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Coercive limits for a subclass of monotone constitutive equations in the theory of inelastic material behaviour of metals

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Abstract. We prove existence and uniqueness of strong global in time solutions for a subclass of monotone constitutive equations in the theory of inelastic material behaviour of metals without the coercivity assumption for the free energy function. We approximate noncoercive models by a sequence of coercive problems and prove the convergence result.

1. Introduction. Systems of equations describing inelastic deformation of metals consist of partial differential equations resulting from mechanical balance laws and of a set of so called constitutive equations, which are always dependent of the material under consideration. One way to derive the constitutive relations is to assume that there exists a finite set of internal variables, whose evolution uniquely determine the state of the material. Therefore in the literature we find constitutive equations in the form of ordinary (often nonlinear) differential equations or differential inclusions for the internal variables. Thus we study here a linear system of partial differential equations coupled with a nonlinear system of ordinary differential equations (or general differential inclusions). Mathematical analysis of such systems of equations began in the seventies with works of J.J. Moreau [21], B. Halphen and Nguyen Quoc Son [14], and G. Duvant and J.L. Lions [12]. Then the research was continued by many authors, for example P. Suquet [25], R. Temam [27], P. LeTallec [18]. All the above mentioned articles or books deal with special constitutive equations or with a “small” class thereof. H.D. Alber in [3] was the first to try to classify constitutive equations applied in the

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theory of inelastic material behaviour of metals. The author defined the class of constitutive relations "of monotone type", for which the natural mathematical tool to study such equations is the theory of evolution equations for monotone operators. With the theory of H. Brezis [6] H.D. Alber proved existence and uniqueness of strong, global in time solutions for all constitutive equations of monotone type provided the coercivity assumption for the free energy function. However there are many examples of constitutive equations for which the above assumption fails. This is the starting point of this article. We describe here a subclass of constitutive equations of monotone type without the coercivity assumption, which we can approximate by coercive problems of monotone type.

In Section 2 using the notations from [3] we formulate the initial-boundary-value problem and define the constitutive equations of monotone type. Section 3 presents the existence theory for models with constitutive equations of monotone type, provided that the free energy function is positive definite. All the results are obtained in [3]. In Section 4 we study a noncoercive example - the monotone model of Bodner-Partom. Using the ideas from Section 4 we define in Section 5 the class of self-controlling models and prove existence of global strong solutions for this class. In section 6 we define another coercive approximation process, which we use in Section 7 to study a noncoercive and nonself-controlling example. In Section 8 we give ideas to extend the method from Section 5.

2. Formulation of the initial-boundary value problem. In this section we give a precise formulation of the initial-boundary value problem, which we are studying, and define the class of constitutive equations of monotone type.

Let a solid body occupy a bounded domain \( \Omega \subseteq \mathbb{R}^3 \) with the smooth boundary \( \partial \Omega \). We denote by \( u(x, t) \in \mathbb{R}^3 \) the displacement vector of the mass point \( x \in \Omega \) at time \( t \geq 0 \). Let us denote by \( S^3 \) the set of symmetric \( 3 \times 3 \) matrices. The Cauchy stress tensor \( T(x, t) \) at the point \( x \in \Omega \) at time \( t \geq 0 \) belongs to \( S^3 \). The balance of momentum in the homogeneous case (with vanishing external forces) yields

\[
\rho u_t(x, t) = \text{div}_x T(x, t),
\]

where \( \rho > 0 \) is the density, which we assume to be constant. The subscripts \( t \) and \( x \) denote the partial derivatives with respect to \( t \) or \( x \). The fundamental idea to describe an inelastic deformation of a solid body with the theory of internal variables is to assume that there exists the inelastic strain tensor \( \varepsilon^p(x, t) \in S^3 \), such that the whole strain tensor \( \varepsilon(x, t) \in S^3 \) is given as the sum

\[
\varepsilon(x, t) = \varepsilon(x, t) - \varepsilon^p(x, t) + \varepsilon^p(x, t),
\]
where the difference $\varepsilon(x, t) - \varepsilon^p(x, t)$ should be represented only by the pure elastic deformation. Thus for the elastic part of the strain tensor we assume that the following linear constitutive relation holds

$$T(x, t) = D(\varepsilon(x, t) - \varepsilon^p(x, t)),$$

where $D : S^3 \rightarrow S^3$ is the elasticity tensor, which we assume to be constant, symmetric and positive definite. Moreover we use in this article the assumption of small deformations, which allows us to write that

$$\varepsilon(x, t) = \frac{1}{2} (\nabla u(x, t) + \nabla^T u(x, t)).$$

The partial differential equation (2.1) with the constitutive equation (2.3), where $\varepsilon$ is given by (2.4), describe with given $\varepsilon^p$ the classical theory of linear elasticity. Obviously the inelastic strain tensor is an unknown function, and to close our system of equations we should additionally derive the evolution of $\varepsilon^p$. Thus let us denote by $z(x, t) = (\varepsilon^p(x, t), \tilde{z}(x, t)) \in S^3 \times \mathbb{R}^{9}$ the vector of internal variables, where the first nine components are defined by the inelastic strain tensor and the other components of the vector $z$ describe various properties of the material under consideration, for example, isotropic or directional hardening of the metal. The evolution of the vector $z$ is given by the inelastic, or sometimes called plastic, constitutive equations

$$z_t(x, t) \in f(\varepsilon(x, t), z(x, t)),$$

where the multivalued function $f : D(f) \subset S^3 \times \mathbb{R}^N \rightarrow \mathcal{P}(\mathbb{R}^N)$ with domain $D(f) = \{ (\varepsilon, z) \in S^3 \times \mathbb{R}^N : f(\varepsilon, z) \neq \emptyset \}$ is given. Let us denote by $B$ the linear operator $B : \mathbb{R}^N \rightarrow S^3$ defined by the formula

$$Bz = \frac{1}{2} (z_p + z_p^T),$$

where $z_p$ is the $3 \times 3$ matrix formed by the first nine components of the vector $z$. Thus the system of equations in the theory of inelastic material behaviour of metals has the form

$$\rho u_{tt}(x, t) = \text{div}_x T(x, t),$$

$$(\text{GS})
\begin{align*}
T(x, t) &= D(\varepsilon(x, t) - Bz(x, t)), \\
z_t(x, t) &\in f(\varepsilon(x, t), z(x, t)).
\end{align*}$$

Moreover we assume that the homogeneous Dirichlet boundary condition

$$u(x, t) = 0 \quad \text{for } x \in \partial \Omega \text{ and } t \geq 0,$$

or the homogeneous Neumann boundary condition

$$T(x, t) n(x) = 0 \quad \text{for } x \in \partial \Omega \text{ and } t \geq 0$$

is satisfied, where $n(x)$ denotes the exterior unit normal vector to $\partial \Omega$ at
Finally the initial conditions are
\[
\begin{align*}
  u(x, 0) & = u^0(x), \\
  u_t(x, 0) & = u^1(x), \\
  z(x, 0) & = z^0(x)
\end{align*}
\]  

with given functions \( u^0, u^1 : \Omega \rightarrow \mathbb{R}^3 \), \( z^0 : \Omega \rightarrow \mathbb{R}^N \).

Before we start with mathematical analysis of the system (GS) we define a class of admissible inelastic constitutive equations. We require that there exists a function \( \psi : D(f) \subset S^3 \times \mathbb{R}^N \rightarrow \mathbb{R}_+ \) satisfying for all \( (\varepsilon, z) \in D(f) \):

\[
\begin{align*}
  (2.9) & \quad \rho \nabla_\varepsilon \psi(\varepsilon, z) = T, \\
  (2.10) & \quad \forall z^* \in f(\varepsilon, z) \quad \rho \nabla_z \psi(\varepsilon, z) \cdot z^* \leq 0.
\end{align*}
\]

The function \( \psi \) is called free energy and the existence of such a function follows from thermodynamic considerations. The inequality (2.10) restricts the class of multivalued functions \( f \) and, as we will see below, yields dissipation to the problem (GS). Therefore in the literature the inequality (2.10) is called the dissipation inequality. Let us assume that \( u : \Omega \times (0, \infty) \rightarrow \mathbb{R}^3 \), \( z : \Omega \times (0, \infty) \rightarrow \mathbb{R}^3 \) is a sufficiently smooth (so that the integrals and derivatives below are well defined) solution of the system (GS) satisfying the boundary condition (2.7) or (2.8) with the free energy \( \psi \) satisfying (2.9) and (2.10). Then we have

\[
\begin{align*}
  (2.11) & \quad \frac{d}{dt} \int_\Omega \rho \left\{ \frac{1}{2} |u_t(x, t)|^2 + \psi(\varepsilon(x, t), z(x, t)) \right\} \, dx \\
  & = \int_\Omega \left\{ \rho u_t(x, t) u_{tt}(x, t) + \rho \nabla_\varepsilon \psi(\varepsilon(x, t), z(x, t)) \varepsilon_t(x, t) \\
  & \quad + \rho \nabla_z \psi(\varepsilon(x, t), z(x, t)) z_t(x, t) \right\} \, dx \\
  & = \int_\Omega \left\{ \rho u_t(x, t) u_{tt}(x, t) + T(x, t) \cdot \nabla u_t(x, t) \\
  & \quad + \rho \nabla_z \psi(\varepsilon(x, t), z(x, t)) z_t(x, t) \right\} \, dx \\
  & = \int_{\partial \Omega} T(x, t) \cdot n(x) \cdot u_t(x, t) \, dS(x) \\
  & \quad + \int_\Omega \rho \nabla_z \psi(\varepsilon(x, t), z(x, t)) z_t(x, t) \, dx \leq 0
\end{align*}
\]

which denotes that the sum of the kinetic energy and the free energy do not increase in time. This dissipation effect is naturally a consequence of existence of a function \( \psi \) satisfying (2.9) and (2.10). Therefore we should describe all such functions. The requirement (2.9), the constitutive equation
Coercive limits for a subclass of monotone constitutive equations

(2.12) \[ \rho \psi(\varepsilon, z) = \frac{1}{2} [\mathcal{D}(\varepsilon - Bz)](\varepsilon - Bz) + \psi_1(z). \]

Thus necessarily \( \psi \) must be quadratic in the variable \( \varepsilon - Bz \), and the function \( \psi_1 \) should be chosen so, that (2.10) holds. But it is not so easy to describe all functions \( \psi \), which satisfy the dissipation inequality (2.10). Therefore in this article we restrict our considerations only to a class of multivalued functions \( f \) for which (2.10) is always satisfied.

Let us assume that there exists a monotone multifunction \( g : D(g) \subset \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N) \) with \( 0 \in g(0) \), such that

(2.13) \[ f(\varepsilon, z) = g(-\rho \nabla_z \psi(\varepsilon, z)) \]

where \( \psi \) is the free energy. To see that such functions \( f \) satisfy (2.10), we obtain from the monotonicity of \( g \)

\[ \forall z^* \in f(\varepsilon, z) \quad \rho \nabla_z \psi(\varepsilon, z) \cdot z^* = -(-\rho \nabla_z \psi(\varepsilon, z) - 0) \cdot (z^* - 0) \leq 0 \]

while \( z^* \in g(-\rho \nabla_z \psi(\varepsilon, z)) \) and \( 0 \in g(0) \).

Next we want to obtain a similar inequality to (2.11) (dissipation of energy) for the difference of two solutions of the problem (GS) with \( f \) satisfying (2.13). This can be done if we additionally assume that the free energy function \( \psi \) is a positive semi-definite quadratic form in all variables. Thus \( \psi \) must be of the form

(2.14) \[ \psi(\varepsilon, z) = \frac{1}{2} [\mathcal{D}(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2} (Lz) \cdot z, \]

where \( L \) is a symmetric positive semi-definite \( N \times N \) matrix. Then the gradient of \( \psi \) with respect to \( z \) is linear and has the form

\[ -\rho \nabla_z \psi(\varepsilon, z) = B^T \mathcal{D}(\varepsilon - Bz) - Lz \overset{df}{=} \bar{L} \varepsilon - Mz \]

with linear mappings \( \bar{L} = B^T \mathcal{D} \) and \( M = B^T DB + L \). Note that the assumption (2.14) restricts again the class of functions \( f \), which we consider here.

Let us assume that \( (u, z) \) and \( (\bar{u}, \bar{z}) \) are two sufficiently smooth solutions of the problem (GS) for the same initial data satisfying the boundary condition (2.7) or (2.8) with the multivalue function \( f \), for which (2.13) holds and with the free energy \( \psi \) satisfying (2.9), (2.10) and (2.14). By analogous computation to the one that led to the inequality (2.11) we obtain

(2.15) \[ \frac{d}{dt} \int_{\Omega} \rho (\frac{1}{2}|u_t - \bar{u}_t|^2 + \psi(\varepsilon - \bar{\varepsilon}, z - \bar{z})) dx = - \int_{\Omega} [(-\rho \nabla_z \psi(\varepsilon, z) + \rho \nabla_z \psi(\bar{\varepsilon}, \bar{z})) \cdot (z_t - \bar{z}_t) dx \leq 0 \]
where we used the linearity of the gradient $\nabla z \psi$ and the monotonicity of the function $g$. The free energy function $\psi$ is positive semi-definite, hence (2.15) yields

\begin{equation}
\int_{\Omega} |(u_t - \bar{u}_t)|^2 \, dx = 0.
\end{equation}

This denotes the uniqueness of the velocity vector field $u_t = \bar{u}_t$ and consequently the uniqueness of the displacement $u = \bar{u}$ and of the strain $\varepsilon = \bar{\varepsilon}$. With this information we use again (2.15) and have

\[ 0 = \int_{\Omega} \rho \psi(0, z - \bar{z}) \, dx = \int_{\Omega} \frac{1}{2} [(B^T DB + L)(z - \bar{z})] \cdot (z - \bar{z}) \, dx = \int_{\Omega} \frac{1}{2} [M(z - \bar{z})] \cdot (z - \bar{z}) \, dx. \]

Thus we see that the uniqueness of the vector can be proved, provided that the $N \times N$ matrix $M$ is positive definite. This statement will give reasons for the definition of the constitutive equations of monotone type. We note additionally that the system (GS) can be studied with a general linear operator $B : \mathbb{R}^N \to S^3$. Then we are ready to define the class of constitutive equations of monotone type.

**Definition 2.1.**

a) We say that a pair $(f, B)$ of a function $f : D(f) \subset S^3 \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ and a linear mapping $B : \mathbb{R}^N \to S^3$ is of **pre-monotone type**, if there exists a symmetric, positive definite $N \times N$ matrix $M$ with the property that the symmetric $N \times N$ matrix $L \overset{df}{=} M - B^T DB$ is positive semi-definite, and there exists a function $g : D(g) \subset \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ such that

\[ f(\varepsilon, z) = g(-\rho \nabla z \psi(\varepsilon, z)) = g(B^T D\varepsilon - Mz) \]

for all $(\varepsilon, z) \in D(f)$, with the positive semi-definite quadratic form

\[ \rho \psi(\varepsilon, z) = \frac{1}{2} [D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2} (Lz) \cdot z. \]

b) If the function $g$ is monotone and satisfies

\[ \forall_{z \in D(g)} \forall_{w \in g(z)} \quad w \cdot z \geq 0, \]

then we say that the pair $(f, B)$ is of **monotone type**.

c) We say that the constitutive equations

\[ T = D(\varepsilon - Bz), \]

\[ z_t \in f(\varepsilon, z) \]

are of pre-monotone type (monotone type) if the pair $(f, B)$ is of pre-monotone type (monotone type).
Hence we have proved above, that smooth solutions of the problem (GS) with constitutive equations of monotone type satisfying the boundary condition (2.7) or (2.8) are unique.

3. Existence theory for constitutive equations of monotone type. In this section we show that for constitutive equations of monotone type not only uniqueness, but also existence of solutions can be proved. The proof of the existence theorem is based on the theory of evolution equations for monotone operators ([6]), and uses stronger assumptions for the free energy function $\psi$ and for the monotone multivalued function $g$. Namely, in this section we require that the quadratic form $\psi$ is positive definite and the function $g$ is maximal monotone with $D(g) = \mathbb{R}^N$. Note that a monotone operator $A : \mathcal{H} \to \mathcal{P} (\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called maximal monotone if it does not have a proper monotone extension ([6]). Thus we will write our initial-boundary value problem as an evolution problem to a maximal monotone operator. Let us set $v = u_t$. Then the initial-boundary value problem with constitutive equations of monotone type has the form

$$
\begin{align*}
&v_t(x,t) = \text{div} \frac{1}{\rho} \mathcal{D}(\varepsilon(x,t) - Bz(x,t)), \\
&\varepsilon_t(x,t) = \frac{1}{2} (\nabla v(x,t) + \nabla^T v(x,t)), \\
&z_t(x,t) \in g(-\rho \nabla \psi(\varepsilon(x,t), z(x,t))),
\end{align*}
$$

with the Dirichlet boundary condition

$$
v(x,t) = 0 \quad \text{for } x \in \partial \Omega
$$

or with the Neumann boundary condition

$$
\mathcal{D}(\varepsilon(x,t) - Bz(x,t)) \cdot n(x) = 0 \quad \text{for } x \in \partial \Omega
$$

and with the initial conditions

$$
v(x,0) = v^0(x), \quad \varepsilon(x,0) = \varepsilon^0(x), \quad z(x,0) = z^0(x),
$$

where $v^0(x) = u^1(x)$ and $\varepsilon^0(x) = \frac{1}{2} (\nabla u^0(x) + \nabla^T u^0(x))$. By $\mathbb{H}_m(\Omega; \mathbb{R}^k)$ we denote the usual Sobolev space of all functions in $L^2(\Omega; \mathbb{R}^k)$, which have weak derivatives in $L^2(\Omega; \mathbb{R}^k)$ up to order $m$. The norm and the scalar product in $\mathbb{H}_m(\Omega; \mathbb{R}^k)$ we denote by $\|\cdot\|_m$ and $(\cdot, \cdot)_m$ respectively. For $m = 0$ we use for simplicity the notation $\|\cdot\|_0 = \|\cdot\|$ and $(\cdot, \cdot)_0 = (\cdot, \cdot)$.

Let us define $W = \mathbb{R}^3 \times S^3 \times \mathbb{R}^N$ and an operator

$$
C : L^2(\Omega; W) \to \mathcal{P}(L^2(\Omega; W))
$$

by $C(v, \varepsilon, z) = \emptyset$ if $v \in L^2(\Omega; \mathbb{R}^3) \setminus \mathbb{H}_1(\Omega; \mathbb{R}^3)$ and
\[ C(v, \varepsilon, z) = \left\{ w = (w_1, w_2, w_3) \in L^2(\Omega; W) \mid w_1 = \frac{1}{\rho} \text{div} \mathcal{D}(\varepsilon - Bz), \right. \]
\[ w_2 = \frac{1}{2}(\nabla v + \nabla^T v), \quad w_3(x) \in g \left( -\rho \nabla_x \psi(\varepsilon(x), z(x)) \right) \quad \text{a.e. in } \Omega \}
\]

if \((v, \varepsilon, z) \in H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; S^3 \times \mathbb{R}^N)\).

Additionally for the Dirichlet boundary value problem we assume that
\[ v \in \hat{H}^1(\Omega; \mathbb{R}^3) \]
and for the Neumann boundary value problem, that the weak divergence \(\text{div} \mathcal{D}(\varepsilon - Bz)\) satisfies
\[ (\text{div} \mathcal{D}(\varepsilon - Bz), u) = -\left( \mathcal{D}(\varepsilon - Bz), \nabla u \right) \quad \forall u \in \hat{H}^1(\Omega; \mathbb{R}^3). \]

Thus the problem (MP) can be written in the form
\[ (3.1) \quad w_t(t) \in C(w(t)), \quad w(0) = (v^0, \varepsilon^0, z^0). \]

The goal of the section is to show that the operator \(-C\) is maximal monotone. This is the statement of the theorem below.

**Theorem 3.1.** Let the quadratic form
\[ \rho \psi(\varepsilon, z) = \frac{1}{2} [\mathcal{D}(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2} (Lz) \cdot z \]
be positive definite on \(S^3 \times \mathbb{R}^N\) and let \(g : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)\) be maximal monotone with \(0 \in g(0)\). Let us assume that, for the Dirichlet boundary condition, \(\Omega \subseteq \mathbb{R}^3\) be an open bounded set, and for the Neumann boundary condition, \(\Omega \subseteq \mathbb{R}^3\) be an open bounded set with Lipschitz boundary. Then the operator \(-C : L^2(\Omega; W) \to \mathcal{P}(L^2(\Omega; W))\) is maximal monotone if the space \(L^2(\Omega; W)\) is equipped with the scalar product
\[ (3.2) \quad \langle (v, \varepsilon, z), (\bar{v}, \bar{\varepsilon}, \bar{z}) \rangle = \int_\Omega \left\{ \rho v \cdot \bar{v} + [\mathcal{D}(\varepsilon - Bz)] \cdot (\varepsilon - B\bar{z}) + (Lz) \cdot \bar{z} \right\} dx. \]

The proof of the Theorem 3.1 is the main result of chapter 4 in the book [3]. Theorem 3.1 shows that the problem (MP) has a natural monotonicity property if we define the scalar product in \(L^2(\Omega; S^3 \times \mathbb{R}^N)\) by the form \(\psi\). This scalar product is equivalent to the standard scalar product, hence the topology defined by the product \(\langle \cdot, \cdot \rangle\) is the standard strong \(L^2\)-topology. It is clear that if \(\psi\) is only positive semi-definite then the monotonicity property of the operator \(-C\) fails.

**Definition 3.2.** We say that the problem (MP) is coercive if and only if the free energy function \(\psi\) connected with the problem is a positive definite quadratic form.

Thus Theorem 3.1 says that the coercive problems with constitutive equations of monotone type can be written in the form (3.1) with a monotone operator \(-C\), and if the right hand side of the inelastic constitutive
Coercive limits for a subclass of monotone constitutive equations

equations is maximal monotone, then the operator \(-C\) is maximal monotone too. Then by the theory presented in the books [4], [6] we have the following global in time existence of large solutions:

**Theorem 3.3.** Let the assumptions of Theorem 3.1 be satisfied. Then to every \(w^0 = (v^0, \varepsilon^0, z^0) \in D(C)\) there exists a unique function \(w \in W^{1,\infty}((0, \infty); L^2(\Omega; W))\) such that

1. \(w(t) \in D(C)\) for all \(t \geq 0\),
2. \(w_t(t) \in C(w(t))\) a.e. in \([0, \infty)\) and \(w(0) = w^0\),
3. \(w\) has everywhere a \(t\)-derivative from the right,
4. if \(w\) and \(\overline{w}\) are two solutions satisfying (i) – (ii), then
   \[
   ||w(t) - \overline{w}(t)||_{\langle \cdot, \cdot \rangle} \leq ||w(0) - \overline{w}(0)||_{\langle \cdot, \cdot \rangle}
   \]
   for all \(t \geq 0\), where \(||\cdot||_{\langle \cdot, \cdot \rangle}\) denotes the norm in \(L^2(\Omega; W)\) generated by the scalar product \(\langle \cdot, \cdot \rangle\).

Theorem 3.3 is the consequence of Theorem 3.1 and the general theory of evolution equations for monotone operators, which is presented, for example, in [4], [6]. All results of Sections 2 and 3 are obtained in the book [3]. In the next sections we want to give an answer to the question:

Have noncoercive problems global in time large solutions?

4. A noncoercive example. The monotone model of Bodner-Partom.

In this and in the next section we study only single-valued functions \(g : \mathbb{R}^N \to \mathbb{R}^N\). Thus the inelastic constitutive equations have now the form of ordinary differential equations. Note that if a single-valued function \(g^* : \mathbb{R}^N \to \mathbb{R}^N\) is monotone and continuous, then the single-valued operator \(g(z) = \{g^*(z)\}\) generated by \(g^*\) is maximal monotone. Before we start with the monotone model of Bodner-Partom we give a general approximation lemma used in this and in the next sections. Let us again state the problem (GS). We want to find functions \(v : \Omega \times [0, T) \to \mathbb{R}^3\), \(e : \Omega \times [0, T) \to S^3\), \(z : \Omega \times [0, T) \to \mathbb{R}^N\) satisfying the system of equations

\[
\begin{align*}
v_t &= \text{div} \frac{1}{\rho} (e - Bz), \\
e_t &= \frac{1}{2} (\nabla v + \nabla^T v), \\
z_t &= f(e, z)
\end{align*}
\]

and the boundary condition

\[
(4.1) \quad v(x, t)|_{\partial \Omega} = 0 \quad \text{or} \quad D(e(x, t) - Bz(x, t))|_{\partial \Omega} = 0,
\]

and the initial conditions

\[
(4.2) \quad v(0) = v^0, \quad e(0) = \varepsilon^0, \quad z(0) = z^0.
\]
We shall approximate this problem with a family of approximate problems

\[ v^k_t = \text{div} \frac{1}{\rho}(\mathcal{D}(\varepsilon^k - Bz^k)) , \]

\[ \varepsilon^k_t = \frac{1}{2}(\nabla v^k + \nabla^T v^k) , \]

\[ z^k_t = f^k(\varepsilon^k, z^k) , \]

(AS)

\[ v^k|_{\partial \Omega} = 0 \quad \text{or} \quad \mathcal{D}(\varepsilon^k - Bz^k)|_{\partial \Omega} = 0 , \]

\[ v^k(0) = v^0_k , \quad e^k(0) = e^0_k , \quad z^k(0) = z^0_k , \]

(4.3)

where \( k \in \mathbb{N} \) and the family of functions \( \{f^k\} \) approximate pointwise the function \( f \), which means that for all sequences \((\varepsilon^k, z^k)\) which converge pointwise for almost every \((x, t) \in \Omega \times [0, T)\) to \((\varepsilon, z)\), we have

\[ f^k(\varepsilon^k(x, t), z^k(x, t)) \xrightarrow{\text{pointwise a.e. in } \Omega \times [0, T)} f(\varepsilon(x, t), z(x, t)) . \]

Moreover let us suppose that \((v^k, \varepsilon^k, z^k) \rightarrow (v^0, \varepsilon^0, z^0)\) in \( L^2(\Omega; W) \).

Lemma 4.1. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded open set with smooth boundary \( \partial \Omega \), \((v^0, \varepsilon^0, z^0) \in L^2(\Omega; W)\) such that \((\text{div } \mathcal{D}(\varepsilon^0 - Bz^0), \frac{1}{2}(\nabla v^0 + \nabla^T v^0), f(\varepsilon^0, z^0)) \in L^2(\Omega; W)\) and \( v^0 \in H_1(\Omega) \) for the Dirichlet boundary value problem and with \((\text{div } \mathcal{D}(\varepsilon^0 - Bz^0), u) = -(\mathcal{D}(\varepsilon^0 - Bz^0), \nabla u) \ \forall u \in H_1(\Omega; \mathbb{R}^3)\) for the Neumann boundary condition. Let us assume that a sequence \((v^k, \varepsilon^k, z^k)\) of solutions for the problem (AS) with boundary condition (4.3) and initial conditions (4.4) satisfies

\[ \text{ess sup} \{||v^k(t)||, ||\varepsilon^k(t)||, ||z^k(t)||\} \leq C , \]

(4.5)

where the positive constant \( C \) does not depend on \( k \). Moreover suppose that for all \( \gamma > 0 \) there exists a natural number \( N(\gamma) \) such that

\[ \text{ess sup} \{||v^k(t) - v^m(t)|| + ||\varepsilon^k(t) - \varepsilon^m(t)|| + ||z^k(t) - z^m(t)||\} \leq \gamma \]

\[ \text{for } k, m \geq N(\gamma) . \]

Then there exists a solution \((v, \varepsilon, z) \in W^{1,\infty}((0, T); L^2(\Omega; W))\) of the problem (GS) satisfying the boundary condition (4.1) and the initial conditions (4.2).

Proof. By (4.6) we have that the sequence \((v^k, \varepsilon^k, z^k)\) converges in the strong topology in \( L^\infty((0, T); L^2(\Omega; W)) \) to the limit \((v, \varepsilon, z)\). Moreover (4.5) yields that there exists a subsequence \((v^{k_\ell}, \varepsilon^{k_\ell}, z^{k_\ell})\) of the sequence \((v^k, \varepsilon^k, z^k)\) which converges weak-* in \( L^\infty_w((0, T); L^2(\Omega; W)) \) to the limes
Coercive limits for a subclass of monotone constitutive equations

where \((v, \varepsilon, z)\) was defined above. This subsequence we denote again with \((v^k_k, \varepsilon^k_k, z^k_k)\). Thus we obtain

\[
\begin{align*}
\dot{v}_t &\overset{\ast}{=} v^k_k = \frac{1}{\rho} \text{div } D(\varepsilon^k_k - B z^k_k) \\
\dot{\varepsilon}_t &\overset{\ast}{=} \varepsilon^k_k = \frac{1}{2} (\nabla v^k_k + \nabla^T v^k_k) \\
\dot{z}_t &\overset{\ast}{=} z^k_k = f^k(\varepsilon^k_k, z^k_k)
\end{align*}
\]

in the space \(L^\infty_w((0,T);L^2(\Omega;W))\). Next we would like to show that \(\chi = f(\varepsilon, z)\). We can assume that the sequence \((\varepsilon^k, z^k)\) converges pointwise for almost every \((x,t)\in\Omega \times [0,T)\) to \((\varepsilon, z)\). (If it is not the case then there exists a subsequence with this property.) Hence using the approximation \(f^k \to f\) we obtain

\[
\begin{align*}
f^k(\varepsilon^k_k, z^k_k) &\overset{a.e.}{\to} f(\varepsilon, z) \\
f^k(\varepsilon^k_k, z^k_k) &\rightharpoonup \chi \quad \text{in } L^2((0,T) \times \Omega)
\end{align*}
\]

while the weak-* convergence in \(L^\infty_w((0,T);L^2(\Omega;W))\) implies weak convergence in \(L^2((0,T) \times \Omega))\). Thus the sequence \(f^k(\varepsilon^k, z^k)\) is bounded in \(L^2((0,T) \times \Omega)\) and uniformly integrable in \(L^p((0,T) \times \Omega)\) for \(1 < p < 2\):

Let \(E \subset (0,T) \times \Omega\) be measurable, then

\[
\sup_k \left\{ \int_E |f^k(\varepsilon^k_k, z^k_k)|^p \, dx \, dt \right\} \leq \left| E \right|^{\frac{2-p}{2}} \sup_k \left( \int_0^T \left( \int_\Omega |f^k(\varepsilon^k_k, z^k_k)|^2 \, dx \right) \, dt \right)^{p/2} \to 0
\]

for \(\left| E \right| \to 0\). Hence by the Vitali theorem we conclude

\[
f^k(\varepsilon^k_k, z^k_k) \to f(\varepsilon, z) \quad \text{in } L^p((0,T) \times \Omega) \quad \text{for } 1 < p < 2.
\]

Moreover \(f^k(\varepsilon^k_k, z^k_k) \rightharpoonup \chi \) in \(L^p((0,T) \times \Omega)\) for \(1 < p < 2\), then \(\chi = f(\varepsilon, z)\) a.e. in \(\Omega \times [0,T)\). To end the proof we must show that the limit \((v,\varepsilon, z)\) satisfies the boundary condition (4.1) and the initial conditions (4.2). For the Dirichlet boundary value problem by the Korn inequality we have \(v^k_k(t) \in H^1_0(\Omega)\) for a.e. \(t \in [0,T)\). Moreover the sequence \(v^k_k(t)\) is uniformly bounded with respect to \(k\) in \(H^1_0(\Omega)\), hence we can assume \(v^k_k(t) \rightharpoonup v(t)\) in \(H^1_0\)-topology for a.e. \(t \in [0,T)\). The space \(H^1_0(\Omega)\) is weakly closed in \(H^1_0(\Omega)\), thus \(v(t) \in H^1_0(\Omega)\) for a.e. \(t \in [0,T)\). For the Neumann boundary condition we have for a.e. \(t \in [0,T)\) and for all \(k\)

\[
(\text{div } D(\varepsilon^k_k(t) - B z^k_k(t)), u) = -(\text{div } D(\varepsilon^k_k(t) - B z^k_k(t)), \nabla u) \quad \forall u \in H^1_0(\Omega;\mathbb{R}^3).
\]

---

\(^{1}\) \(y^n \overset{\ast}{\rightharpoonup} y \) in \(L^p_w((0,T);L^2(\Omega)) \iff \forall \phi \in L^1(0,T) \forall \psi \in L^2(\Omega)
\]

\[
\int_0^T \left( \int_\Omega y^n \phi \, dx \right) \phi \, dt \to \int_0^T \left( \int_\Omega y \psi \, dx \right) \phi \, dt
\]
The sequence \( \text{div} \mathcal{D}(\varepsilon^k(t) - Bz^k(t)) \) converges weakly to \( \text{div} \mathcal{D}(\varepsilon(t) - Bz(t)) \) in \( L^2(\Omega; \mathbb{R}^3) \) for a.e. \( t \in [0, T) \) and the sequence \( \mathcal{D}(\varepsilon^k(t) - Bz^k(t)) \) converges in the strong topology to \( \mathcal{D}(\varepsilon(t) - Bz(t)) \) in \( L^2(\Omega; S^3) \) for a.e. \( t \in [0, T) \). Thus for a.e. \( t \in [0, T) \)

\[
(\text{div} \mathcal{D}(\varepsilon(t) - Bz(t)), u) = - (\mathcal{D}(\varepsilon(t) - Bz(t)), \nabla u) \quad \forall u \in H_0(\Omega; \mathbb{R}^3).
\]

We see that the limit \((v, \varepsilon, z) \in W^{1,\infty}((0, T); L^2(\Omega; W))\), hence the trace operator \((v, \varepsilon, z) \to (v|_{t=0}, \varepsilon|_{t=0}, z|_{t=0})\) is well defined. The sequence \( (v^k, \varepsilon^k, z^k) \sim (v, \varepsilon, z) \) in \( W^{1,\infty}((0, T); L^2(\Omega; W)) \) then the weak continuity of the trace operator yields

\[
(v^k|_{t=0}, \varepsilon^k|_{t=0}, z^k|_{t=0}) \rightarrow (v|_{t=0}, \varepsilon|_{t=0}, z|_{t=0})
\]

in \( L^2(\Omega; W) \). The sequence \( (v^k|_{t=0}, \varepsilon^k|_{t=0}, z^k|_{t=0}) \) converges to \((v^0, \varepsilon^0, z^0)\) in \( L^2(\Omega; W) \) then \((v|_{t=0}, \varepsilon|_{t=0}, z|_{t=0}) = (v^0, \varepsilon^0, z^0)\).}

Now we are ready to study noncoercive problems with constitutive equations of monotone type. The main idea is to approximate such problems by a family of coercive problems, for which Theorem 3.1 holds. We start with the monotone model of Bodner-Partom. In this model the vector \( z \) of internal variables consists of the six components of the inelastic strain tensor \( \varepsilon^p \), and of one positive variable \( y \), which describes isotropic hardening. Thus \( N = 7 \) and \( z(x, t) = (\varepsilon^p(x, t), y(x, t)) \) with \( y(x, t) > 0 \). The inelastic constitutive equations are in the form

\[
\varepsilon_t^p = F\left(\frac{\sigma}{y}\right) \frac{\sigma}{|\sigma|},
\]

\[
y_t = \frac{\sigma}{y} F'\left(\frac{\sigma}{y}\right),
\]

where \( F : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), is a given function and \( \sigma = T - \frac{1}{3} (\text{tr} T) \cdot I \) is the stress deviator. It is proved in [3], that the model is of monotone type only for \( F(s) = ds^\beta \), \( d > 0 \), \( \beta > 1 \). Therfore the system of equations for the monotone model of Bodner-Partom has the form

\[
\left\{
\begin{array}{l}
\rho v_t = \text{div} \mathcal{D}(\varepsilon - \varepsilon^p), \\
\varepsilon_t = \frac{1}{2} (\nabla v + \nabla^T v), \\
\varepsilon_t^p = d\left(\frac{\sigma}{y}\right)^\beta \frac{\sigma}{|\sigma|}, \\
y_t = d\left(\frac{\sigma}{y}\right)^\beta+1
\end{array}
\right.
\]

(MBP)

with the free energy

\[
\rho \psi(\varepsilon, \varepsilon^p, y) = \frac{1}{2} [\mathcal{D}(\varepsilon - \varepsilon^p)] \cdot (\varepsilon - \varepsilon^p) + \frac{\beta}{2(\beta + 1)} y^2.
\]
The simple calculation
\[
\rho \nabla (\varepsilon^p, y) \psi(\varepsilon, \varepsilon^p, y) \cdot (\varepsilon^p, y_t)
= -D(\varepsilon - \varepsilon^p) \cdot d\left(\frac{|\sigma|}{y}\right)^\beta \frac{|\sigma|}{|\sigma| + \frac{\beta}{\beta + 1} y} \cdot d\left(\frac{|\sigma|}{y}\right)^{\beta + 1}
= -d\left(\frac{|\sigma|}{y}\right)^\beta |\sigma| + \frac{\beta}{\beta + 1} d\left(\frac{|\sigma|}{y}\right)^\beta \cdot |\sigma| \leq 0
\]
shows that the dissipation inequality is satisfied. Moreover the operator
\[L(\varepsilon^p, y) = (0, \frac{\beta}{\beta + 1} y)\] is only positive semi-definite and the operator
\[M(\varepsilon^p, y) = (B^T DB + L)(\varepsilon^p, y) = (D\varepsilon^p, \frac{\beta}{\beta + 1} y)\] is positive definite. Thus the problem (MBP) is noncoercive. We want now to approximate this noncoercive model with a sequence of coercive models. First of course, we should choose a "good" approximation process. Here the mechanical analysis helps us to solve the problem. The system (MBP) is connected with the so-called rheological diagram:

which shows that the properties of the metal under consideration are described by a connection in series of a pure elastic element (the spring) with an inelastic element (the daphot). The coercivity of the problem fails because the inelastic deformation is fully irreversible. This statement yields the main idea in the theory of coercive limits. We will try to approximate the problem (MBP) with coercive models, for which the rheological diagram is given by the figure below:
The diagram shows that the stress deviator $\sigma_k$, responsible for the inelastic deformation, is now "reduced" by the stress from the inserted spring. Therefore we require that the approximation satisfies the system:

\[ \rho v^k_t = \text{div} \mathcal{D}(\varepsilon^k - \varepsilon^{p,k}), \]

\[ \varepsilon^{p,k}_t = \frac{1}{2}(\nabla v^k + \nabla^T v^k), \]

(CBP)

\[ \varepsilon^{p,k}_t = d\left(\frac{|\sigma^k - \frac{1}{k} \sigma^{p,k}|}{y^k}\right) \beta \frac{\sigma^k - \frac{1}{k} \sigma^{p,k}}{y^k}, \]

\[ y^k_t = d\left(\frac{|\sigma^k - \frac{1}{k} \sigma^{p,k}|}{y^k}\right)^{\beta + 1}. \]

(here $\sigma^{p,k} = \mathcal{D}\varepsilon^{p,k} - \frac{1}{3}(\text{tr} \mathcal{D}\varepsilon^{p,k}) \cdot I$)

and the homogeneous Dirichlet or Neumann boundary condition, and initial conditions

\[ v^k(0) = v^0_k, \quad \varepsilon^k(0) = \varepsilon^0_k, \quad \varepsilon^{p,k}(0) = \varepsilon^{p,0}_k, \quad y^k(0) = y^0_k, \]

where the initial data should be suitably chosen.

**Definition 4.2.** We say that the initial data $(v^0, \varepsilon^0, \varepsilon^{p,0}, y^0) \in L^2(\Omega; W)$ is admissible for the system (MBP) $\Leftrightarrow$

\[ \text{tr} \varepsilon^{p,0} = 0 \quad \text{a.e. in } \Omega, \quad y^0 \geq \alpha > 0 \quad \text{a.e. in } \Omega, \]

\[ (\text{div} \mathcal{D}(\varepsilon^0 - \varepsilon^{p,0}), \frac{1}{2}(\nabla v^0 + \nabla^T v^0), (\frac{|\sigma^0|}{y^0})^\beta \sigma^0, (\frac{|\sigma^0|}{y^0})^{\beta + 1}) \in L^2(\Omega; W) \]

where $\sigma^0 = T^0 - \frac{1}{3}(\text{tr} T^0)I$ and $T^0 = D(\varepsilon^0 - \varepsilon^{p,0})$, and additionally for the Dirichlet boundary condition it holds that:

\[ v^0 \in \hat{H}_1(\Omega; \mathbb{R}^3), \]

and for the Neumann boundary condition we require that:

\[ \forall u \in \hat{H}_1(\Omega; \mathbb{R}^3) \quad (\text{div} \mathcal{D}(\varepsilon^0 - \varepsilon^{p,0}), u) = -(\mathcal{D}(\varepsilon^0 - \varepsilon^{p,0}), \nabla u). \]

**Remark.** We say that the initial data (4.7) is admissible for the system (CBP) if and only if the data belongs to the domain of the operator on the right hand side of the system (CBP) (a similar condition to (4.9)) and satisfies (4.8) and (4.10) or (4.11), respectively.

**Lemma 4.3.** Let the initial data $(v^0, \varepsilon^0, \varepsilon^{p,0}, y^0) \in L^2(\Omega; W)$ be admissible for the system (MBP), $\text{div} \mathcal{D}e^0 \in L^2(\Omega; \mathbb{R}^3)$ and for the Neumann boundary value problem $\mathcal{D}e^0(x) \cdot n(x)|_{x \in \partial \Omega} = 0$, then $(v^0, \varepsilon^0, \frac{k}{k+1} \varepsilon^{p,0}, y^0)$ is admissible for the system (CBP).
Proof. Let us denote by $P$ the linear operator $P : S^3 \to S^3$ defined by $PT = T - \frac{1}{3} (\text{tr} T) \cdot I$. Hence using this notation we have $\sigma^k = P \mathcal{D}(\varepsilon^k - \varepsilon^{p,k})$ and $\sigma^{p,k} = P \mathcal{D}(\varepsilon^{p,k})$. Then for the initial data the difference

$$\sigma^k(0) - \frac{1}{k} \sigma^{p,k}(0) = P \mathcal{D}(\varepsilon^k(0) - \varepsilon^{p,k}(0)) - \frac{1}{k} P \mathcal{D}(\varepsilon^{p,k}(0))$$

$$= P \mathcal{D}(\varepsilon^0 - \left(1 + \frac{1}{k}\right) \cdot \frac{k}{k+1} \varepsilon^{p,0}) = P \mathcal{D}(\varepsilon^0 - \varepsilon^{p,0})$$

does not depend on $k$, and the right hand side of the inelastic constitutive equation in the system (CBP) is well defined for the initial data $(v^0, \varepsilon^0, \frac{k}{k+1}, \varepsilon^{p,0}, y^0)$. Moreover the weak divergence

$$\text{div} \mathcal{D}(\varepsilon^k(0) - \varepsilon^{p,k}(0)) = \text{div} \left(\varepsilon^0 - \frac{k}{k+1} \varepsilon^{p,0}\right) =$$

$$= \frac{k}{k+1} \text{div} \mathcal{D}(\varepsilon^0 - \varepsilon^{p,0}) + \frac{1}{k+1} \text{div} \mathcal{D}\varepsilon^0$$

belongs to $L^2(\Omega; \mathbb{R}^3)$ while $\text{div} \mathcal{D}\varepsilon^0 \in L^2(\Omega; \mathbb{R}^3)$. Finally for the Neumann boundary condition the assumption $\mathcal{D}\varepsilon^0 \cdot n|_{\partial \Omega} = 0$ yields (4.11) for the data $(\varepsilon^0, \frac{k}{k+1} \varepsilon^{p,0})$.

**Lemma 4.4.** Let $(v^0, \varepsilon^0, \varepsilon^{p,0}, y^0) \in L^2(\Omega; W)$ be admissible for the system (MBP), $\text{div} \mathcal{D}\varepsilon^0 \in L^2(\Omega; \mathbb{R}^3)$ and for the Neumann boundary problem $\mathcal{D}\varepsilon^0 \cdot n|_{\partial \Omega} = 0$. Then for all $k \in \mathbb{N}$ the problem (CBP) has a unique, strong, global in time solution $(v^k, \varepsilon^k, \varepsilon^{p,k}, y^k)$ to the initial data $(v^0, \varepsilon^0, \frac{k}{k+1} \varepsilon^{p,0}, y^0)$.

**Proof.** It is easy to see that the problem (CBP) possesses the free energy function of the form

$$\rho \psi^k(\varepsilon^k, \varepsilon^{p,k}, y^k) = \frac{1}{2} [\mathcal{D}(\varepsilon^k - \varepsilon^{p,k})] \cdot (\varepsilon^k - \varepsilon^{p,k})$$

$$+ \frac{\beta}{2(\beta + 1)} (y^k)^2 + \frac{1}{2k} (\mathcal{D}\varepsilon^{p,k}) \cdot \varepsilon^{p,k}.$$ 

Thus the quadratic form $\psi^k$ is positive definite and the operator $L$ is defined by

$$L(\varepsilon^{p,k}, y^k) = \left(\frac{1}{k} \mathcal{D}\varepsilon^{p,k}, \frac{\beta}{\beta + 1} y^k\right).$$

Moreover the operator

$$M(\varepsilon^{p,k}, y^k) = (B^T \mathcal{D} B + L)(\varepsilon^{p,k}, y^k) = \left(\mathcal{D}\left(1 + \frac{1}{k}\right) \varepsilon^{p,k}, \frac{\beta}{\beta + 1} y^k\right)$$

is positive definite. Then the problem (CBP) is coercive and of monotone type (compare with the monotone problem (MBP)). The inelastic constitutive equations are of the form $g(-\rho \nabla_z \psi(\varepsilon, z))$ with a monotone func-
This function has a maximal monotone extension \( \tilde{g} \) with the property \( 0 \in \tilde{g}(0) \). Moreover for admissible initial data all solutions must belong to \( D(g) \) and \( \text{tr} \varepsilon^p(x,t) = 0 \) for all \( t \) and for a.e. \( x \in \Omega \). If \( D(\tilde{g}) = \mathbb{R}^N \) then we can use Theorem 3.1 and if \( D(\tilde{g}) \neq \mathbb{R}^N \) then the results from Section 8 allow us to use Theorem 3.3. (4.2) yields that the initial data \( (v^0, \varepsilon^0, \frac{k}{k+1} \varepsilon^p, 0, y^0) \in D(C) \), where \( C \) is the operator from Theorem 3.1.

Theorem 3.1 and Theorem 3.3 complete the proof. ■

Next we will study the limit case \( k \to \infty \). This denotes that the inserted spring becomes weaker and weaker, and we can expect that on finite time intervals the sequence \( (v^k, \varepsilon^k, \varepsilon^p, y^k) \) converges to a solution of the problem (MBP).

**Theorem 4.5.** Let \( T > 0 \) be fixed and \( (v^0, \varepsilon^0, \varepsilon^p, 0, y^0) \in L^2(\Omega; W) \) be admissible for (MBP) with \( \text{div} \varepsilon^0 \in L^2(\Omega, \mathbb{R}^3) \) and for the Neumann boundary problem \( \varepsilon^0 \cdot n \big|_{\partial \Omega} = 0 \). The approximate sequence \( (v^k, \varepsilon^k, \varepsilon^p, y^k) \) from Lemma 4.4 satisfies

\[
\begin{align*}
(4.12) \quad & (i) \quad \mathop{\text{ess sup}}_{t \in [0,T]} \{ ||v^k(t)||, ||\varepsilon^k(t)||, ||\varepsilon^p(t)||, ||y^k(t)|| \} \leq C \\
& \text{and the positive constant } C \text{ does not depend on } k \text{ and } T ,
\end{align*}
\]

\[
\begin{align*}
(4.13) \quad & (ii) \quad \forall \gamma > 0 \exists N(\gamma) \quad \mathop{\text{ess sup}}_{t \in [0,T]} \{ ||v^k(t) - v^m(t)|| \\
& + ||\varepsilon^k(t) - \varepsilon^p(t) - \varepsilon^p(t) + \varepsilon^m(t)|| \\
& + ||y^k(t) - y^m(t)|| \} \leq \gamma \quad \text{for } k, m \geq N(\gamma) .
\end{align*}
\]

**Proof.** By the existence theorem for coercive monotone problems we have for all \( k > 0 \)

\[
(v^k, \varepsilon^k, \varepsilon^p, y^k) \in W^{1,\infty}((0,T); \mathcal{H}_k) ,
\]

where \( \mathcal{H}_k \) denotes the space \( L^2(\Omega, W) \) equipped with the scalar product defined by the usual \( L^2 \)-scalar product for the first component \( v \) and by the free energy \( \psi^k \) for the other components. Let us denote by \( C^k \) the operator from Theorem 3.1 associated with the system (MBP), then Theorem 3.3 yields

\[
(4.14) \quad \mathop{\text{ess sup}}_{t \in [0,T]} ||(v^k_t(t), \varepsilon^k_t(t), \varepsilon^p, y^k(t))||_{\mathcal{H}_k} \\
\leq \left\| C^k \left( v^0, \varepsilon^0, \frac{k}{k+1} \varepsilon^p, 0, y^0 \right) \right\|_{\mathcal{H}_k}.
\]

Inequality (4.14) says that the left hand side of the system (MBP) is bounded in the space \( L^\infty((0,T), \mathcal{H}_k) \) by the right hand side evaluated for the initial data. This is the classical result from the theory of evolution.
equations for monotone operators. First we will prove that the right hand side of (4.14) is bounded:

\[(4.15) \| C^k \left( \varepsilon^0, \varepsilon^0, \frac{k}{k+1} \varepsilon^{p,0}, y^0 \right) \| \leq C \left\{ \text{div} \mathcal{D} \left( \varepsilon^0 - \frac{k}{k+1} \varepsilon^{p,0} \right) \right. \]
\[+ \frac{1}{2} \left( \nabla \varepsilon^0 + \nabla T \varepsilon^0 \right) - d \left( \frac{\sigma^0}{y^0} \right)^\beta \]
\[+ \left. \left\| d \left( \frac{\sigma^0}{y^0} \right)^{\beta+1} \right\| + \frac{1}{\sqrt{k}} \left\| d \left( \frac{\sigma^0}{y^0} \right)^\beta \right\| \right\} ,\]

where \( \sigma^0 = P \mathcal{D} (\varepsilon^0 - \varepsilon^{p,0}) \) and \( P \) is the projector from Lemma 4.3. It is easy to see that for the initial data \( (v^0, \varepsilon^0, \varepsilon^{p,0}, y^0) \) admissible for the system (MBP) with \( \text{div} \mathcal{D} \varepsilon^0 \in L^2(\Omega; \mathbb{R}^3) \), there exists a positive constant \( C \) independent of \( k \) and \( T \) such that

\[ \text{ess sup}_{t \in [0,T]} \left\{ \| u^k(t) \|, \| \varepsilon^k(t) - \varepsilon^{p,k}(t) \|, \| y^k(t) \|, \frac{1}{\sqrt{k}} \| \varepsilon^{p,k}(t) \| \right\} \leq C . \]

To end the proof of (i) we must only show that the sequence \( \varepsilon^{p,k}(t) \) is bounded in \( L^\infty((0,T); L^2(\Omega; S^3)) \). We can do this using the information that the sequence \( y^k(t) \) is bounded in \( L^\infty((0,T); L^2(\Omega; R)) \) and the special form of the system (MBP).

\[(4.16) \| \varepsilon^{p,k}(t) \|^2 = d^2 \int_\Omega \left( \frac{|\sigma^k(t) - \frac{1}{k} \sigma^{p,k}(t)|}{y^k(t)} \right)^{2\beta} dx \]
\[\leq d^2 \left\{ \int_\Omega \left( \frac{|\sigma^k(t) - \frac{1}{k} \sigma^{p,k}(t)|}{y^k(t)} \right)^{2\beta+2} dx \right\}^{\frac{\beta}{\beta+1}} \cdot |\Omega|^{\frac{1}{\beta+1}} \]
\[= d^{\beta+1} \| y^k(t) \|^\frac{2\beta}{\beta+1} \cdot |\Omega|^{\frac{1}{\beta+1}} . \]

Inequality (4.16) shows that we can control the right hand side of the equation for the inelastic deformation by the right hand side of the equation for the isotropic hardening. Then (4.16) completes the proof of (i).

To prove (ii) we use the energy method. Let us define by

\[ E^{k,m}(t) = \frac{\rho}{2} \| v^k(t) - v^m(t) \|^2 + \frac{1}{2} \int_\Omega \left[ \mathcal{D}(\varepsilon^k(t) - \varepsilon^m(t)) - \varepsilon^{p,k}(t) + \varepsilon^{p,m}(t) \right] \]
\[\cdot (\varepsilon^k(t) - \varepsilon^m(t) - \varepsilon^{p,k}(t) + \varepsilon^{p,m}(t)) dx + \frac{\beta}{2(\beta+1)} \| y^k(t) - y^m(t) \|^2 \]

the energy for the difference \( (v^k, \varepsilon^k, \varepsilon^{p,k}, y^k) - (v^m, \varepsilon^m, \varepsilon^{p,m}, y^m) \). Then for the time derivative of the function \( E^{k,m}(t) \) (the derivative is well defined)
we have

\[
\frac{d}{dt} \mathcal{E}^{k,m}(t) = \int_{\Omega} \rho(v^k(t) - v^m(t)) \cdot (v^k_t(t) - v^m_t(t)) \, dx \\
+ \int_{\Omega} \left[ D(e^k(t) - e^m(t) - e^{p,k}(t) + e^{p,m}(t)) \right] \\
\cdot (e^k_t(t) - e^m_t(t) - e^{p,k}_t(t) + e^{p,m}_t(t)) \, dx \\
+ \frac{\beta}{\beta + 1} \int_{\Omega} (y^k(t) - y^m(t)) \cdot (y^k_t(t) - y^m_t(t)) \, dx
\]

= \int_{\Omega} \rho(v^k(t) - v^m(t)) \cdot (v^k_t(t) - v^m_t(t)) \, dx \\
- \int_{\Omega} \left( \text{div} D(e^k(t) - e^m(t) - e^{p,k}(t) + e^{p,m}(t)) \right) \cdot (v^k(t) - v^m(t)) \, dx \\
- \int_{\Omega} (\sigma^k(t) - \sigma^m(t)) (e^{p,k}_t(t) - e^{p,m}_t(t)) \, dx \\
+ \frac{\beta}{\beta + 1} \int_{\Omega} (y^k(t) - y^m(t)) (y^k_t(t) - y^m_t(t)) \, dx.
\]

We have used here that \( \text{tr} \ e^{p,m}_t(t) = \text{tr} \ e^{p,m}_t(t) = 0 \) and that \((v^k, e^k, e^{p,k}, y^k)\) and \((v^m, e^m, e^{p,m}, y^m)\) satisfy homogeneous boundary conditions. By equation (CBP1) we obtain

\[
(4.17) \quad \frac{d}{dt} \mathcal{E}^{k,m}(t) \\
=: - \int_{\Omega} \left( \frac{1}{k} \sigma^{p,k}(t) - \sigma^m(t) + \frac{1}{m} \sigma^{p,m}(t) \right) (e^{p,k}_t(t) - e^{p,m}_t(t)) \, dx \\
- \int_{\Omega} \left( \frac{1}{k} D e^{p,k}(t) - \frac{1}{m} D e^{p,m}(t) \right) \cdot (e^{p,k}_t(t) - e^{p,m}_t(t)) \, dx \\
+ \frac{\beta}{\beta + 1} \int_{\Omega} (y^k(t) - y^m(t)) (y^k_t(t) - y^m_t(t)) \, dx
\]

(monotonicity of (BP))

\[
\leq - \int_{\Omega} \left( \frac{1}{k} D e^{p,k}(t) - \frac{1}{m} D e^{p,m}(t) \right) (e^{p,k}_t(t) - e^{p,m}_t(t)) \, dx
\]

\[
\leq C \cdot \left( \frac{1}{m} + \frac{1}{k} \right) (\|e^{p,k}(t)\|^2 + \|e^{p,m}(t)\|^2 + \|e^{p,k}_t(t)\|^2 + \|e^{p,m}_t(t)\|^2).
\]

The uniformly boundedness of the sequence \(|e^{p,k}_t(t)|\) implies for fixed \( T \), \(|e^{p,k}(t)|\) \( C(T) \) where the constant \( C(T) \) does not depend on \( k \). (Note that the sequence of initial data \( e^{p,k}(0) = 1_{k+1} e^{p,0} \) is bounded in \( L^2(\Omega; S^3) \).)
Then there exists a positive constant $C'(T)$ independent of $k$ and $m$ such that

$$E^{k,m}(t) \leq E^{k,m}(0) + C'(T) \left( \frac{1}{m^2} + \frac{1}{k} \right) \cdot T$$

Thus the right hand side of (4.18) for fixed $T$ can be arbitrarily small for sufficiently large $k$ and $m$. ■

**Corollary 4.6.** For initial data $(v^0, \varepsilon^0, \varepsilon^{p,0}, y^0) \in L^2(\Omega; W)$ satisfying assumptions of Theorem 4.5 there exists a subsequence $(v^{k_t}, \varepsilon^{k_t}, \varepsilon^{p,k_t}, y^{k_t})$ of the sequence $(v^k, \varepsilon^k, \varepsilon^{p,k}, y^k)$ from Lemma 4.4, which converges weak-* in the space $W^{1,\infty}_w((0,T); L^2(\Omega; W))$ to a solution $(v, \varepsilon, \varepsilon^p, y)$ of the system (MBP). Moreover the sequences $(v^{k_t}, \varepsilon^{k_t} - \varepsilon^{p,k_t}, y^{k_t})$ converge in the strong $L^2$-topology to $(v, \varepsilon - \varepsilon^p, y)$.

**Proof.** From Theorem 4.5 we conclude that (for a selected subsequence)

$$(v^k, \varepsilon^k, \varepsilon^{p,k}, y^k) \rightharpoonup^* (v, \varepsilon, \varepsilon^p, y) \quad \text{in} \quad W^{1,\infty}_w((0,T); L^2(\Omega; W))$$

and moreover

$$(v^k, \varepsilon^k - \varepsilon^{p,k}, y^k) \rightarrow (v, \varepsilon - \varepsilon^p, y) \quad \text{in} \quad L^\infty((0,T); L^2(\Omega; \mathbb{R}^3 \times S^3 \times \mathbb{R})).$$

Thus the sequence $\sigma^k - \frac{1}{k^2} \sigma^{p,k} = PD(\varepsilon^k - \varepsilon^{p,k}) - \frac{1}{k} PD\varepsilon^{p,k} \rightarrow PD(\varepsilon - \varepsilon^p) \overset{df}{=} \sigma$ in $L^\infty((0,T); L^2(\Omega; S^3))$ while the sequence $\|\varepsilon^{p,k}\|$ is bounded. Then we can assume that $\sigma^k - \frac{1}{k} \sigma^{p,k} \rightarrow \sigma$ pointwise for a.e. $(x,t) \in \Omega \times [0,T)$. Lemma 4.1 ends this proof. ■

**Remark 4.7.** With the energy method presented in Section 2 we can show that strong solutions of the problem (MBP) are unique. (The operator $M(\varepsilon^p, y) = (D\varepsilon^p, \beta \varepsilon^p)$ is positive definite.)

**Remark 4.8.** We have proved existence of strong, global in time solutions for the problem (MBP) using additionally that $\text{div} D\varepsilon^0$ exists and belongs to $L^2(\Omega; \mathbb{R}^3)$. This is naturally equivalent to $\text{div} D\varepsilon^{p,0} \in L^2(\Omega; \mathbb{R}^3)$. This assumption is a consequence of the type of approximation process we have used. We can try to approximate the initial data $\varepsilon^0 \in L^2(\Omega; S^3)$ by a sequence of smooth data $\varepsilon^{0,n}$ for which $\text{div} D\varepsilon^{0,n} \in L^2(\Omega; \mathbb{R}^3)$. (4.16) shows that such approximations must have the following property: the sequence $\|(|\varepsilon^{0,n}_{y^3}|)^{\beta+1}\|$ should be bounded. This can be done, for example, for $\varepsilon^0 \in L^{2\beta+2}(\Omega)$.

**Remark 4.9.** For the Neumann boundary value problem the condition $D\varepsilon^0 \cdot n|_{\partial \Omega} = 0$ can be omitted if we require that the approximation process
satisfies the nonhomogeneous boundary condition
\[ D(\varepsilon^k(x,t) - \varepsilon^{n,k}(x,t)) \cdot n(x)|_{x \in \partial \Omega} = \frac{1}{k+1} D\varepsilon^0(x) \cdot n(x)|_{x \in \partial \Omega}. \]
This condition is independent of \( t \) and we can try to prove Theorem 3.1 with the inhomogeneous, constant in time Neumann boundary data.

5. Self-controlling models. In this section we want to generalize the phenomenon of the monotone Bodner–Partom model. Let us consider a system with general constitutive equations of monotone type
\[
\begin{align*}
\varepsilon_t &= \text{div} \frac{1}{\rho} D(\varepsilon - Bz), \\
(\text{MP}) \\
\varepsilon_t &= \frac{1}{2} (\nabla v + \nabla^T v), \\
\varepsilon_t &= g(-\rho \nabla z \psi(\varepsilon, z))
\end{align*}
\]
with a continuous maximal monotone function \( g : \mathbb{R}^N \to \mathbb{R}^N \) and with the free energy function
\[
\rho \psi(\varepsilon, z) = \frac{1}{2} [D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2} (Lz) \cdot z.
\]
We assume that the homogeneous Dirichlet boundary condition \( v(x, t)|_{x \in \partial \Omega} = 0 \) or the Neumann boundary condition \( D(\varepsilon(x, t) - Bz(x, t))|_{x \in \partial \Omega} = 0 \) is satisfied and the initial conditions are
\[
(5.1) \quad v(x, 0) = v^0(x), \quad \varepsilon(x, 0) = \varepsilon^0(x), \quad z(x, 0) = z^0(x).
\]
Moreover suppose that the linear operator
\[
(5.2) \quad M = B^T DB + L \quad \text{is positive definite}
\]
and the operator \( L \) is only positive semi–definite and our model is noncoercive. We want to approximate the problem (MP) by a sequence of coercive problems. First we define the class of admissible initial data for the system (MP).

DEFINITION 5.1. We say that the initial data \( (v^0, \varepsilon^0, z^0) \in \mathbb{L}^2(\Omega; \mathbb{W}) \) is admissible for the problem (MP) if and only if:
\[
\left( \text{div} \ D(\varepsilon^0 - Bz^0), \frac{1}{2} (\nabla v^0 + \nabla^T v^0), g(-\rho \nabla z \psi(\varepsilon^0, z^0)) \right) \in \mathbb{L}^2(\Omega; \mathbb{W})
\]
and additional \( v^0 \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \) for the Dirichlet boundary problem or
\[
\forall u \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \quad (\text{div} \ D(\varepsilon^0 - Bz^0), u) = -(D(\varepsilon^0 - Bz^0), \nabla u)
\]
for the Neumann boundary condition.
We define now a coercive approximation of the problem (MP): for all $N \ni k > 0$. Let $(v_k, \varepsilon_k, z_k)$ be the unique solution of the system

\begin{align*}
v^k_t &= \text{div} \frac{1}{\rho} D(\varepsilon^k - B z^k), \\
\varepsilon^k &= \frac{1}{2} (\nabla v^k + \nabla^T v^k), \\
z^k &= g(-\rho \nabla \psi^k(\varepsilon^k, z^k))
\end{align*}

(CMP)

with the same function $g : \mathbb{R}^N \to \mathbb{R}^N$ and with the free energy in the form

\begin{equation}
\rho \psi^k(\varepsilon^k, z^k) = \frac{1}{2} [D(\varepsilon^k - B z^k)] : (\varepsilon^k - B z^k) + \frac{1}{2} \quad (L z^k) \cdot z^k + \frac{1}{2k} \quad (D B_L z^k) \cdot B_L z^k
\end{equation}

(5.3)

where the linear operator $B_L : \mathbb{R}^N \to S^3$ is defined by

\begin{equation}
B_L z = \begin{cases} 
B z & \text{if } z \in \text{ker } L, \\
0 & \text{if } z \in (\text{ker } L)^\perp,
\end{cases}
\end{equation}

(5.4)

satisfying the homogeneous Dirichlet or Neumann boundary condition and initial conditions

\begin{equation}
v^k(x,0) = v^0(x), \quad \varepsilon^k(x,0) = \varepsilon^0(x), \quad z^k(x,0) = z^0_k(x)
\end{equation}

(5.5)

where $L z^0_k(x) = L z^0(z)$ and $B_L z^0_k(x) = \frac{k}{k+1} B_L z^0(x)$. Thus we change the initial conditions only for $\pi z$ where $\pi : \mathbb{R}^N \to \mathbb{R}^N$ is the orthogonal projection on $\text{ker } L$. For all other components of the vector $(v, \varepsilon, z)$ the initial conditions are independent of $k$ and coincide with conditions 5.1. The definition of the operator $B_L$ and assumption (5.2) yield that the free energy function (5.3) is positive definite and the problem (CMP) is coercive.

Remark. We say that the initial data are admissible for the system (CMP) if the right hand side of the system (CMP) evaluated for the initial data belongs to $L^2(\Omega; W)$ and the data satisfies the boundary condition.

Lemma 5.2. Let $(v^0, \varepsilon^0, z^0) \in L^2(\Omega; W)$ be admissible for the system (MP), $\text{div } D B_L z^0 \in L^2(\Omega; \mathbb{R}^3)$ and for the Neumann boundary problem $D B_L z^0 \cdot n|_{\partial \Omega} = 0$, then the initial data $(v^0, \varepsilon^0, z^0)$ where $z^0_k = \frac{k}{k+1} \pi z^0 + (I - \pi) z^0$ is admissible for the system (CMP). (Here $\pi : \mathbb{R}^N \to \mathbb{R}^N$ is the orthogonal projection on $\text{ker } L$).

Proof. The initial data $z^0_k$ is so chosen, that the argument of the nonlinear function $g$ for $t = 0$ is independent of $k$:

\begin{align*}
-\rho \nabla \psi^k(\varepsilon^k(0), z^k(0)) &= -\rho \nabla \psi^k(\varepsilon^0, z^0_k) \\
&= B^T D e^0 - \left( B^T D B + \frac{1}{k} B_L^T D B_L \right) \left( \frac{k}{k+1} \pi z^0 + (I - \pi) z^0 \right)
\end{align*}
\[ B^T D \varepsilon^0 - L z^0 - B^T D B ((I - \pi) z^0) \]
\[ - B^T D B \left( \frac{k}{k + 1} \pi z^0 \right) - \frac{1}{k + 1} B_L^T D B_L (\pi z^0) \]
\[ = B^T D \varepsilon^0 - L z^0 - B^T D B z^0 = -\rho \nabla_z \psi (\varepsilon^0, z^0). \]

Moreover for the weak divergence \( \text{div} \ D (\varepsilon^k (0) - B z^k (0)) \) we have
\[ \text{div} \ D (\varepsilon^0 - B z^k) = \text{div} \ D (\varepsilon^0 - B z^0) + \frac{1}{k + 1} \text{div} \ D B \pi z^0 \]
\[ = \text{div} \ D (\varepsilon^0 - B z^0) + \frac{1}{k + 1} \text{div} \ D B L z^0 \in L^2 (\Omega; \mathbb{R}^3). \]

For the Neumann boundary problem the assumption \( \text{div} \ D B_L z^0 \cdot n|_{\partial \Omega} = 0 \) yields that the initial data \((\varepsilon^0, z^0)\) satisfies the homogeneous Neumann boundary condition.

**Lemma 5.3.** For the initial data \((v^0, \varepsilon^0, z^0) \in L^2 (\Omega; W)\) satisfying the assumption of Lemma 5.2 for all \( N \ni k > 0 \) there exists the unique, strong, global in time solution for the problem (CMP).

**Proof.** This lemma is a consequence of Lemma 5.2, Theorem 3.1 and Theorem 3.3.

Before we study the limit case \( k \to \infty \), we define a subclass of constitutive equations having a similar property to (4.16).

**Definition 5.4.** We say that the model (MP) is self-controlling if and only if there exists a function \( F : \mathbb{R}^3_+ \to \mathbb{R}_+ \) mapping sets, bounded in at least one variable, into bounded sets such that for all functions \( y \in L^2 (\Omega; \mathbb{R}^N) \) with \( |y(x)| \geq 1 \) for a.e. \( x \in \Omega \) it holds that
\[ (5.6) \quad \| B_L g (y) \| \leq F (\| L g (y) \|, \| y \|), \]
where the operator \( B_L \) is defined by (5.4).

Inequality (5.6) denotes that for self-controlling models we can control the right hand side of the inelastic constitutive equation for the variables \( B_L z \) by the right hand side for the variables \( L z \) or by the argument \( \nabla z \psi \) of the nonlinear function \( g \). Note that for the monotone Bodner–Partom model the function \( F \) is given by (4.16):
\[ F (a, b) = c a^{\frac{\beta}{\beta + 1}} \]
where \( c \) is a positive constant.

**Theorem 5.5.** Let us assume that the model (MP) is self-controlling and the initial data \((v^0, \varepsilon^0, z^0) \in L^2 (\Omega; W)\) is admissible for the system (MP) with \( \text{div} \ D B_L z^0 \in L^2 (\Omega; \mathbb{R}^3) \) and for the Neumann boundary condition with \( \text{div} \ D B_L z^0 \cdot n|_{\partial \Omega} = 0 \). Then from the sequence \((v^k, \varepsilon^k, z^k)\) defined in Lemma
we can select a subsequence \((v^{k_t}, \varepsilon^{k_t}, z^{k_t})\), which converges weak-* in 
\(W^{1,\infty}_w([0,T]; L^2(\Omega; W))\) for arbitrary \(T > 0\) to a solution \((v, \varepsilon, z)\) of the 
problem \((MP)\). Moreover it holds that

\[(v^{k_t}, \varepsilon^{k_t} - Bz^{k_t}, Lz^{k_t}) \rightarrow (v, \varepsilon - Bz, Lz) \text{ in the } L^2\text{-strong topology.}\]

**Proof.** Similar to Theorem 4.5 we obtain

\[(5.7) \quad \text{ess sup}_{t \in [0,T]} \|(v^k(t), \varepsilon^k(t), z^k(t))\|_{H_k} \leq \|C^k(v^0, \varepsilon^0, z^0)\|_{H_k}\]

where the Hilbert space \(H_k\) and the operator \(C^k\) are defined similar to those
in the proof of Theorem 4.5. For the right hand side of \((5.7)\) we have

\[
\|C^k(v^0, \varepsilon^0, z^0)\|_{H_k} \\
\leq C\{\|\text{div } D(\varepsilon^0 - Bz^0)\| + \frac{1}{2} (\nabla v^0 + \nabla v^0) - Bg(-\rho \nabla \varepsilon^k(\varepsilon^0, z^0))\| \\
+ \|Lg(-\rho \nabla \varepsilon^k(\varepsilon^0, z^0))\| + \frac{1}{\sqrt{k}} \|B_Lg(-\rho \nabla \varepsilon^k(\varepsilon^0, z^0))\|\}
\]

\[
\leq C\{\|\text{div } D(\varepsilon^0 - Bz^0)\| + \frac{1}{k+1} \|\text{div } B_Lz^0\| \\
+ \frac{1}{2} (\nabla v^0 + \nabla v^0) - Bg(-\rho \nabla \varepsilon^0(\varepsilon^0, z^0))\| + \|Lg(-\rho \nabla \varepsilon^0(\varepsilon^0, z^0))\| \\
+ \frac{1}{\sqrt{k}} \|B_Lg(-\rho \nabla \varepsilon^0(\varepsilon^0, z^0))\|\}.
\]

Therefore there exists a positive constant \(C\) independent of \(k\) such that

\[(5.8) \quad \text{ess sup}_{t \in [0,T]} \left\{\|v^k(t)\|, \|\varepsilon^k(t) - Bz^k(t)\|, \|Lz^k(t)\|, \frac{1}{\sqrt{k}} \|B_Lz^k(t)\| \right\} \leq C.
\]

Now using that the model \((MP)\) is self-controlling we obtain

\[
\|B_Lz^k(t)\|^2 = \|B_Lg(-\rho \nabla \varepsilon^k(\varepsilon^k(t), z^k(t)))\|^2 \\
= \int_{\Omega \cap \{|y^k(t)| \leq 1\}} |B_Lg(-\rho \nabla \varepsilon^k(\varepsilon^k(x,t), z^k(x,t)))|^2 dx \\
+ \int_{\Omega \cap \{|y^k(t)| > 1\}} |B_Lg(-\rho \nabla \varepsilon^k(\varepsilon^k(x,t), z^k(x,t)))|^2 dx \\
\leq C^2_*|\Omega| + C^2(\|Lg(y^k(t))\|, \|y^k(t)\|)
\]

where \(y^k(t) = -\rho \nabla \varepsilon^k(\varepsilon^k(t), z^k(t)))\), \(C_* = \sup_{y \in B_N(1)} |B_Lg(y)|\) and \(B_N(1)\) denotes the unit ball in \(\mathbb{R}^N\). Inequality \((5.8)\) yields that the arguments
\(\|Lg(y^k(t))\|\) and \(\|y^k(t)\|\) are bounded for a.e. \(t \in [0,T]\). Then we can remove
the factor \(\frac{1}{\sqrt{k}}\) in inequality \((5.8)\). Next we show with the energy method that
the sequence \((v^k, \varepsilon^k - Bz^k, Lz^k)\) is a Cauchy sequence. Let us denote by

\[
\mathcal{E}^{k,m}(t) = \frac{\rho}{2} \|v^k(t) - v^m(t)\|^2 + \frac{1}{2} \int_\Omega \left[ [D(\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t))] \\
\cdot (\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) - Bz^m(t)) \, dx \right.
\]

\[
+ \frac{1}{2} \int_\Omega \left[ [L(z^k(t) - z^m(t))] \cdot (z^k(t) - z^m(t)) \, dx \right.
\]

the energy for the difference \((v^k, \varepsilon^k - Bz^k, Lz^k) - (v^m, \varepsilon^m - Bz^m, Lz^m)\).

Then for the time derivative of the function \(\mathcal{E}^{k,m}(t)\) we obtain similarly to (4.17)

\[
\frac{d}{dt} \mathcal{E}^{k,m}(t) = \int_\Omega \rho(v^k(t) - v^m(t))(v^k_t(t) - v^m_t(t)) \, dx
\]

\[
+ \int_\Omega \left[ [D(\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t))] \\
\cdot (Bz^k_t(t) - Bz^m_t(t)) \, dx \right.
\]

\[
+ \int_\Omega \left[ [L(z^k(t) - z^m(t))] \cdot (z^k_t(t) - z^m_t(t)) \, dx \right.
\]

(using the equation CMP1)

\[
= - \int_\Omega \left[ [B^T D(\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t))] \\
\cdot (Bz^k_t(t) - Bz^m_t(t)) \, dx \right.
\]

\[
+ \int_\Omega \left[ [B^T D(\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t))] \\
\cdot (Bz^k_t(t) - Bz^m_t(t)) \, dx \right.
\]

\[
= \int_\Omega \left[ g(-\rho \nabla z^k(\varepsilon^k(t), z^k(t))) - g(-\rho \nabla z^m(\varepsilon^m(t), z^m(t))) \right]
\]

\[
\cdot (-\rho \nabla z^k(\varepsilon^k(t), z^k(t)) + \rho \nabla z^m(\varepsilon^m(t), z^m(t))) \, dx
\]

\[
+ \int_\Omega \left[ [D(\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t))] \cdot (Bz^k_t(t) - Bz^m_t(t)) \, dx \right.
\]

(the monotonicity of \(g\)
Coe
crative limits for a subclass of monotone constitutive equations

\[ \leq C \cdot \left( \frac{1}{k} + \frac{1}{m} \right) \cdot (\|B_L z^k(t)\|^2 + \|B_L z^m(t)\|^2 + \|B_L z^k_t(t)\|^2 + \|B_L z^m_t(t)\|^2). \]

The boundedness of the sequence \( \|B_L z^k(t)\|^2 + \|B_L z^m(t)\|^2 \) uniformly for a.e. \( t \in [0, T) \) and the following estimate for the initial value of the function \( \mathcal{E}^{k,m}(t) \)

\[ \mathcal{E}^{k,m}(0) = \frac{1}{2} \int_\Omega \left[ \mathcal{D}(B_0 z^0_k - B_0 z^0_m) \cdot (B_0 z^0_k - B_0 z^0_m) \right] dx \]

\[ \leq C \cdot \left( \frac{1}{k+1} + \frac{1}{m+1} \right)^2 \|B_L z^0\|^2 \]

yields that the function \( \mathcal{E}^{k,m}(t) \) is arbitrarily small for sufficiently large \( k, m \).

Finally we can select from the sequence \( (v^k, \varepsilon^k, z^k) \) a subsequence, which we again denote by \( (v^k, \varepsilon^k, z^k) \) satisfying

\[ (v^k, \varepsilon^k, z^k) \overset{*}{\rightharpoonup} (v, \varepsilon, z), \quad (v^k_t, \varepsilon^k_t, z^k_t) \overset{*}{\rightharpoonup} (v_t, \varepsilon_t, z_t) \]

in \( L^\infty_0((0, T); L^2(\Omega; W)) \) and

\[ (v^k, \varepsilon^k - B z^k, L z^k) \rightharpoonup (v, \varepsilon - B z, L z) \quad \text{in the space} \quad L^\infty((0, T); L^2(\Omega; W)). \]

Moreover, perhaps again passing to a subsequence, we can assume that

\[ \nabla_z \psi^k(\varepsilon^k(x, t), z^k(x, t)) \rightharpoonup \nabla_z \psi(\varepsilon(x, t), z(x, t)) \]

pointwise for almost every \( (x, t) \in \Omega \times [0, T) \). Thus using Lemma 4.1 the limit \( (v, \varepsilon, z) \) is a strong solution to the noncoercive problem (MP).

Similar to section 4 we can show using the energy method that strong solutions of the problem (MP) are unique. From the definition of the self-controlling models it is clear that for Lipschitz operators \( B_L \) we can prove convergence of the coercive approximation process (CMP) to the solution of the problem (MP).

6. Another coercive approximation. In this section we define and study another type of coercive approximation. The idea of the approximation process is again a consequence of mechanical considerations. Let us consider again a general noncoercive model of monotone type. We want to approximate this model by a sequence of coercive models, for which the
Let $k$ be a natural number greater than zero. Thus the approximate system of equations is of the form

$$v_k^t = \text{div} \left( \frac{1}{\rho} \mathcal{D} \left( \varepsilon_k - B z_k + \frac{1}{k} \varepsilon_k \right) \right),$$

(ACA)

$$\varepsilon_k^t = \frac{1}{2} (\nabla v_k + \nabla^T v_k),$$

$$z_k^t \in g(-\rho \nabla_z \psi^k(\varepsilon^k, z^k))$$

with the homogenous Dirichlet boundary condition

$$v_k^t \big|_{x \in \partial \Omega} = 0,$$

or with the homogeneous Neumann boundary condition

$$\mathcal{D} \left( \varepsilon_k - B z_k + \frac{1}{k} \varepsilon_k \right) \cdot n \big|_{x \in \partial \Omega} = 0,$$

and with the initial conditions

$$v(0) = v^0, \quad \varepsilon(0) = \varepsilon^0, \quad z(0) = z^0.$$

Here $g : \mathbb{R}^N \rightarrow \mathcal{P}(\mathbb{R}^N)$ is a maximal monotone mapping with $0 \in g(0)$, and the free energy function $\psi^k$ is defined by

$$(6.1) \quad \rho \psi^k(\varepsilon^k, z^k) = \frac{1}{2} [\mathcal{D}(\varepsilon^k - B z^k)] \cdot (\varepsilon^k - B z^k)$$

$$+ \frac{1}{2} (L z^k) z^k + \frac{1}{2k} [\mathcal{D}(\varepsilon^k)] \cdot \varepsilon^k$$

with the given linear operators $B : \mathbb{R}^N \rightarrow S^3$ and $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$. If the operator $L$ is symmetric and positive semi-definite and the operator $M = B^T \mathcal{D} B + L$ is positive definite then the free energy $\psi^k$ is a positive definite quadratic form on $S^3 \times \mathbb{R}^N$. Thus we see that the problem (ACA) is coercive, but this system of equations is not of the form which we studied in Section 3. The first equation (ACA 1) contains the perturbation term $\frac{1}{k} \text{div} \frac{1}{\rho} \mathcal{D} \varepsilon^k$. Therefore we must prove an analogous theorem to Theorem 3.1 for the
problem (ACA). Using the notation of Section 3 let us define the operator
\[ C^k : \mathbb{L}^2(\Omega; W) \to \mathcal{P}(\mathbb{L}^2(\Omega; W)); \quad W = \mathbb{R}^3 \times S^3 \times \mathbb{R}^N \]
by \[ C^k(v, \varepsilon, z) = \emptyset \text{ if } v \in \mathbb{L}^2(\Omega; \mathbb{R}^3) \setminus \mathbb{H}_1(\Omega; \mathbb{R}^3) \text{ and} \]
\[ C^k(v, \varepsilon, z) = \begin{cases} 
  w = (w_1, w_2, w_3) \in \mathbb{L}^2(\Omega; W) & | w_1 = \text{div} \left( \frac{1}{\rho} \mathcal{D} \left( \varepsilon - Bz + \frac{1}{k} \varepsilon \right) \right) \\
  w_2 = \frac{1}{2} (\nabla v - \nabla^T v), & w_3(x) \in g(-\rho \nabla_x \psi^k(\varepsilon^k(x), z^k(x))) \quad \text{a.e. in } \Omega 
\end{cases} \]
if \( v \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \). Moreover for the Dirichlet boundary condition we require additionally that \( v \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \) and for the Neumann boundary value problem that the weak divergence satisfies
\[ \text{div} \mathcal{D} \left( \varepsilon - Bz + \frac{1}{k} \varepsilon \right), u \right) = \left( \mathcal{D} \left( \varepsilon - Bz + \frac{1}{k} \varepsilon \right), \nabla u \right) 
\]
for all \( u \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \).

**Theorem 6.1.** Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded set, and for the Neumann boundary value problem let additionally \( \partial \Omega \) be Lipschitz. Then the operator \(-C^k\) is maximal monotone, if the space \( \mathbb{L}^2(\Omega; W) \) is equipped with the scalar product
\[ \langle (v, \varepsilon, u), (\overline{v}, \overline{\varepsilon}, \overline{z}) \rangle = \int_{\Omega} \left( \rho u \overline{v} + [\mathcal{D}(\varepsilon - Bz)](\overline{\varepsilon} - B\overline{z}) \right. \\
\left. + \frac{1}{k} [\mathcal{D} \varepsilon] \cdot \overline{\varepsilon} + (Lz) \cdot \overline{z} \right) dx \]

**Remark.** The proof of Theorem 3.1 presented in [3] uses that the operator \( L \) is positive definite. In the problem (ACA) we have that \( L \) is only positive semi–definite, but the first equation (ACA1) yields coerciveness of the problem. Therefore the proof presented below is only a modification of the proof of Theorem 3.1 from [3].

**Proof.** First we show that the operator \(-C^k\) is monotone in the space \( \mathbb{L}^2(\Omega; W) \) with the scalar product defined by (6.4). Let \((v, \varepsilon, z), (\overline{v}, \overline{\varepsilon}, \overline{z}) \in \mathcal{D}(C^k)\) and \( w = (w_1, w_2, w_3) \in C^k(v, \varepsilon, z), \overline{w} = (\overline{w}_1, \overline{w}_2, \overline{w}_3) \in C^k(\overline{v}, \overline{\varepsilon}, \overline{z})\). Then we compute that
\[ \langle w - \overline{w}, (v, \varepsilon, u) - (\overline{v}, \overline{\varepsilon}, \overline{z}) \rangle = \int_{\Omega} \left[ \rho (w_1 - \overline{w}_1)(v - \overline{v}) \\
+ [\mathcal{D}(w_2 - \overline{w}_2 - Bw_3 + B\overline{w}_3)](\varepsilon - \overline{\varepsilon} - Bz - B\overline{z}) \right. \\
\left. \right. \left. + \frac{1}{k} [\mathcal{D}(w_2 - \overline{w}_2)](\varepsilon - \overline{\varepsilon}) + [L(w_3 - \overline{w}_3)] \cdot (z - \overline{z}) \right] dx \]
(\text{using definiton (6.2)})
\[
\begin{align*}
&= \int_{\Omega} \left\{ \text{div } D \left( \frac{\varepsilon - \bar{\varepsilon}}{k} - Bz + B\bar{z} \right) \cdot (v - \bar{v}) \\
&+ \left[ D \left( \frac{1}{2} (\nabla (v - \bar{v}) + \nabla^T (v - \bar{v})) \right) - B(w_3 - \bar{w}_3) \right] (\varepsilon - \bar{\varepsilon} - Bz - B\bar{z}) \\
&+ \frac{1}{k} \left[ D \left( \frac{1}{2} (\nabla (v - \bar{v}) + \nabla^T (v - \bar{v})) \right) \right] (\varepsilon - \bar{\varepsilon}) + [L(w_3 - \bar{w}_3)](z - \bar{z}) \right\} dx \\
&= \int_{\Omega} \left\{ -[DB(w_3 - \bar{w}_3)] \cdot (\varepsilon - \bar{\varepsilon} - Bz + B\bar{z}) + L(w_3 - \bar{w}_3)(z - \bar{z}) \right\} dx \\
&= \int_{\Omega} (w_3 - \bar{w}_3) \cdot [B^T D(\varepsilon - \bar{\varepsilon} - B(z - \bar{z})) - L(z - \bar{z})] dx \\
&= -\int_{\Omega} (w_3 - \bar{w}_3) \cdot [-\rho \nabla_z \psi^k(\varepsilon, z) + \rho \nabla_z \psi^k(\varepsilon, \bar{z})] dx \leq 0
\end{align*}
\]

Since \( w_3(x) \in g(-\rho \nabla_z \psi^k(\varepsilon(x), z(x))) \) and \( \bar{w}_3(x) \in g(-\rho \nabla_z \psi^k(\bar{\varepsilon}(x), \bar{z}(x))) \) for a.e. \( x \in \Omega \).

To prove that the monotone operator \(-C^k\) is maximal monotone we use the theorem of Minty. We show that for all \( \lambda > 0 \) the operator \( \lambda I - C^k \) is surjective. Let \( r = (r_1, r_2, r_3) \in L^2(\Omega; W) \). We should prove that there exists \( w = (v, \varepsilon, z) \in D(C^k) \) such that

\[
\begin{align*}
(6.5) \quad & -\text{div} \frac{1}{\rho} D \left( \varepsilon - Bz + \frac{1}{k} \varepsilon \right) + \lambda v = r_1 , \\
(6.6) \quad & -\frac{1}{2} (\nabla v + \nabla^T v) + \lambda \varepsilon = r_2 , \\
(6.7) \quad & -g(-\rho \nabla_z \psi^k(\varepsilon, z)) + \lambda z \ni r_3 .
\end{align*}
\]

First we study the differential inclusion (6.7). We assume that \( \varepsilon \) is given and solve (6.7) for \( z \). Using the relation \(-\rho \nabla_z \psi^k(\varepsilon, z) = B^T D\varepsilon - Mz\) we obtain

\[
(6.8) \quad -r_3 + \lambda M^{-1} B^T D\varepsilon \in g(B^T D\varepsilon - Mz) + \lambda M^{-1} (B^T D\varepsilon - Mz) .
\]

Thus the solvability of (6.8) is ensured by the maximal monotonicity of \( g \). Moreover the solution \( z(x) \) is unique and depends continuously on \( r_3(x) \) and \( \varepsilon(x) \). Let us denote by \( G(\varepsilon, r_3) \) the solution \( z \) of (6.8). We consider now the case \( z = G(\varepsilon, r_3) \) and \( \bar{z} = G(\bar{\varepsilon}, r_3) \). Then we have

\[
-r_3 + \lambda z \in g(B^T D\varepsilon - Mz) \quad \text{and} \quad -r_3 + \lambda \bar{z} \in g(B^T D\bar{\varepsilon} - M\bar{z}) .
\]

By the monotonicity of \( g \) we obtain

\[
0 \leq [(B^T D\varepsilon - Mz) - (B^T D\bar{\varepsilon} - M\bar{z})] \cdot \lambda (z - \bar{z}) \\
= -\lambda [M(z - \bar{z})](z - \bar{z}) + \lambda [D(\varepsilon - \bar{\varepsilon})] \cdot B(z - \bar{z}) .
\]
Then the assumption \( \lambda > 0 \) yields
\[
[M(z - \bar{z})] \cdot (z - \bar{z}) = [B^T \mathcal{D} B(z - \bar{z})](z - \bar{z}) + [L(z - \bar{z})](z - \bar{z}) 
\leq [\mathcal{D}(\varepsilon - \bar{\varepsilon})] \cdot B(z - \bar{z}).
\]
(6.9) and \( L \geq 0 \) imply
\[
\int_{\Omega} [\mathcal{D} B(z - \bar{z})] B(z - \bar{z}) \, dx \leq \left( \int_{\Omega} [\mathcal{D}(\varepsilon - \bar{\varepsilon})] \cdot (\varepsilon - \bar{\varepsilon}) \, dx \right)^{1/2} \left( \int_{\Omega} [\mathcal{D}(Bz - B\bar{z})] \cdot B(z - \bar{z}) \, dx \right)^{1/2}
\]
Let us denote by \( \|\varepsilon\|_D = (\int_{\Omega} (\mathcal{D} \varepsilon) \cdot \varepsilon \, dx)^{1/2} \) and by \( \|\varepsilon\|_{D_k} = (\int_{\Omega} (1 + \frac{1}{k})(\mathcal{D} \varepsilon) \cdot \varepsilon \, dx)^{1/2} \). Then using the notation we have
\[
\|B \mathcal{G}(\varepsilon, r_3) - B \mathcal{G}(\bar{\varepsilon}, r_3)\|_D \leq \|\varepsilon - \bar{\varepsilon}\|_D \leq \left( \frac{k}{k + 1} \right)^{1/2} \|\varepsilon - \bar{\varepsilon}\|_{D_k}.
\]
Next we solve (6.5) and (6.6) for all \((r_1, r_2) \in L^2(\Omega; \mathbb{R}^3 \times S^3)\), assuming that \( z \in L^2(\Omega; \mathbb{R}^N) \) is given. We solve (6.6) for \( \varepsilon \)
\[
\varepsilon = \frac{1}{\lambda} \left( \frac{1}{2}(\nabla u + \nabla^T) + r_2 \right)
\]
and insert the result into (6.5). Thus we must find \( v \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \), which satisfies
\[
\int_{\Omega} \left\{ \frac{1}{\lambda \rho} \left[ \left( 1 + \frac{1}{k} \right) \mathcal{D} \left( \frac{1}{2}(\nabla u + \nabla^T u) \right) \cdot \left( \frac{1}{2}(\nabla u + \nabla^T u) \right) + \lambda v \cdot u \right] \right\} \, dx
\]
\[
= \int_{\Omega} \left\{ \frac{1}{\rho} \left[ \mathcal{D} \left( Bz - \frac{1}{\lambda} \left( 1 + \frac{1}{k} \right) r_2 \right) \right] \cdot \nabla u + r_1 \cdot u \right\} \, dx
\]
for all \( u \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \) in the case of the Dirichlet boundary condition, and for all \( u \in \mathbb{H}_1(\Omega; \mathbb{R}^3) \) in the case of the Neumann boundary value problem. The solvability of (6.11) follows from the theorem of Lax-Milgram. Moreover for \((r_1, r_2) = 0\) we obtain from (6.11) setting \( u = v \)
\[
\int_{\Omega} \left\{ \frac{1}{\rho} \left[ \left( 1 + \frac{1}{k} \right) \mathcal{D} \varepsilon \right] \cdot \varepsilon + |v|^2 \right\} \, dx
\]
\[
= \int_{\Omega} \frac{1}{\rho} (\mathcal{D} Bz) \cdot \nabla v \, dx = \lambda \int_{\Omega} \frac{1}{\rho} (\mathcal{D} Bz) \cdot \varepsilon \, dx.
\]
Let us define the linear operator \( \mathcal{R} : L^2(\Omega; \mathbb{R}^N \times \mathbb{R}^3 \times S^3) \to L^2(\Omega; S^3) \) by \( \mathcal{R}(z, r_1, r_2) = \varepsilon \), where \( \varepsilon \) is the solution of (6.5) and (6.6) with given functions \( z, r_1, r_2 \). Then (6.12) implies
\[
\|\mathcal{R}(z, 0, 0)\|_{D_k} \leq \|Bz\|_D.
\]
Now for given \((r_1, r_2, r_3) \in \mathbb{L}^2(\Omega; W)\) we define the mapping \(\mathcal{P} : \mathbb{L}(\Omega; S^3) \to \mathbb{L}^2(\Omega; S^3)\) by

\[
\mathcal{P}(\varepsilon) = \mathcal{R}(G(\varepsilon, r_3), r_1, r_2).
\]

The inequalities (6.10) and (6.13) and the linearity of the operator \(\mathcal{R}\) yield

\[
\|\mathcal{P}(\varepsilon) - \mathcal{P}(\varepsilon^*)\|_{\mathcal{D}_k} = \|\mathcal{R}(G(\varepsilon, r_3) - G(\varepsilon^*, r_3), 0, 0\|_{\mathcal{D}_k} \leq \|BG(\varepsilon, r_3) - BG(\varepsilon^*, r_3)\|_{\mathcal{D}} \leq \left(\frac{k}{k+1}\right)^{1/2} \|\varepsilon - \varepsilon^*\|_{\mathcal{D}_k}.
\]

Therefore the mapping \(\mathcal{P}\) is a contraction in \(\mathbb{L}^2(\Omega; S^3)\) equipped with the norm \(\|\cdot\|_{\mathcal{D}_k}\). Then \(\mathcal{P}\) has a unique fixed point \(\varepsilon^* \in \mathbb{L}^2(\Omega; S^3)\). If we define \(z = G(\varepsilon^*, r_3)\) and \((v, \varepsilon)\) as the solution of (6.5) and (6.6) with \(z = G(\varepsilon^*, r_3)\) then \(\varepsilon = \mathcal{R}(G(\varepsilon^*, r_3), r_1, r_2) = \mathcal{P}(\varepsilon^*) = \varepsilon^*\).

Thus \(w = (v, \varepsilon^*, z)\) is the solution of (6.5), (6.6) and (6.7) and the proof is complete. \(\blacksquare\)

**Corollary 6.2.** Let the initial data \((v^0, \varepsilon^0, z^0)\) be admissible for the problem (MP) and additionally \(\text{div} \mathcal{D}v_0^0 \in \mathbb{L}^2(\Omega; \mathbb{R}^3)\) and for the Neumann boundary condition \(\mathcal{D}v_0^0 \cdot n|_{x \in \partial \Omega} = 0\). Then the sequence of solutions \((v^k, \varepsilon^k, z^k)\) of the problem (ACA) converges weak-* in the space \(W^{1,\infty}((0, T); \mathbb{L}^2(\Omega; W))\) to a solution of the problem (MP) for all \(T > 0\) provided the model (MP) is self-controlling.

**Proof.** The proof of this result is similar to the proof of Theorem 5.5. Only the estimation for differences of two solutions \((v^k, \varepsilon^k, z^k) - (v^m, \varepsilon^m, z^m)\) is different. Thus with the notation

\[
\mathcal{E}^{k,m}(t) = \frac{\rho}{2} \|v^k(t) - v^m(t)\|^2 + \frac{1}{2} \int_\Omega \left[\mathcal{D}(\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t))\right] \\
\times (\varepsilon^k(t) - Bz^k(t) - \varepsilon^m(t) + Bz^m(t)) \, dx
\]

we obtain

\[
\frac{d}{dt} \mathcal{E}^{k,m}(t) = -\int_\Omega \left[\mathcal{D}(\varepsilon^k(t) - Bz^k - \varepsilon^m(t) + Bz^m(t))\right] \cdot (Bz^k(t) - Bz^m(t)) \, dx
\]

\[
+ \int_\Omega \left[L(z^k(t) - z^m(t))\right] \cdot (z^k(t) - z^m(t)) \, dx
\]

\[
- \int_\Omega \left(\frac{1}{k} \varepsilon^k(t) - \frac{1}{m} \varepsilon^m(t)\right) (\varepsilon^k(t) - \varepsilon^m(t)) \, dx
\]
Coercive limits for a subclass of monotone constitutive equations

\[\begin{align*}
-\int_{\Omega} \left[ g(-\rho \nabla z^k(x), y^k(x)) - g(-\rho \nabla z^m(x), y^m(x)) \right] \\
\cdot (-\rho \nabla z^k(x), y^k(x)) + \rho \nabla z^m(x), y^m(x)) \, dx \\
- \int_{\Omega} \left( \frac{1}{k} \epsilon^k(t) - \frac{1}{m} \epsilon^m(t) \right) (\epsilon^k(t) - \epsilon^m(t)) \, dx \\
(\text{the monotonicity of } g) \\
\leq C \cdot \left( \frac{1}{k} + \frac{1}{m} \right) (\|\epsilon^k(t)\|^2 + \|\epsilon^m(t)\|^2 + \|\epsilon^k(t)\|^2 + \|\epsilon^m(t)\|^2)
\end{align*}\]

For the self-controlling models we have

\[\begin{align*}
\|\epsilon^k(t)\|^2 + \|\epsilon^m(t)\|^2 &\leq \|\epsilon^k(t) - B z^k(t)\|^2 + \|\epsilon^m(t) - B z^m(t)\|^2 \\
&+ \|B z^k(t)\|^2 + \|B z^m(t)\|^2 \\
&\leq C + \|B z^k(t)\|^2 + \|B z^m(t)\|^2 \leq C(T).
\end{align*}\]

Therefore the function \(\epsilon^{k,m}(t)\) is arbitrarily small for sufficiently large \(k, m\).

7. Another noncoercive example. We want to apply the approximation process (ACA) to a model which is not self-controlling. The system of equations is

\[\begin{align*}
v_t &= \text{div} \left( \frac{1}{\rho} (D(\varepsilon - \varepsilon^p)) \right), \\
(NSCP) \\
\varepsilon_t &= \frac{1}{2} (\nabla v + \nabla^T v), \\
\varepsilon_t^p &= d|\sigma|^{\beta} \sigma
\end{align*}\]

where \(\sigma = PD(\varepsilon - \varepsilon^p) = D(\varepsilon - \varepsilon^p) - \frac{1}{3} \text{tr} (D(\varepsilon - \varepsilon^p)) \cdot I\) and \(\beta > 0\). As usual we assume that the homogeneous Dirichlet or Neumann boundary condition is satisfied, and the initial conditions are

\[\begin{align*}
v(0) &= v^0, \ \varepsilon(0) = \varepsilon^0, \ \varepsilon^p(0) = \varepsilon^{0,0} \ \text{with } \text{tr} \varepsilon^{p,0}(x) = 0 \ \text{for a.e. } x \in \Omega.
\end{align*}\]

In this model the vector \(z\) of internal variables contains only \(\varepsilon^p\). Then the free energy function is defined by

\[\begin{align*}
\rho \psi(\varepsilon, \varepsilon^p) &= \frac{1}{2} \left[ D(\varepsilon - \varepsilon^p) \right] : (\varepsilon - \varepsilon^p)
\end{align*}\]

and the operator \(M : S^3 \to S^3\) is given by \(M\varepsilon^p = D\varepsilon^p\), since \(B\varepsilon^p = \varepsilon^p\) and \(L\varepsilon^p = 0\). Thus the function \(\psi\) is only positive semi-definite and the model (NSCS) is noncoercive. Moreover the right hand side of the equation (NSCS 3) is not controlled by the argument \(\sigma\) in the \(L^2\)-topology (note \(\beta > 0\)), and the operator \(L \equiv 0\). This denotes that the model (NSCP) does not belong to the class of self-controlling models. We approximate the problem
(NSCP) using the process (ACA)

\[
v^k_t = \text{div} \left( \frac{1}{\rho} (D(\varepsilon^k - \varepsilon^{p,k} + \frac{1}{k}\varepsilon^k)) \right),
\]

(ACA)

\[
\varepsilon^k_t = \frac{1}{2} (\nabla v^k + \nabla^T v^k),
\]

\[
\varepsilon^{p,k} = d|\sigma^k|^\beta \sigma^k
\]

where \( \sigma^k = P D(\varepsilon^k - \varepsilon^{p,k}) \) with the homogeneous Dirichlet or Neumann boundary condition and with the initial conditions given by (7.1) for all natural numbers \( k > 0 \). The free energy function

\[
\rho_\beta^k(\varepsilon^k, \varepsilon^{p,k}) = \frac{1}{2} [D(\varepsilon^k - \varepsilon^{p,k})] \cdot (\varepsilon^k - \varepsilon^{p,k}) + \frac{1}{2k} (D\varepsilon^k) \cdot \varepsilon^k.
\]

is connected with this approximate problem. This function is positive definite on \( S^3 \times S^3 \) and by Theorem 6.1 there exists the unique global in time strong solution \((v^k, \varepsilon^k, \varepsilon^{p,k})\), provided that the initial data \((v^0, \varepsilon^0, \varepsilon^{p,0})\) satisfies:

A1. \((\text{div} (D(\varepsilon^0 - \varepsilon^{p,0} + \frac{1}{k}\varepsilon^0)), \frac{1}{2} (\nabla v^0 + \nabla^T v^0), |\sigma^0|^\beta \sigma^0) \in L^2(\Omega; W)\), where \( W = \mathbb{R}^3 \times S^3 \times S^3 \)

and \( \sigma = PD(\varepsilon^0 - \varepsilon^{p,0}) \),

A2. for the Dirichlet–boundary condition \( v^0 \in \mathcal{H}_1(\Omega; \mathbb{R}^3) \) and for the Neumann boundary value problem it holds that

\[
\forall u \in \mathcal{H}_1(\Omega; \mathbb{R}^3) \quad \left( \text{div} D(\varepsilon^0 - \varepsilon^{p,0} + \frac{1}{k}\varepsilon^0), u \right) = - \left( D(\varepsilon^0 - \varepsilon^{p,0} + \frac{1}{k}\varepsilon^0), \nabla u \right).
\]

Now we want to study the limit case of \( k \to \infty \). Thus we assume additionally that the initial data \((v^0, \varepsilon^0, \varepsilon^{p,0})\) satisfies

A3. \( \text{div} D\varepsilon_0 \in L^2(\Omega; \mathbb{R}^3) \) and for the Neumann boundary condition \( D\varepsilon_0 \cdot n|_{x \in \partial \Omega} = 0 \).

**Theorem 7.1.** Let the initial data \((v^0, \varepsilon^0, \varepsilon^{p,0}) \in L^2(\Omega; W)\) satisfy A1 – A3. Then the sequence \((v^k, \varepsilon^k, \varepsilon^{p,k})\) of solutions for the problem (ACA) contains a subsequence which converges to the solution \((v, \varepsilon, \varepsilon^p)\) of the problem (NSCP) in the following topology:

\[
v^k \rightharpoonup v \quad \text{in} \quad W^{1,\infty}((0,T); L^2(\Omega; \mathbb{R}^3)),
\]

\[
\varepsilon^k \rightharpoonup \varepsilon \quad \text{in} \quad W^{1,\infty}((0,T); L^p(\Omega; S^3)),
\]

\[
\varepsilon^{p,k} \rightharpoonup \varepsilon^p \quad \text{in} \quad W^{1,\infty}((0,T); L^p(\Omega; S^3)),
\]

\[
\sigma^k \rightharpoonup \sigma \quad \text{in} \quad L^{\infty}((0,T); L^{\beta+2}(\Omega; S^3)),
\]

where \( p = \frac{\beta+2}{\beta+1} \) is the conjugate exponent to \( \beta + 2 \).
Proof. We start similarly as in the proof of Theorem 4.5 and obtain

\[
\text{ess sup}_{t \in [0,T]} \left\{ \|v^k_t(t)\|, \|\varepsilon^k_t - \varepsilon^{p,k}_t\|, \frac{1}{\sqrt{k}} \|\varepsilon^k_t\| \right\} \leq C,
\]

where \( C \) is a positive constant, which does not depend on \( k \) and \( T \). This result allows us to prove that the sequence \((v^k, \varepsilon^k - \varepsilon^{p,k})\) contains a subsequence, which converges weak-* in the space \( W_{W}^{1,\infty}((0,T); L^2(\Omega; \mathbb{R}^3 \times S^3)) \) to a solution \((v, \varepsilon - \varepsilon^p)\) of the first equation (NSCP 1). (Here \( \varepsilon - \varepsilon^p \) is only a symbol and does not denote the difference of the limits of the sequences \( \{\varepsilon^k\} \) and \( \{\varepsilon^{p,k}\}\).) To prove a similar result for the equations (NSCP 2) and (NSCP 3) we need some estimates for the sequence \( \{\varepsilon^k\} \) or for the sequence \( \{\varepsilon^{p,k}\} \), which are independent of \( k \). For the self-controlling models this additional information gave us the special form of the inelastic constitutive equations. For the system (NSCP) we must use another method. If we define by

\[
E^k(t) = \frac{\rho}{2} \|v^k(t)\|^2 + \frac{1}{2} \int_{\Omega} [D(\varepsilon^k - \varepsilon^{p,k})] \cdot (\varepsilon^k - \varepsilon^{p,k}) dx + \frac{1}{2k} \int_{\Omega} (D\varepsilon^k) \cdot \varepsilon^k dx
\]

the energy function for the problem (ACA), then we obtain for the time derivative

\[
\frac{d}{dt} E^k(t) = \int_{\Omega} \rho v^k v^k_t dx + \int_{\Omega} [D(\varepsilon^k - \varepsilon^{p,k})](\varepsilon^k_t - \varepsilon^{p,k}_t) dx
\]

\[
+ \frac{1}{k} \int_{\Omega} (D\varepsilon^k) \cdot \varepsilon^k_t dx
\]

\[
= - \int_{\Omega} [D(\varepsilon^k - \varepsilon^{p,k})] \cdot \varepsilon^{p,k}_t dx = - \int_{\Omega} d|\sigma^k|^{\beta+2} dx.
\]

Integrating (7.4) we have

\[
E^k(t) + \int_0^t \int_{\Omega} d|\sigma^k|^{\beta+2} dx d\tau = E^k(0)
\]

\[
= \frac{\rho}{2} \|v^0\|^2 + \frac{1}{2} \int_{\Omega} [D(\varepsilon^0 - \varepsilon^{p,0})] \cdot (\varepsilon^0 - \varepsilon^{p,0}) dx + \frac{1}{2k} \int_{\Omega} (D\varepsilon^0) \cdot \varepsilon^0 dx.
\]

Thus the sequence \( \{\sigma^k\} \) is bounded in the space \( L^{\beta+2}((0,t) \times \Omega; S^3) \) for all \( t \in [0,T] \). Moreover from (7.4) we conclude that

\[
\text{ess sup}_{t \in [0,T]} \int_{\Omega} |\sigma^k|^{\beta+2} dx \leq c \cdot \text{ess sup}_{t \in [0,T]} \left\{ \|v^k\| \cdot \|v^k_t\| + \|\varepsilon^k - \varepsilon^{p,k}\| \cdot \|\varepsilon^k_t - \varepsilon^{p,k}_t\| + \frac{1}{\sqrt{k}} \|\varepsilon^k\| \cdot \frac{1}{\sqrt{k}} \|\varepsilon^k_t\| \right\}
\]

\[
\leq (\text{by (7.3)}) \leq C(T)
\]
where \( C(T) \) is a positive constant independent of \( k \). This yields that the sequence \( \{\sigma^k\} \) is bounded in the space \( L^\infty((0,T); L^{\beta+2}(\Omega;S^3)) \). Now for the right hand side of the equation (ACA 3) holds

\[
(7.7) \quad \text{ess sup}_{t \in [0,T]} \int_\Omega |\varepsilon^p_{t,k}|^{\beta+2} \frac{1}{\beta+1} \, dx = \text{ess sup}_{t \in [0,T]} \int_\Omega \frac{d^{\beta+2}}{d^{\beta+1}} |\sigma^k|^{\beta+2} \leq C(T) \cdot d^{\beta+2}.
\]

Inequality (7.7) denotes that the sequence \( \{\varepsilon^p_{t,k}\} \) is bounded in \( L^\infty((0,T); L^{\beta+2}(\Omega;S^3)) \) where \( p = \frac{\beta+2}{\beta+1} \) and by (7.3) the same result holds for the sequence \( \{\varepsilon^k_t\} \). Therefore the sequence \( (v^k, \varepsilon^k, \varepsilon^p_k) \) contains a subsequence \( (v^{k_i}, \varepsilon^{k_i}, \varepsilon^{p_i,k_i}) \), which we denote again by \( (v^k, \varepsilon^k, \varepsilon^p_k) \) satisfying

\[
v_t^k \overset{*}{\rightharpoonup} v^k = \text{div} \frac{1}{\rho} D(\varepsilon^k - \varepsilon^{p,k} + \frac{1}{k} \varepsilon^k) \overset{*}{\rightharpoonup} \text{div} \frac{1}{\rho} D(\varepsilon - \varepsilon^p)
\]

in \( L^\infty((0,T); L^2(\Omega; \mathbb{R}^3)) \),

\[
\varepsilon^k_t \overset{*}{\rightharpoonup} \varepsilon^k_t = \frac{1}{2} (\nabla v^k + \nabla^T v^k) \overset{*}{\rightharpoonup} \frac{1}{2} (\nabla v + \nabla^T v) \text{ in } L^\infty((0,T); L^p(\Omega; S^3))
\]

and moreover \( \sigma^k \overset{*}{\rightharpoonup} \sigma \text{ in } L^\infty((0,T); L^{\beta+2}(\Omega; S^3)) \). Here \( v = w^* - \lim v^k \text{ in } L^\infty((0,T); L^2(\Omega; \mathbb{R}^3)) \), \( \varepsilon = w^* - \lim \varepsilon^k \text{ in } L^\infty((0,T); L^p(\Omega; S^3)) \) and \( \varepsilon^p = w^* - \lim \varepsilon^{p,k} \text{ in } L^\infty((0,T); L^p(\Omega; S^3)) \). To end the proof we must show that \( \chi = d \cdot |\sigma| \beta \sigma \cdot \). For self-controlling models we have proved that the argument \( -\rho \nabla \psi^k(\varepsilon^k, z^k) \) of the nonlinear function \( g \) converges in the strong \( L^2 \)-topology. Here we have only the weak convergence of the sequence \( \{\sigma^k\} \text{ in } L^{\beta+2}(\Omega; S^3) \) and the weak convergence of the nonlinearity \( |\sigma^k| \beta \cdot \sigma^k \text{ in } L^p(\Omega; S^3) \). To prove that in fact the sequence \( |\sigma^k| \beta \cdot \sigma^k \text{ converges weakly to } |\sigma| \beta \cdot \sigma \) we use the method presented by J. L. Lions in [19]. In the book [13] this method is called the Minty–Browder method. The limit functions \( (v, \varepsilon, \varepsilon^p) \) satisfy the system

\[
v_t = \text{div} \frac{1}{\rho} D(\varepsilon - \varepsilon^p) \in L^2(\Omega; \mathbb{R}^3),
\]

\[
\varepsilon_t = \frac{1}{2} (\nabla v + \nabla^T v) \in L^p(\Omega; S^3),
\]

\[
\varepsilon^p_t = \chi \in L^p(\Omega; S^3).
\]

Moreover for the trace of the strain velocity we obtain \( \text{tr } \varepsilon_t = \text{tr } (\varepsilon^k_t - \varepsilon^p_t) \in L^2(\Omega; \mathbb{R}) \). Therefore the integral \( \int_\Omega D(\varepsilon - \varepsilon^p) \varepsilon_t dx = \int_\Omega P D(\varepsilon - \varepsilon^p) P \varepsilon_t dx + \frac{1}{3} \int_\Omega \text{tr } D(\varepsilon - \varepsilon^p) \cdot \text{tr } \varepsilon_t dx \) is well defined. (Note \( P D(\varepsilon - \varepsilon^p) = \sigma \in L^{\beta+2}(\Omega; S^3) \)).

Thus if we denote by

\[
\mathcal{E}(t) = \frac{\rho}{2} \|v(t)\|^2 + \frac{1}{\Omega} \left[ \int_\Omega D(\varepsilon(t) - \varepsilon^p(t)) \cdot (\varepsilon(t) - \varepsilon^p(t)) dx \right]
\]
the energy function for the system (7.8) then similar to (7.4) and (7.5) we obtain
\begin{equation}
\mathcal{E}(t) + \int_0^t \int_\Omega \chi \cdot \sigma \, dx \, dt = \mathcal{E}(0)
= \frac{\rho}{2} \| \mathbf{v}^0 \|^2 + \frac{1}{2} \int_\Omega [\mathcal{D}(\mathbf{e}^0) - \mathbf{e}^{p,0}] (\mathbf{e}^0 - \mathbf{e}^{p,0}) \, dx.
\end{equation}

The equalities (7.9) and (7.5) yield
\begin{equation}
\begin{aligned}
\mathcal{E}(t) + \frac{\rho}{2} \| \mathbf{v}^k(t) \|^2 + \frac{1}{2} \int_\Omega [\mathcal{D}(\mathbf{e}^k(t) - \mathbf{e}^{p,k}(t))] \cdot (\mathbf{e}^k(t) - \mathbf{e}^{p,k}(t)) \, dx \\
+ \int_0^t \int_\Omega d|\sigma^k|^\beta + 2 \, dx \, dt - \frac{1}{2k} \int_\Omega (\mathcal{D}\mathbf{e}^0) \cdot \mathbf{e}^0 \, dx \\
\leq \mathcal{E}(t) + \int_0^t \int_\Omega \chi \cdot \sigma \, dx \, dt.
\end{aligned}
\end{equation}

The convexity of the function $\mathcal{E}(t)$ implies
\begin{equation}
\liminf_{k \to \infty} \left( \frac{\rho}{2} \| \mathbf{v}^k(t) \|^2 + \frac{1}{2} \int_\Omega [\mathcal{D}(\mathbf{e}^k(t) - \mathbf{e}^{p,k}(t))] \cdot (\mathbf{e}^k(t) - \mathbf{e}^{p,k}(t)) \, dx \right) \geq \mathcal{E}(t).
\end{equation}

Then from inequality (7.10) we obtain for all $t \in [0, T]$
\begin{equation}
\liminf_{k \to \infty} \left( \int_0^t \int_\Omega d|\sigma^k|^\beta + 2 \, dx \, dt - \frac{1}{2k} \int_\Omega (\mathcal{D}\mathbf{e}^0) \cdot \mathbf{e}^0 \, dx \right)
= \liminf_{k \to \infty} (\int_0^t \int_\Omega d|\sigma^k|^\beta + 2 \, dx \, dt) \leq \int_0^t \int_\Omega \chi \cdot \sigma \, dx \, dt.
\end{equation}

Let us choose a function $\alpha \in L^{\beta+2}((0, t) \times \Omega; S^3)$. Then the monotonicity of the nonlinear operator $|\sigma^k|^\beta \sigma^k$ yields
\begin{equation}
\liminf_{k \to \infty} \int_0^t \int_\Omega (d|\sigma^k|^\beta \sigma^k - d|\alpha|^\beta \alpha) (\sigma^k - \alpha) \, dx \, dt \geq 0.
\end{equation}

Using (7.11) and the weak convergence $d|\sigma^k|^\beta \sigma^k \rightharpoonup \chi$ in $L^p((0, t) \times \Omega; S^3)$ and $\sigma^k \rightharpoonup \sigma$ in $L^{\beta+2}((0, t) \times \Omega; S^3)$ we have
\begin{equation}
\int_0^t \int_\Omega (\chi - d|\alpha|^\beta \alpha) (\sigma - \alpha) \, dx \, dt \geq 0.
\end{equation}

If we now choose $\alpha$ in the form $\sigma - \lambda w$ with $\lambda > 0$ and $w \in L^{\beta+2}((0, t) \times \Omega; S^3)$ then from (7.12) follows
\begin{equation}
\lambda \int_0^t \int_\Omega (\chi - d|\sigma - \lambda w|^\beta (\sigma - \lambda w)) \cdot w \, dx \, dt \geq 0.
\end{equation}
Upon canceling $\lambda$ we pass to the limit $\lambda \to 0^+$ and obtain
\[
\int_0^t \int_\Omega (\chi - d|\sigma|^{\beta}\sigma) \cdot wx dx d\tau \geq 0.
\]
Replacing $w$ by $-w$ yields $\chi = d|\sigma|^{\beta}\sigma$ for a.e. $(x, t') \in \Omega \times (0, t)$.

**Corollary 7.2.** The sequence $(v_k, \varepsilon_k, \varepsilon^{p,k})$ of solutions for the problem (ACA) contains a subsequence $(v^{k_l}, \varepsilon^{k_l}, \varepsilon^{p,k_l})$ such that for all $t \in [0, T]$
\[
v^{k_l}(t) \to v(t) \quad \text{in } L^2(\Omega; \mathbb{R}^3),
\]
\[
\varepsilon^{k_l}(t) - \varepsilon^{p,k_l}(t) \to \varepsilon(t) - \varepsilon^p(t) \quad \text{in } L^2(\Omega; S^3),
\]
where $(v, \varepsilon, \varepsilon^p)$ is the solution from Theorem 7.1.

**Proof.** In the proof of Theorem 7.1 we have obtained that
\[
|\sigma^k|^{\beta}\sigma^k \rightharpoonup |\sigma|^{\beta}\sigma \quad \text{in } L^\infty((0, T); L^p(\Omega; S^3))
\]
(here $\sigma = PD(\varepsilon - \varepsilon^p)$ and $p = \frac{\beta+2}{\beta+1}$). The convexity of the norm in $L^{\beta+2}((0, t) \times \Omega; S^3)$ yields
\[
\liminf_{k \to \infty} \left( \int_0^t \int_\Omega d|\sigma^k|^{\beta+2} dxd\tau \right) \geq \int_0^t |\sigma|^{\beta+2} dxd\tau.
\]
Then (7.11) and (7.13) imply for all $t \in [0, T]$
\[
\liminf_{k \to \infty} \left( \int_0^t \int_\Omega d|\sigma^k|^{\beta+2} dxd\tau \right) = \int_0^t |\sigma|^{\beta+2} dxd\tau.
\]
Thus passing to the limit inferior for $k \to \infty$ in inequality (7.10) and using the convexity of $\mathcal{E}(t)$ we obtain
\[
\liminf_{k \to \infty} \left( \frac{\rho}{2} \|v^k(t)\|^2 + \frac{1}{2} \int_\Omega [D(\varepsilon^k(t) - \varepsilon^{p,k}(t))] \cdot (\varepsilon^k(t) - \varepsilon^{p,k}(t)) dx \right) = \mathcal{E}(t).
\]
The last result and the weak convergence $v^k(t) \rightharpoonup v(t)$ in $L^2(\Omega; \mathbb{R}^3)$ and $\varepsilon^k(t) - \varepsilon^{p,k}(t) \rightharpoonup \varepsilon(t) - \varepsilon^p(t)$ in $L^2(\Omega; S^3)$ complete the proof.

**8. End remarks.** In this section we discuss briefly the case $g : D(g) \subset \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$. We assume that the mapping $g$ is maximal monotone with $0 \in g(0)$, $g$ generates a maximal monotone operator in the space $L^2(\Omega; \mathbb{R}^N)$ by $w \in g(v)$ iff $v(x) \in D(g)$ and $w(x) \in g(v(x))$ for a.e. $x \in \Omega$ (see [4] p.146). We denote the operator by $g$ again. Let us rewrite the general problem (MP)
from Section 3.

\[
v_t(x, t) = \text{div} \frac{1}{\rho} D(\varepsilon(x, t) - Bz(x, t)),
\]

\[(MP)\]

\[
\varepsilon_t(x, t) = \frac{1}{2}(\nabla v(x, t) + \nabla^T v(x, t)),
\]

\[
z_t(x, t) \in g(-\rho \nabla \psi(\varepsilon(x, t), z(x, t)))
\]

with the homogeneous Dirichlet or Neumann boundary condition and with the initial conditions

\[
v(x, 0) = v^0(x), \quad \varepsilon(x, 0) = \varepsilon^0(x), \quad z(x, 0) = z^0(x).
\]

The free energy function \( \psi \) is defined by

\[
\rho \psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2}(Lz) \cdot z
\]

where \( B : \mathbb{R}^N \to S^3 \), \( L : \mathbb{R}^N \to \mathbb{R}^N \) are given linear operators (\( L \) is symmetric and semi-positive definite), for which the operator \( M = B^T D B + L \) is positive definite. Let us denote by \( A_\lambda \) the Yosida approximation of the operator \( g \). \( A_\lambda \) is generated by the pointwise Yosida approximation for the mapping \( g : D(g) \subset \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N) \). We want to approximate the problem (MP) by a sequence of problems, for which we can use Theorem 3.1

\[
v_\lambda^\alpha(x, t) = \text{div} \frac{1}{\rho} D(\varepsilon_\lambda^\alpha(x, t) - Bz_\lambda^\alpha(x, t)),
\]

\[(YA)\]

\[
\varepsilon_t^\alpha(x, t) = \frac{1}{2}(\nabla v^\alpha(x, t) + \nabla^T v^\alpha(x, t)),
\]

\[
z_t^\alpha(x, t) = A_\lambda(-\rho \nabla \psi(\varepsilon^\alpha(x, t), z^\alpha(x, t))) - A_\lambda(0)
\]

with the same boundary and initial conditions as in the problem (MP). Moreover the function \( \psi \) is defined by (8.1) with the same linear operators \( B \) and \( L \). Since \( A_\lambda - A_\lambda(0) : \mathbb{R}^N \to \mathbb{R}^N \) is a maximal monotone operator with \( (A_\lambda - A_\lambda(0))(0) = 0 \) then this operator satisfies the assumptions of Theorem 3.1. (Note that for \( \lambda \to 0^+ \), \( A_\lambda(0) \to 0 \), provided \( 0 \in g(0) \).)

A. The case \( \psi \) - positive definite

Let us assume that the function \( \psi \) is positive definite. If we denote by \( \mathcal{H} \) the space \( L^2(\Omega; S^3 \times S^3 \times \mathbb{R}^N) \) with the scalar product from Theorem 3.1

\[
\langle (v, \varepsilon, z), (\bar{v}, \bar{\varepsilon}, \bar{z}) \rangle = \int_\Omega \{ \rho v \cdot \bar{v} + [D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + (Lz) \cdot \bar{z} \} \, dx.
\]

then we obtain

\[
\text{ess sup}_{t \in [0, T]} \| (v_t^\lambda(t), \varepsilon_t^\lambda(t), z_t^\lambda(t)) \|_{\mathcal{H}} \leq \| C^\lambda(v^0, \varepsilon^0, z^0) \|_{\mathcal{H}}
\]

where \( C^\lambda \) is the operator defined by the right hand side of the system (YA).
Thus we obtain

\[(8.2) \quad \| C^\lambda(v^0, \varepsilon^0, z^0) \|_H \leq C \left\{ \| \text{div}D(\varepsilon^0 - Bz^0)\| + \| LA_\lambda(-\rho \nabla_z \psi(\varepsilon^0, z^0))\| \\
\quad + \left\| \frac{1}{2}(\nabla v^0 + \nabla^T v^0) - B(\lambda\nabla_{\psi}(\varepsilon^0, z^0) - \lambda_0) \right\| \right\}. \]

If we assume that $-\rho \nabla_z \psi(\varepsilon^0, z^0) \in D(g)$ then there exists a positive constant $C$ independent of $\lambda$ such that

\[(8.3) \quad \text{ess sup}_{t \in [0,T]} (\|v^\lambda(t)\|, \|\varepsilon^\lambda(t)\|, \|z^\lambda(t)\|) \leq C \]

(note that the operator $L$ is positive definite and the boundedness of $\|Lz^\lambda\|$ implies boundedness of $\|z^\lambda\|$). The second important step is an estimate for the difference of two approximation steps. Let us define the energy for such differences by:

\[
E^{\lambda,\mu}(t) = \frac{\rho}{2} \| v^\lambda(t) - v^\mu(t) \|^2 + \frac{1}{2} \int_\Omega [D(\varepsilon^\lambda(t)) - Bz^\lambda(t) - \varepsilon^\mu(t) + Bz^\mu(t)] \\
\quad \cdot (\varepsilon^\lambda(t) - Bz^\lambda(t) - \varepsilon^\mu(t) - Bz^\mu(t)) \, dx \\
+ \frac{1}{2} \int_\Omega [L(z^\lambda(t) - z^\mu(t))] \cdot (z^\lambda(t) - z^\mu(t)) \, dx.
\]

Then for the time derivative of the function $E^{\lambda,\mu}(t)$ we have

\[
\frac{d}{dt} E^{\lambda,\mu}(t) = - \int_\Omega (-\rho \nabla_z \psi(\varepsilon^\lambda(t), z^\lambda(t)) + \rho \nabla_z \psi(\varepsilon^\mu(t), z^\mu(t)))) \cdot \\
(\lambda\nabla_{\psi}(\varepsilon^\lambda(t), z^\lambda(t))) - \lambda(\varepsilon^\mu(t), z^\mu(t))) \, dx \\
- \int_\Omega (-\rho \nabla_z \psi(\varepsilon^\lambda(t), z^\lambda(t)) + \rho \nabla_z \psi(\varepsilon^\mu(t), z^\mu(t))) \\
\cdot (\lambda(0) - \lambda(0)) \, dx
\]

(similar to the proof of Theorem 3.1 from [6] p. 54)

\[
\leq \frac{\lambda}{4} \| A_\lambda(-\rho \nabla_z \psi(\varepsilon^\lambda(t), z^\lambda(t))) \|^2 + \frac{\mu}{4} \| A_\mu(-\rho \nabla_z \psi(\varepsilon^\mu(t), z^\mu(t))) \|^2 \\
+ \| -\rho \nabla_z \psi(\varepsilon^\lambda(t), z^\lambda(t)) + \rho \nabla_z \psi(\varepsilon^\mu(t), z^\mu(t)) \| \cdot \| A_\lambda(0) - A_\mu(0) \|.
\]

Therefore using (8.3) we conclude that $E^{\lambda,\mu}(t)$ is arbitrarily small for $\lambda, \mu \to 0^+$. This implies that the sequence $-\rho \nabla_z \psi(\varepsilon^\lambda, z^\lambda) \to -\rho \nabla_z \psi(\varepsilon, z)$ in $L^\infty((0,T); L^2(\Omega; \mathbb{R}^N))$ and the right hand side

\[
A_\lambda(-\rho \nabla_z \psi(\varepsilon^\lambda, z^\lambda)) \to \chi \in g(-\rho \nabla_z \psi(\varepsilon, z))
\]

in $L^\infty((0,T); L^2(\Omega; \mathbb{R}^N))$. Thus we have proved that the approximation process (YA) converges to a solution of the problem (MP).
B. The case $\psi$ - semi-positive definite

Now let us suppose that the free energy $\psi$ is only semi-positive definite, which is equivalent to the operator $L$ being only semi-positive definite. In this case we can try to apply the approximation process (ACA) from Section 6:

$$v_t^{\lambda,k}(x, t) = \text{div} \left( \frac{1}{\rho} D(\varepsilon_t^{\lambda,k}(x, t) - Bz_t^{\lambda,k}(x, t) + \frac{1}{k} \varepsilon_t^{\lambda,k}(x, t)) \right),$$

(AYA) $$\varepsilon_t^{\lambda,k}(x, t) = \frac{1}{2}(\nabla v_t^{\lambda,k}(x, t) + \nabla^T v_t^{\lambda,k}(x, t)),$$

$$z_t^{\lambda,k}(x, t) = A_{\lambda} \left( -\rho \nabla_z \psi^{\lambda,k}(\varepsilon_t^{\lambda,k}(x, t), z_t^{\lambda,k}(x, t)) \right) - A_{\lambda}(0)$$

with the same boundary and initial conditions as in the problem (MP). The free energy function is now defined by

$$\rho \psi(\varepsilon^{\lambda,k}, z^{\lambda,k}) = \frac{1}{2} \left[ D(\varepsilon^{\lambda,k} - Bz^{\lambda,k}) \right] \cdot (\varepsilon^{\lambda,k} - Bz^{\lambda,k})$$

$$+ \frac{1}{2} (Lz^{\lambda,k}) \cdot z^{\lambda,k} + \frac{1}{2k} D(\varepsilon^{\lambda,k}) \cdot \varepsilon^{\lambda,k}.$$ 

To prove that the process (AYA) converges to a solution of the problem (YA) we use only that the Yosida approximation is a family of lipschitz operators but then to obtain that the process (YA) converges to a solution of the problem (MP) we need a control of the right hand side $B_L A_{\lambda} \left( -\rho \nabla_z \psi^{\lambda,k}(\varepsilon^{\lambda,k}, z^{\lambda,k}) \right)$, where the operator $B_L$ was defined in Section 5. Hence we try to extend the definition of the self-controlling models:

there exists a positive constant $\lambda^*$ such that for all $\lambda \leq \lambda^*$ and for all functions $y \in L^2(\Omega; \mathbb{R}^N)$ with $|y(x)| \geq 1$ for a.e. $x \in \Omega$ it holds that

\begin{equation}
\|B_L A_{\lambda}(y)\| \leq \mathcal{F}(\|L A_{\lambda}(y)\|, \|y\|)
\end{equation}

where $\mathcal{F}$ is a function satisfying the assumptions of Definition 5.4. If we now suppose that our model satisfies (8.4), then in the same manner as in Section 6 we can prove that for fixed $\lambda \leq \lambda^*$ the process (AYA) converges to a solution of the problem (YA), provided that the initial data $\varepsilon^0$ satisfies

\begin{equation}
\text{div } D\varepsilon^0 \in L^2(\Omega; \mathbb{R}^3)
\end{equation}

and for the Neumann boundary condition $D\varepsilon^0 \cdot n_{|\partial\Omega} = 0$.

Using that the domain of $A_{\lambda}$ is equal to $L^2(\Omega; \mathbb{R}^N)$ and $A_{\lambda}$ is continuous we can omit the assumption (8.5) (see Remark 4.7). Also similar to the case A we show that the process (YA) converges to a solution of the problem (MP).

REFERENCES


Coercive limits for a subclass of monotone constitutive equations