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Aerodynamic Torque exhibits non-resonance oscillation in satellite motion

Abstract This paper deals with the non-linear oscillation of a satellite in an elliptic orbit around the Earth under the influence of aerodynamic and gravitational torque. It is assumed that the orbital plane coincides with the equatorial plane of the Earth. Using Bogoliubov–Krylov–Mitropolsky (BKM) methods of nonlinear oscillations, it is observed that the amplitude of the oscillation remains constant up to the second order of approximation. Numerically time series, 2D and 3D phase spaces are plotted for Earth Moon system using Matlab. The existence of main and parametric resonance concludes the different frequency states which transit the motion from regular to an attractor that leads to chaotic state.

2010 Mathematics Subject Classification: 70K42.

Key words and phrases: Aerodynamic torque, Gravitational torque, Non-linear oscillation, Non-Resonance, Resonance.


None of the authors studied the non-resonance motion of a satellite under the influence of aerodynamic torque in an almost elliptic orbit.

2. Equation of Motion. Let us consider a rigid satellite $S$ moving in an elliptic orbit around the Earth $E$ such that the orbital plane of the satellite coincides with the equatorial plane of the Earth. The satellite is assumed to be a tri-axial body with principal moments of inertia $A < B < C$ at its centre of mass and $C$ is the moment of inertia about the spin axis which is perpendicular to the orbital plane. Let $\vec{r}$ be the radius vector of the centre of mass of the satellite, $\nu$ be the true anomaly, $\theta$ be the angle that the long axis of the satellite make with a fixed line $EF$ lying in the orbital plane and $\frac{\eta}{2}$ be the angle between the radius vector and the long axis as shown in Figure 1.

Figure 1: Motion of Satellite S in an elliptic orbit around the Earth
The equation of motion for the system has been obtained by Bhardwaj and Kaur (2014) ([10]) as

\[
\frac{d^2 \eta}{d\nu^2} + n^2 \eta = -e \cos \nu \frac{d^2 \eta}{d\nu^2} + 2e \sin \nu \frac{d\eta}{d\nu} + 4e \sin \nu + n^2(\eta - \sin \eta) + e(A_\ast \nu^2 \sin \nu + B_\ast \nu \sin \nu + C_\ast \sin \nu + D_\ast \nu + E_\ast),
\]

(1)

where

\[
n^2 = \frac{3(B - A)}{C},
\]

\[
\epsilon = \rho SC_d l^2 \Omega^2 C = \text{parameter due to aerodynamic torque},
\]

\[
A_\ast = \frac{a^2(1 - e)}{\Omega^2 l},
\]

\[
B_\ast = \left[ \frac{\omega a(2e - 1) \cos i}{\Omega} + \frac{2V_1 a(1 - 2e)}{\Omega l} \right],
\]

\[
C_\ast = \left[ \omega V_1 (2e - 1) \cos i + \frac{V_1^2 \omega^2 (1 - 2e)}{l} + \frac{\omega e \sin i}{2\Omega} \right],
\]

\[
D_\ast = \left[ \frac{\omega a(2e - 1)}{\Omega} \sin i \right],
\]

\[
E_\ast = \omega V_1 (2e - 1) \sin i
\]

are all constants.

In the equation (1), the non-linearity \((\eta - \sin \eta)\) is sufficiently weak and therefore it can be taken of the order of \(e\), so by taking \(n^2 = ce\) and \(\epsilon = e\epsilon_1\), we get

\[
\frac{d^2 \eta}{d\nu^2} + n^2 \eta = ef(\nu, \eta, \eta', \eta'')
\]

(2)

where

\[
f(\nu, \eta, \eta', \eta'') = -\cos \nu \frac{d^2 \eta}{d\nu^2} + 2 \sin \nu \frac{d\eta}{d\nu} + 4 \sin \nu + c(\eta - \sin \eta) + e_1(A_\ast \nu^2 \sin \nu + B_\ast \nu \sin \nu + C_\ast \sin \nu + D_\ast \nu + E_\ast)
\]

As \(e\) is very small, so the solution of the equation can be obtained by BKM method.

For \(e = 0\) the generating solution of the zeroth order is given by

\[
\eta = a \cos \psi, \quad \psi = n \nu + \psi^*
\]

where amplitude \(a\) and phase \(\psi^*\) are constants, which can be determined from initial conditions.
The solution of equation (2) is obtained in the form

\[ \eta = a \cos \psi + e u_1(a, \nu, \psi) + e^2 u_2(a, \nu, \psi) + \ldots \]  

(3)

where

\[ \frac{da}{d\nu} = e A_1(a) + e^2 A_2(a) + \ldots \]  

(4)

\[ \frac{d\psi}{d\nu} = n + e B_1(a) + e^2 B_2(a) + \ldots \]  

(5)

Substituting the values of \( \eta, \frac{d\eta}{d\nu} \) and \( \frac{d^2\eta}{d\nu^2} \) in the equation (2), and comparing the coefficients of like powers of \( e \) and \( e^2 \), we get

\[ n^2 \frac{\partial^2 u_1}{\partial \psi^2} + 2n \frac{\partial^2 u_1}{\partial \nu \partial \psi} + \frac{\partial^2 u_1}{\partial \nu^2} - 2n A_1 \sin \psi - 2n a B_1 \cos \psi + n^2 u_1 \]  

(6)

\[ = 4 \sin \nu + e_1 \left( A_* \nu^2 \sin \nu + B_* \nu \sin \nu + C_* \sin \nu + D_* \nu + E_* \right) \]

\[ + c [a \cos \psi - \sin(a \cos \psi)] - na [\cos(\nu - \psi) - \cos(\nu + \psi)] \]

\[ + \frac{n^2 a}{2} [\cos(\nu + \psi) + \cos(\nu - \psi)] \]

and

\[ n^2 \frac{\partial^2 u_2}{\partial \psi^2} + 2n \frac{\partial^2 u_2}{\partial \nu \partial \psi} + \frac{\partial^2 u_2}{\partial \nu^2} - \left( A_1 \frac{dA_1}{da} \right) \cos \psi \]

\[ - \left( 2 A_1 B_1 + a A_1 \frac{dB_1}{da} \right) \sin \psi + 2 A_1 \frac{\partial^2 u_1}{\partial a \partial \nu} + 2 n B_1 \frac{\partial^2 u_1}{\partial \psi^2} \]

\[ + 2 n A_1 \frac{\partial^2 u_1}{\partial a \partial \psi} + 2 B_1 \frac{\partial^2 u_1}{\partial \nu \partial \psi} - 2 A_2 \sin \psi - 2 A_1 A_2 \cos \psi + n^2 u_2 \]

\[ = 2 \sin \nu \left( A_1(a) \cos \psi - a B_1(a) \sin \psi + \frac{\partial u_1}{\partial \nu} + n \frac{\partial u_1}{\partial \psi} \right) \]

\[ - \cos \nu \left( n^2 \frac{\partial^2 u_1}{\partial \psi^2} + 2 n \frac{\partial^2 u_1}{\partial \nu \partial \psi} + \frac{\partial^2 u_1}{\partial \nu^2} \right) \]

\[ - 2 n A_1(a) \sin \psi - 2 A_1 A_2 \cos \psi \]

\[ + c (u_1 - u_1 \cos(a \cos \psi)) \]

Using Fourier expansion

\[ \sin(a \cos \psi) = 2 \sum_{k=0}^{\infty} (-1)^k J_{(2k+1)}(a) \cos(2k + 1) \psi \]

\[ \cos(a \cos \psi) = J_0(a) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(a) \cos(2k) \psi \]
where $J_k(a)$ is Bessel’s function of order $k$ in the equation (6) and equating the coefficients of like powers of $\sin \psi$ and $\cos \psi$, so that $u_1(a, \nu, \psi)$ do not contain the resonant terms, we get

$$A_1(a) = 0 \text{ and } B_1(a) = \frac{c}{2an} \left[ 2J_1(a) - a \right].$$

Substituting the values of $A_1(a)$ and $B_1(a)$ in the equation (6) and solving, we get

$$u_1(a, \nu, \psi) = 4 \sin \nu \frac{n^2}{n^2 - 1} - \frac{1}{2n + 1} \left( na + \frac{n^2 a}{2} \right) \cos(\nu + \psi)$$

$$+ \frac{1}{2n - 1} \left( \frac{n^2 a}{2} - na \right) \cos(\nu - \psi)$$

$$+ A_s \epsilon_1 \frac{1}{n^2 - 1} \left[ \nu^2 \sin \nu - \frac{4\nu \cos \nu}{n^2 - 1} - \frac{2(n^2 + 3)}{(n^2 - 1)^2} \sin \nu \right]$$

$$+ B_s \epsilon_1 \frac{1}{n^2 - 1} \left[ \nu \sin \nu - \frac{2 \cos \nu}{n^2 - 1} \right]$$

$$+ C_s \epsilon_1 \frac{\sin \nu}{n^2 - 1} + \nu \frac{\nu}{n^2} D_s \epsilon_1 + \frac{1}{n^2} E_s \epsilon_1$$

$$+ \frac{c}{2n^2} \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(a) \frac{\cos(2k + 1)\psi}{k(k + 1)} \right)$$

Thus, we have

$$\frac{dA_1}{da} = 0$$

$$\frac{dB_1}{da} = \frac{c}{2an} \left[ 2J_1'(a) - 1 \right] - \frac{c}{4a^2 n} \left[ 2J_1(a) - a \right].$$

Putting the values of $A_1(a)$, $B_1(a)$, $\frac{\partial u_1}{\partial \psi}$, $\frac{\partial^2 u_1}{\partial \psi^2}$, $\frac{\partial^2 u_1}{\partial \nu \partial \psi}$, $\frac{\partial^2 u_1}{\partial \nu^2}$, $\frac{\partial^2 u_1}{\partial a \partial \psi}$, $\frac{dA_1}{da}$, $\frac{dB_1}{da}$ and $u_1$ in the equation (7), and then equating the coefficients of $\cos \psi$ and $\sin \psi$ to zero, to avoid the resonant term, we obtain

$$n^2 \frac{\partial^2 u_2}{\partial \psi^2} + 2n \frac{\partial^2 u_2}{\partial \nu \partial \psi} + \frac{\partial^2 u_2}{\partial \nu^2} - aB_1^2 \cos \psi - 2nA_2 \sin \psi - 2naB_2 \cos \psi$$

$$+ n^2 u_2 + \frac{c}{a} \left[ 2J_1(a) - a \right] \left[ \frac{(n + 1)(2na + n^2 a)}{2n(2n + 1)} \cos(\nu + \psi) \right]$$

$$+ \frac{(1 - n)(n^2 a - 2na)}{2(2n - 1)} \cos(\nu - \psi)$$

$$- \frac{c}{2n^2} \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(a) (2k + 1)^2 \frac{\cos(2k + 1)\psi}{k(k + 1)} \right]$$

$$= \frac{c}{a} \left[ 2J_1(a) - a \right] (- \sin \nu \sin \psi + \sin \nu \cos \psi) + \frac{6 \sin 2\nu}{n^2 - 1} \right]$$
Equating the coefficient of $\sin \psi$, solution is obtained as

$$\eta = \frac{n^2 - 1}{2(2n + 1)} \left( \frac{n^2 a}{2} - na \right) - \frac{n^2 - 1}{2(2n - 1)} \left( \frac{na + n^2 a}{2} \right) \cos \psi$$

$$- \frac{n^2 + 4n + 3}{2(2n + 1)} \left( na + \frac{n^2 a}{2} \right) \cos(2\nu + \psi) - \frac{n^2 + 4n - 3}{2(2n - 1)} \left( \frac{n^2 a}{2} - na \right) \cos(2\nu - \psi)$$

$$+ A_1 \epsilon_1 \left[ \frac{\sin \nu \cos \nu}{n^2 - 1} \left\{ -2 + 3\nu^2 - \frac{16}{n^2 - 1} - \frac{6(n^2 + 3)}{(n^2 - 1)^2} - \frac{4n^2}{n^2 - 1} \nu \cos \nu \right\} \right.$$  

$$+ \frac{8\nu \sin^2 \nu}{n^2 - 1} \left\{ \frac{n^2 + 1}{n^2 - 1} \right\} + B_* \epsilon_1 \frac{2\sin \nu}{n^2 - 1} \left[ \frac{3}{2} \nu + \tan \nu + \frac{2 \tan \nu}{n^2 - 1} - \frac{n^2}{n^2 - 1} \cot \nu \right]$$

$$+ C_* \epsilon_1 \frac{3 \sin 2\nu}{2(n^2 - 1)} + 2D_* \epsilon_1 \frac{2 \sin^2 \nu}{n^2 - 1}$$

$$- \frac{c}{n} \sin \nu \sum_{k=1}^{\infty} (-1)^k \frac{J_{2k+1}(a)(2k + 1)}{k(k + 1)} \sin(2k + 1) \psi \frac{k(k + 1)}{n^2 - 1} \cos(\nu + \psi) + \frac{1}{2n - 1} \left( \frac{n^2 a}{2} - na \right) \cos(\nu - \psi)$$

$$+ A_* \epsilon_1 \frac{1}{n^2 - 1} \left[ \nu^2 \sin \nu - \frac{4\nu \cos \nu}{n^2 - 1} - \frac{2(n^2 + 3)}{(n^2 - 1)^2} \sin \nu \right]$$

$$+ B_* \epsilon_1 \frac{1}{n^2 - 1} \left[ \nu \sin \nu - \frac{2 \cos \nu}{n^2 - 1} \right] + C_* \epsilon_1 \frac{\sin \nu}{n^2 - 1} + \frac{\nu}{n^2} D_* \epsilon_1 + \frac{1}{n^2} E_* \epsilon_1$$

$$+ \frac{c}{2n^2} \sum_{k=0}^{\infty} (-1)^k \frac{J_{2k+1}(a)}{k(k + 1)} \cos(2k + 1) \psi$$

$$\times \left[ 1 - \left\{ J_0(a) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(a) \cos(2k) \psi \right\} \right]$$

Equating the coefficient of $\sin \psi$ and $\cos \psi$, we get $A_2(a) = 0$ and

$$B_2(a) = - \frac{c^2}{8a^2 n^3} (2J_1(a) - a)^2 + \frac{3n(n^2 - 1)}{4(4n^2 - 1)} + \frac{c^2}{2n^3 a} \sum_{k=1}^{\infty} \frac{J_{2k+1}(a) J'_{2k+1}(a)}{k(k + 1)},$$

where $J'_{2k+1}(a) = \frac{1}{2} [J_{2k}(a) - J_{2k+2}(a)]$. Thus in the first approximation, the solution is obtained as $\eta = a \cos \psi$

$$\frac{da}{d\nu} = 0 \quad (\text{since } A_1(a) = 0) \Rightarrow a = \text{constant} \quad (10)$$

$$\frac{d\psi}{d\nu} = n + eB_1(a) = n + \frac{n}{2a} (2J_1(a) - a) \quad (\text{since } n^2 = ce) \quad (11)$$
and in the second approximation, the solution is obtained as

\[ \eta = a \cos \psi \]

\[ + e \left[ \frac{4 \sin \nu}{n^2 - 1} - \frac{1}{2n + 1} \left( na + \frac{n^2 a}{2} \right) \cos(\nu + \psi) + \frac{1}{2n - 1} \left( \frac{n^2 a}{2} - na \right) \cos(\nu - \psi) \right. \]

\[ + A_\epsilon e_1 \frac{1}{n^2 - 1} \left[ \nu \sin \nu - \frac{2 \cos \nu}{n^2 - 1} \right] + B_\epsilon e_1 \frac{1}{n^2 - 1} \left[ \nu \frac{\sin \nu}{n^2 - 1} + \frac{\nu D_\epsilon e_1}{n^2} + \frac{1}{n^2} E_\epsilon e_1 \right. \]

\[ + \frac{c}{2n^2} \sum (-1)^k J_{2k+1}(a) \cos(2k + 1) \psi \]

\[ \frac{da}{d\nu} = 0 \quad (\because A_2(a) = 0) \Rightarrow a = \text{constant} \]  

(13)

and

\[ \frac{d\psi}{d\nu} = n + \frac{n}{2a} (2J_1(a) - a) - \frac{n}{8a^2} (2J_1(a) - a)^2 + e_2 \frac{3n(n^2 - 1)}{4(4n^2 - 1)} \]

\[ + \frac{n}{2a} \sum_{k=1}^{\infty} \frac{J_{2k+1}(a) J_{2k+1}(a)}{k(k+1)} \]  

(14)

From the equations (10) and (13), it is observed that the amplitude of the oscillation remains constant up to the second order of approximation and the main resonance occurs at \( n = \pm 1 \) and parametric resonance at \( n = \pm 1/2 \).

3. Results and discussions. The 2D and 3D phase spaces are plotted for Earth Moon system. For natural satellite Moon, it is assumed that

- Semi major axis = \( a = 3.8 \times 10^5 \) Km,
- Eccentricity = \( e = 0.0549 \),
- Inclination = \( i = 5.1450^{\circ} \),
- Angular velocity = \( \omega = 2.425 \times 10^{-6} \) rad/sec.

The effect of mass parameter and aerodynamic torque parameter is studied on the non linear oscillation of a satellite in an elliptic orbit. The phase spaces for different values of mass parameter, aerodynamic torque parameter and constants are plotted as described in tables and figures.

Table 1 gives the behavior for Earth-Moon system at fixed values of \( A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon, E_\epsilon, e \) and for the variable parameters \( \epsilon, c \) with variation in the values of \( n \).

Table 2 gives the behavior for Earth-Moon system at fixed values of \( A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon, E_\epsilon, e \) and for the variable parameters \( n, c \) with variation in the values of \( \epsilon \).

Table 3 gives the behavior for Earth-Moon system at fixed values of \( A_\epsilon, C_\epsilon, D_\epsilon, E_\epsilon, e, \epsilon, c, n \) with variation in the values of \( B_\epsilon \).
Table 4 gives the behavior for Earth-Moon system at fixed values of $A_*$, $B_*$, $D_*$, $E_*$, $e$, $c$, $n$ with variation in the values of $C_*$. Table 5 gives the behavior for Earth-Moon system at fixed values of $A_*$, $B_*$, $C_*$, $E_*$, $e$, $c$, $n$ with variation in the values of $D_*$. Table 6 gives the behavior for Earth-Moon system at fixed values of $A_*$, $B_*$, $C_*$, $D_*$, $e$, $c$, $n$ with variation in the values of $E_*$. The time series and 2D phase plots are plotted for the values of $n = 0.1$, $c = 0.182149$, $e = 0.00000001$, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$ which is shown in Fig. 2 and 3D phase plots are plotted for the values of $n = 0.1$, $c = 0.182149$, $e = 0.00000001$ and 0.005, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$ which is shown in Fig. 3.

Fig. 4 gives time series and 2D phase plots for the values of $n = 0.5$, $c = 4.553734$, $e = 0.0001$, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$ and Fig. 5 gives 3D phase plots for the values of $n = 0.5$, $c = 4.553734$, $e = 0.0001$ and 0.005, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$.

Fig. 6 gives time series and 2D phase plots for the values of $n = 0.8$, $c = 11.657559$, $e = 0.000005$, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$ and Fig. 7 gives 3D phase plots for the values of $n = 0.8$, $c = 11.657559$, $e = 0.000005$ and 0.001, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$.

From tables and figures, it is observed that the phase space plots almost remain spiral for different values of constants, mass parameter, eccentricity and aerodynamic torque parameter.

4. Tables and figures

Table 1: For fixed values of $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$, $e = 0.0549$ variable $c$, $e$ and variation in $n$ the graphical behavior are spiral.

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Table 2: For fixed values of $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$, $e = 0.0549$ variable $c$, $n$ and variation in $\epsilon$ the graphical behavior are spiral.

<table>
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Table 3: For fixed values of $A^* = 3.1E+16$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$, $e = 0.0549$ variable $c$, $\epsilon$, $n$ and variation in $B^*$ the graphical behavior are spiral.

<table>
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Table 4: For fixed values of $A^* = 3.1E+16$, $B^* = -8.12389$, $D^* = -1.3E+12$, $E^* = 0$, $e = 0.0549$ variable $c$, $\epsilon$, $n$ and variation in $C^*$ the graphical behavior are spiral.

<table>
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<td>11.657559</td>
<td>0.8</td>
<td>-5.5E+11</td>
</tr>
<tr>
<td>3.</td>
<td></td>
<td>11.657559</td>
<td>0.8</td>
<td>-8.7E+10</td>
</tr>
</tbody>
</table>
Table 5: For fixed values of $A_\star = 3.1E+16$, $B_\star r = -8.12389$, $C_\star = -5.4E+11$, $E_\star = 0$, $e = 0.0549$ variable $c$, $\epsilon$, $n$ and variation in $D_\star$ the graphical behavior are spiral.

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>$\epsilon$</th>
<th>$c$</th>
<th>$n$</th>
<th>$D_\star$</th>
</tr>
</thead>
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<td>-1.3E+12</td>
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<tr>
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<td>-7.7E+12</td>
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<tr>
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<td>11.657559</td>
<td>0.8</td>
<td>-1E+13</td>
</tr>
</tbody>
</table>

Table 6: For fixed values of $A_\star = 3.1E+16$, $B_\star = -8.12389$, $C_\star = -5.4E+11$, $D_\star = -1.3E+12$, $E_\star = 0$, $e = 0.0549$ variable $c$, $n$, and variation in $E_\star$ the graphical behavior are spiral.

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>$\epsilon$</th>
<th>$c$</th>
<th>$n$</th>
<th>$E_\star$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
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<td>11.657559</td>
<td>0.8</td>
<td>0</td>
</tr>
<tr>
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<td>11.657559</td>
<td>0.8</td>
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</table>

Figure 2: 2D and time series Plot of $n = 0.1$, $c = 0.182149$, $\epsilon = 0.00000001$, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$. 
Figure 3: 3D Plots of $n = 0.1$, $c = 0.182149$, $\epsilon = 0.00000001$ and 0.005, $A^* = 3.1E+16$, $B^* = -8.12389$, $C^* = -5.4E+11$, $D^* = -1.3E+12$, $E^* = 0$ and $e = 0.0549$.

Figure 4: 2D, 3D and time series. Plot of: $n = 0.8$, $c = 11.657559$, $\epsilon = 0.000005$, $A_* = 3.1E+16$, $B_* = -8.12389$, $C_* = -5.4E+11$, $D_* = -1.3E+12$, $E_* = 0$ and $e = 0.0549$.

Figure 5: 2D, 3D and time series. Plot of: $n = 0.8$, $c = 11.657559$, $\epsilon = 0.000005$, $A_* = 3.1E+16$, $B_* = -8.12389$, $C_* = -5.4E+11$, $D_* = -1.3E+12$, $E_* = 0$ and $e = 0.0549$. 
Figure 6: 2D, 3D and time series. Plot of: $n = 0.1$, $c = 0.182149$, $\epsilon = 0.0000001$, $A_* = 3.1E + 16$, $B_* = -8.12389$, $C_* = -5.4E + 11$, $D_* = -1.3E + 12$, $E_* = 0$ and $e = 0.0549$.

Figure 7: 2D, 3D and time series. Plot of: $n = 0.8$, $c = 11.657559$, $\epsilon = 0.000005$ and $0.001$, $A_* = 3.1E + 16$, $B_* = -8.12389$, $C_* = -5.4E + 11$, $D_* = -1.3E + 12$, $E_* = 0$ and $e = 0.0549$. 
5. Conclusion. Using BKM method, it is observed that the amplitude of the oscillation remains constant up to the second order of approximation. The main resonance occurs at \( n = \pm 1 \) and parametric resonance at \( n = \pm 1/2 \). Using time series, 2D and 3D phase plots it is concluded that mass parameter, eccentricity, aerodynamic torque parameter, variable constant almost have the same effect and motion almost remains constant only with change in the size of oscillations.

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REFERENCES


Nierezonansowe oscylacje aerodynamicznego momentu obrotowego dla satelity okołoziemskiego

Rashmi Bhardwaj, Manjeet Kaur

**Streszczenie** Artykuł poświęcony jest nieliniowej oscylacji satelity w eliptycznej orbicie wokół Ziemi pod wpływem grawitacji i aerodynamicznego momentu obrotowego. Przyjmuje się, że płaszczyzna orbity pokrywa się z płaszczyzną równikową Ziemi. Po zastosowaniu metody nieliniowych oscylacji Bogoliubova–Krylov–Mitropolsky’ego (BKM) obserwujemy, że amplituda oscylacji jest stała przy aproksymacji rzędu drugiego. Ilustracje szeregów czasowych w przestrzeni fazowej 2- i 3-wymiarowej wykonano z wykorzystaniem procedur zaimplementowanych w MatLabe. Istnienie głównej składowej rezonansu i składowych parametrycznych wyjaśnia chaotyczny charakter częstotliwości.


Słowa kluczowe: moment aerodynamiczny • moment grawitacyjny • oscylacje nieliniowe • oscylacje swobodne • rezonans.
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