' + \mu_2(-)''$ is a decomposition analogous to (3).

We see that $N$ can be covered by two local charts, $m_1$ and $m_2$ being the corresponding mappings. The boundary of $N$ is $\{p(x^1, x^2)|x^1(x^2)^{-1} \in C\} \cup \cup \{\infty\}$. This completes the proof.

2. Note that $N$ is homeomorphic to a manifold the points of which are circles in the $C$-plane, including circles with radius equal to 0 and $\infty$. These singular circles constitute the boundary.
Denote by \( L \) the group of non-singular complex \( 2 \times 2 \)-matrices. We map \( L \) onto a transformation group \( T \) which acts on \( N \) as follows:

(4) if \( a = [a_{k,k}, k=1,2] \in L \) and \( u = p(x^1, x^2) \in N \), then

\[ \tau_a u := p(a_1^1 x^1 + a_1^2 x^2, a_2^1 x^1 + a_2^2 x^2). \]

Observe that \( \tau_u \) does not depend on the choice of the initial point \( (x^1, x^2) \) on the orbit \( u \). Thus (4) defines the action correctly and we denote by \( T \) the image of \( L \) by \( \tau \).

We have \( \tau_a = \tau_b \) if and only if \( b = \lambda a \) for some \( 0 \neq \lambda \in \mathbb{C} \). This implies the following

3. **Proposition.** \( T \) is isomorphic to the group of complex \( 2 \times 2 \)-matrices with the determinant equal to 1. The real dimension of \( T \) is 6.

4. **Theorem.** Let \( c \) denote the point in \( N \) with the \( m_1 \)-coordinates \((0, 0, 1)\). Then a stationary subgroup \( S \subset T \) of \( c \) consists of the matrices of the form

\[
\begin{bmatrix}
  a & b \\
  -b & \bar{a}
\end{bmatrix}
\]

such that \( a\bar{a} + b\bar{b} = 1 \).

**Proof.** Consider the equation \((aj + b)(cj + d)^{-1} = j\) and split it into the \( ' \) and \( '' \) parts according to (3). We obtain \( c = -\bar{b} \) and \( d = \bar{a} \). Then we normalize the obtained matrices according to proposition 3.

Let us denote by \( \Lambda \) the closure of the set \( \{u \in N | \mu_1(u) = (0, 0, t)\} \). Denote by \( T_{\Lambda} \) the stationary group of \( \Lambda \).

5. **Theorem.** \( T_{\Lambda} \) consists of the matrices of the form

\[
\begin{bmatrix}
  a & 0 \\
  0 & 1/a
\end{bmatrix}
\text{ or } \begin{bmatrix}
  0 & a \\
  -1/a & 0
\end{bmatrix}, \text{ where } a \in \mathbb{R}.
\]

The proof is analogous to that of theorem 4.

6. **Proposition.** The group \( S \) is isomorphic to the group of rotations of the Euclidean 3-dimensional space.

This fact can be proved by checking that the Lie algebras of both groups are isomorphic.

7. **Lemma.** Let \( h = h' + h'' j \in H \) be such that \( h' \neq 0 \) and \( h'' \neq 0 \). Then there exist two complex numbers \( g_1 \) and \( g_2 \) and two complex singular matrices \( G_1 \) and \( G_2 \) such that

(i) \( |g_1| \neq 1, |g_1g_2| = 1, \) and \( \arg g_1 = \arg g_2 = \arg h' \);
(ii) each \( G_1 \) sends \( h \) to \( g_1 \) and \( j \) to itself for \( \nu = 1, 2 \).
Proof. In view of theorem 4, we have to find \( g \) and the required matrices from the equation
\[
a(h' + h''j) + b = gj(-b(h' + h''j) + \bar{u}).
\]

After performing some simple calculations and splitting both member into their ' and '' parts, we obtain the following system of equations
\[
h'a + (1 - g\bar{h}'')b = 0, \quad (h'' - g)a + g\bar{h}'b = 0.
\]

Its determinant must be 0 and for \( g \) we have the equation
\[
-\bar{h}''g^2 + (1 + \bar{h}'h' + \bar{h}''h'')g - h'' = 0
\]
which can be written in the form
\[
(\gamma + \gamma^{-1})/2 = (1 + \bar{h}'h' + \bar{h}''h'')/(2|h''|),
\]
where \( \gamma = g \exp(-i \arg h'') \). Since \( \gamma \) is real and it satisfies the equation
\[
\text{ch} \log \gamma = (1 + |h|^2)/(2|h''|),
\]
there exist two distinct roots of equation (5), namely \( g_1 = \gamma_1 e^{i\alpha} \) and \( g_2 = \gamma_2 e^{i\alpha} \), where \( \alpha = \arg h'' \). These roots yield the two matrices
\[
\begin{bmatrix}
g_1h' & g_1 - h'' \\
\bar{h}'' - \bar{g}_2 & \bar{g}_1 - h'
\end{bmatrix}
\quad (v = 1, 2)
\]
which satisfy (ii).

8. Theorem. For any two points \( u, v \in \text{int} N \), there exist two elements \( a_1 \) and \( a_2 \) in \( T \) such that each \( a_v \) sends \( u \) to \( p(j, 1) \) and \( v \) to \( p(gj, 1) \), where \( g \), are complex and \( |g_1g_2| = 1 \).

Proof. Write \( u = p(w' + w''j, 1) \). Then the matrix
\[
a_v = \begin{bmatrix}
1 & w' \\
0 & w''
\end{bmatrix}
\]
sends \( u \) to \( p(j, 1) \). We choose an \( h \) such that \( a_u v = p(h, 1) \) and apply lemma 7. This implies the existence of \( G_1 \) and \( G_2 \) which send \( a_u u \) to itself and \( a_v v \) to \( p(g_1j, 1) \) (or, respectively, to \( p(g_2j, 1) \)), where \( g_1 \) and \( g_2 \) satisfy the conditions of the theorem. Then we put \( a_1 = G_1 \circ a_v \) and \( a_2 = G_2 \circ a_v \).

9. Proposition. The stationary group of the pair \( (p(j, 1), p(gj, 1)) \), where \( 1 \neq g \in C \), is represented by matrices of the form
\[
\begin{bmatrix}
e^{ri} & 0 \\
0 & e^{-ri}
\end{bmatrix}.
\]
The action of this group can be expressed also by
\[p(h, 1) \mapsto p(e^{ri}h e^{-ri}, 1).\]
This proposition can be easily obtained by lemma 7. The same lemma allows us to prove the following

10. PROPOSITION. The group $T$ acts transitively on a bundle of directions on $N$.

II. THE CROSS-RATIO AND THE METRIC IN $N$

We recall that $A$ is the orbit of the point $p(j, 1)$ under the action of the group of matrices of the form

\[
\begin{bmatrix}
\sqrt{s} & 0 \\
0 & 1/\sqrt{s}
\end{bmatrix},
\]

where $s$ varies in the half-line of positive numbers (cf. theorem 5). We observe that these matrices constitute a connected component of $T_A$. However, $T_A$ contains another topologically connected component to which matrices of the form

\[
\begin{bmatrix}
0 & -\sqrt{s} \\
\sqrt{s} & 0
\end{bmatrix}
\]

belong. Each such matrix can be represented as a product

\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
\sqrt{s} & 0 \\
0 & -1/\sqrt{s}
\end{bmatrix}.
\]

An action of the first matrix of this decomposition is nothing but a change of orientation on $N$. More precisely, it sends $p(j, 0)$ to $p(0, j)$ and vice versa.

Then we can define the cross-ratio on int$N$ which is invariant under $T_A$. We denote by $(u, v; r, q)$ the value of the cross-ratio of the quadruple $(u, v, r, q)$. If $u = p(a_1j, 1)$, $v = p(a_2j, 1)$, $r = p(\beta_2j, 1)$ and $q = p(\beta_2j, 1)$, then we have

\[
(u, v; r, q) = \frac{a_1-\beta_1}{a_2-\beta_1} / \frac{a_1-\beta_2}{a_2-\beta_2}.
\]

We extend this function onto cl$A$ by continuity. In particular, we have

\[
(8) \quad (u, v; p(0, 1), p(1, 0)) = a_1/a_2.
\]

Thus we obtain

11. PROPOSITION. We have

\[
(u, v; p(0, 1), p(1, 0)) (v, w; p(0, 1), p(1, 0)) = (u, w; p(0, 1), p(1, 0))
\]

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and, if \( v \) lies between \( u \) and \( w \), then
\[
|\log(u, v; p(0, 1), p(1, 0))| + |\log(v, w; p(0, 1), p(1, 0))| = |\log(u, w; p(0, 1), p(1, 0))|.
\]

Now we are able to define a metric in \( N \).

12. **Definition.** We say that three distinct points in \( N \) are **\( N \)-collinear** if there exists \( t \in T \) which maps these three points to points of \( \Lambda \). The **\( N \)-line** through points \( u \) and \( v \) is the set of points which are collinear with \( u \) and \( v \).

13. **Proposition.** Every \( N \)-line is a curve in \( N \). Its boundary consists of two points in the boundary of \( N \).

14. **Definition.** Fix a positive real \( K \). We define a **\( T \)-invariant distance** \( \delta \) in \( N \) as follows. If \( u \) and \( v \) are distinct points in \( \text{int} N \), then we choose \( a \in T \) such that \( au = p(j, 1) \) and \( av = p(gj, 1) \), where \( g \in C \), and we set
\[
\delta(u, v) := K |\log |g||.
\]

In view of sections 8 and 9, the function \( \delta \) is uniquely determined. We have to express it in \( m_1 \)-coordinates. If \( u = p(c, 1) \) and \( v = p(h, 1) \), then we construct a transformation \( a \) according to the proof of theorem 8. Then we use formula (5) and theorem 8. After some calculations we obtain
\[
\frac{\chi(K^{-1} \delta(u, v))}{(\delta')^2} = \left( |f' - h'|^2 + |f''|^2 + |h|^2 \right)^2 / |2f'' h'|
\]

This formula implies immediately \( \delta(u, v) = \delta(v, u) \). Additivity of \( \delta \) on any \( N \)-line follows from proposition 9.

15. **Theorem.** The infinitesimal form of the metric \( \delta \) is
\[
\frac{\delta s^2}{|p(h, 1)|} = \frac{K^2}{(x')^2} \left( (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right),
\]
where \( x^1 = \text{re} h', x^2 = \text{im} h', x^3 = |h'| \).

**Proof.** Consider a curve which is parametrized by the mapping \( t \mapsto p(h + tx, 1) \). Let \( X \) be a 1-jet of this mapping, its source being \( 0 \). Thus \( X \) is a vector which is tangent to \( N \) at \( u = p(h, 1) \). We have to calculate the norm \( |X| \) of \( X \), which is induced by \( \delta \). We have
\[
|X| = \lim_{t \to 0} \frac{1}{t} \delta(p(h + tx, 1), p(h, 1)).
\]
After some elementary calculations we obtain
\[ |X| = \frac{K^2}{|h''|^2} \left((\text{re} x')^2 + (\text{im} x')^2 + |x'|^2 \right) \]
which is consistent with (9). So the following theorem is a corollary to the just obtained formulas:

16. **Theorem.** \( \delta \) is a hyperbolic distance.

**III. Final Remarks**

Let us denote by \( B \) the boundary of \( N \). This boundary is homeomorphic to the complex projective line (which is isomorphic to the real Moebius sphere). Each \( N \)-line has exactly two points in common with \( B \). These are the so-called infinite points of the line. Conversely, each pair of distinct points on \( B \) determines exactly one \( N \)-line.

The group \( T \) acts as a group of projective transformations. The proper and improper circles in \( B \) are traces of \( N \)-planes according to the following definition:

17. **Definition.** A subset \( \Pi \subset N \) is called an \( N \)-plane if there exists \( a \in T \) which maps \( \Pi \) to
\[ \Pi_0 = \text{cl}\{w \in N \mid w = p(h, 1), \text{ where } \text{im} h' = 0\} . \]

Thus each \( N \)-plane is a 2-dimensional submanifold of \( N \).

18. **Theorem.** For any \( N \)-lines \( a \) and \( \beta \) with the unique point of coincidence \( v \in \text{int} N \), there exists a unique \( N \)-plane \( \Sigma \) such that \( a \subset \Sigma \) and \( \beta \subset \Sigma \).

**Proof.** By theorem 8, there exists \( a \in T \) which sends \( a \) to \( \Lambda \). Let \( z_1 \) and \( z_2 \) be the infinite points of the \( N \)-line \( a \beta \). Apply proposition 9 and perform a transformation \( r \) such that \( r \Lambda = \Lambda \) and \( \text{im} z_1 = \text{im} z_2 = 0 \). Hence \( r \circ a \) sends \( a \) and \( \beta \) into \( \Lambda \). Thus \( a^{-1} \circ r^{-1} \Lambda \) is the \( N \)-plane through \( a \) and \( \beta \).

The following two theorems are easy to prove.

19. **Theorem.** The stationary subgroup \( T_0 \) of \( \Pi_0 \) consists of those transformations which have real matrices and determinants equal to 1.

20. **Theorem.** If we restrict the Klein space \((N, T)\) to \((\Pi_0, T_0)\), then we obtain the plane hyperbolic geometry.

Let \( a \) and \( \beta \) be two \( N \)-lines as in theorem 18. We denote by \( z_1, z_2 \) and \( s_1, s_2 \), respectively, pairs of their infinite points. Thus \( z_1, z_2, s_1, s_2 \) are situated on a circle in \( B \). We denote by \( \vartheta \) hyperbolical measure of
the angle between $\alpha$ and $\beta$. Observe that, in view of proposition 6, the mapping $m_1|\text{int} \mathcal{N}$ is conformal. Then the following relation holds between $\vartheta$ and the cross-ratio of the chosen pairs of infinite points:

$$(x_1, x_2; s_1, s_2) = -\cotg \frac{\vartheta}{2}.$$  

This is proved in [3] in the case where $\alpha$ and $\beta$ are both in $\Pi_0$, but remains true in general because of the invariance with respect to $T$ of both members of this equality.

REFERENCES


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