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On a class of limit reliability functions of some regular homogeneous series-parallel systems

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Abstract

In the investigation of large-scale systems the problem of the complexity of their reliability functions arises. This problem may be approximately solved by the assuming that the number of the system components tends to infinity and finding the limit reliability function. In this paper a four-element class of limit reliability functions for regular homogeneous series-parallel system is presented. The number of series components of the system has the order of the logarithm of the number of its parallel components. The result is obtained under the assumption that the lifetimes of the particular components are independent identically distributed random variables. The class presented is different from the known class of limit distributions of minimax statistics of independent random variables with the same distribution. Moreover, some examples of the considered systems and their limit reliability functions are given. The results can be useful in the reliability investigation of large systems.

1. Introduction. Limit theorems for distributions of extreme statistics included in [1], [3] and [7], and some results on limit distributions of minimax statistics contained in [1] and [2], provided the motivation for this paper. The results obtained in [2] and also presented in [1] refer to the limit reliability functions of the series-parallel system with identical components and equal numbers of series and parallel components.

In a natural way the problem of the existence of limit reliability functions for series-parallel systems with different numbers of series and parallel components arises. This problem is partly solved in [4] for the case when the number of series components is much less than the logarithm of the numbers of parallel components, and in [5] for the case when the number of series components is much greater that the logarithm of the number of par-
allel components. The present paper deals with the remaining case when the number of series components has the order of the logarithm of the number of parallel components of the system.

2. Basic notions. Suppose that $E_{ij}$, where $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l_i$, are components of a system $S$ and $X_{ij}$ are the lifetimes of $E_{ij}$. Moreover, suppose that $X_{ij}$ are independent random variables. We call the system $S$ series-parallel if its lifetime $X$ is given by

$$X = \max_{i \leq i \leq k} \min_{1 \leq j \leq l_i} X_{ij},$$

and regular if $l_1 = l_2 = \ldots = l_k = l, l \in N$. A regular system $S$ is called homogeneous if the random variables $X_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$, have the same distribution function $F(x) = P(X_{ij} \leq x)$, i.e. if the components $E_{ij}$ have the same reliability function $R(x) = 1 - F(x)$. Assuming $k = k_n$ and $l = l_n$, where $n$ tends to infinity and $k_n$ and $l_n$ are sequences of natural numbers such that at least one of them tends to infinity, we obtain a sequence of regular homogeneous systems corresponding to the sequence $(k_n, l_n)$. Next, replacing $n$ by a positive real number $t$ and assuming that $k_t$ and $l_t$ are positive real numbers, we obtain a family of regular systems corresponding to the pair $(k_t, l_t)$. The reliability functions for this family systems are given by

$$R_t(x) = 1 - [1 - (R(x))]^{l_t k_t}, \ x \in (-\infty, \infty), \ t \in (0, \infty).$$

We do not assume that the lifetime distributions are necessarily concentrated on $[0, \infty)$, so that a reliability function need not satisfy the usual condition $R(x) = 1$ for $x \in (-\infty, 0)$. This is a generalization of the usual concept of a reliability function which turns out to be convenient in theoretical considerations. At the same time, from our results on generalized reliability functions one gets, as particular cases, the same properties of the ordinary reliability functions. Under the above definition, the following corollary is true.

**Corollary 1.** A reliability function $R(x)$ is nonincreasing and right-continuous and $R(-\infty) = 1, R(+\infty) = 0$.

**Definition 1.** A reliability function $R(x)$ is called degenerate if there exists $x_0 \in (-\infty, \infty)$ such that $R(x) = 1$ for $x < x_0$ and $R(x) = 0$ for $x \geq x_0$.

**Corollary 2.** A function

$$R(x) = 1 - \exp[-V(x)], x \in (-\infty, \infty),$$

is a reliability function if and only if $V(x)$ is a nonnegative, nonincreasing,
right-continuous function, $V(-\infty) = \infty, V(\infty) = 0$. Besides that $V(x)$ may be identically $\infty$ in an interval.

**Assumption 1.** In our further considerations, $V(x)$ always stands for a function with the properties of Corollary 2. If $V(x)$ is identically $\infty$ in an interval, we assume that $\exp[-\infty] = 0$. The statement that $V(x)$ is nonnegative, nonincreasing and right-continuous, only concerns the interval where $V(x) \neq \infty$. Moreover, we denote the set of continuity points of the reliability function $R(x)$ by $C_R$ and the set of continuity points of $V(x)$ and points of such that $V(x) = \infty$ by $C_V$.

According to Definition 1, Corollary 2 and Assumption 1, we make the following definition.

**Definition 2.** A function $V(x)$ defined for $x \in (-\infty, \infty)$, nonnegative, nonincreasing, right-continuous and such that $V(-\infty) = \infty$ and $V(\infty) = 0$ is called degenerate if there exists $x_0 \in (-\infty, \infty)$ such that $V(x) = \infty$ for $x < x_0$ and $V(x) = 0$ for $x \geq x_0$.

Now, the following corollary is clear.

**Corollary 3.** The reliability function $R(x)$ given by (2) is degenerate if and only if $V(x)$ is degenerate.

**Definition 3.** A reliability function $R(x)$ is called a limit reliability function of the regular homogeneous series-parallel system if there exist functions $a_t = a(t) > 0$ and $b_t = b(t) \in (-\infty, \infty)$ such that

$$R_t(a_t x + b_t) \rightarrow R(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for} \quad x \in C_R.$$

The pair $(a_t, b_t)$ is called a norming function pair.

### 3. Limit reliability functions of a regular homogeneous series-parallel system.

We look for nondegenerate limit reliability functions of the form (2). Our considerations, according to Corollary 3, resolve into the assignment of possible types of $V(x)$. We first give several auxiliary theorems.

From (1) and (2), we immediately get the following lemma ([6]).

**Lemma 1.** If $R(x)$ is given by (2) and the family $R_t(x)$ is given by (1), $k_t \rightarrow \infty$ as $t \rightarrow -\infty$ and $a_t > 0, b_t \in (-\infty, \infty)$ are some functions, then

$$R_t(a_t x + b_t) \rightarrow R(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for} \quad x \in C_R$$

if and only if

$$k_t(T(a_t x + b_t))^t \rightarrow V(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for} \quad x \in C_V.$$

We shall need the following slight extension of Gnedenko's lemma ([4]).
Lemma 2. If \( R_t(x), t \in (0, \infty), \) is a family of reliability functions such that for some functions \( a_t > 0, b_t \in (-\infty, \infty), \)
\[
R_t(a_t x + b_t) \to R_0(x) \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in C_{R_0},
\]
where \( R_0(x) \) is a nondegenerate reliability function, then
\[
R_t(\alpha_t x + \beta_t) \to G_0(x) \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in C_{G_0},
\]
where \( G_0(x) \) is a nondegenerate reliability function and \( \alpha_t > 0, \beta_t \in (-\infty, \infty) \)
are some functions, holds if and only if there exist constants \( a > 0, b \in (-\infty, \infty) \) such that \( \frac{\alpha_t}{a_t} = a \) and \( \frac{\beta_t - b_t}{a_t} = b \) as \( t \to \infty. \) Moreover,
\[
G_0(x) = R_0(ax + b) \quad \text{for} \quad x \in (-\infty, \infty).
\]

Lemma 3. Suppose that \( k_t \to \infty \) as \( t \to \infty \) and there exist functions \( a_t > 0 \) and \( b_t \in (-\infty, \infty) \) such that
\[
k_t(R(a_t x + b_t))^t \to V_\nu(x) \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in C_V
\]
and for every natural \( \nu \geq 2 \)
\[
k_t(R(a_{\nu t} x + b_{\nu t}))^t \to V_\nu(x) \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in C_{V_\nu},
\]
where \( \tau_{\nu t} = \tau_\nu(t), t \in (0, \infty), \tau_\nu \in (0, \infty), R(x) \) is the reliability function of
a particular component of a regular homogeneous series-parallel system, and \( V(x) \) and \( V_\nu(x) \) are nondegenerate functions in the sense of Definition 2. Then there exist \( \alpha_\nu > 0 \) and \( \beta_\nu \in (-\infty, \infty) \) such that
\[
V_\nu(x) = V(\alpha_\nu x + \beta_\nu) \quad \text{for} \quad x \in (-\infty, \infty).
\]

Proof. From (1) and (3), by Lemma 1, it follows that
\[
R_t(a_t x + b_t) \to R(x) \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in C_R,
\]
where \( R(x) \) is given by (2); according to Corollary 3, it is a nondegenerate
function.

In an analogous way, from (1) and (4), it follows that
\[
R_t(a_{\nu t} x + b_{\nu t}) \to R_\nu(x) \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in C_{R_\nu},
\]
where
\[
R_\nu(x) = 1 - \exp[-V_\nu(x)]
\]
and \( R_\nu(x) \) is a nondegenerate reliability function. Therefore, by Lemma 2, there exist \( \alpha_\beta > 0 \) and \( \beta_\nu \in (-\infty, \infty) \) such that for all \( x \in (-\infty, \infty) \) and \( \nu \geq 2, \) we have
\[
R_\nu(x) = R(\alpha_\nu x + \beta_\nu).
\]
Now (6) and (2) imply (5). ■

Lemmas 2 and 3 justify the following definitions.
Definition 4. The reliability functions $R_0(x)$ and $R(x)$ are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, \infty)$ such that $R_0(x) = R(ax + b)$ for $x \in (-\infty, \infty)$.

Definition 5. The functions $V_0(x)$ and $V(x)$ with the properties given in Definition 2 are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, \infty)$ such that $V_0(x) = V(ax + b)$ for all $x \in (-\infty, \infty)$.

Lemma 4. If $k_t \to \infty$ as $t \to \infty$ and $a_t > 0$, $b_t \in (-\infty, \infty)$, are some functions, then from the assertion

$$R_t(a_t x + b_t) \to R(x) \text{ as } t \to \infty \text{ for } x \in C_R,$$

where $R(x)$ is a reliability function given by (2) it follows that for every natural $\nu \geq 2$ and all $x \in C_V$,

$$k_{\tau_{\nu}} k_t^{-t_{\nu}/l_{\nu}} [k_t(R(a_{\tau_{\nu}} x + b_{\tau_{\nu}}))^{l_{\nu}}]^{t_{\nu}/l_{\nu}} \to V(x) \text{ as } t \to \infty,$$

where $\tau_{\nu} = \tau_{\nu}(t)$, $t \in (0, \infty)$, and $\tau_{\nu} \to \infty$ as $t \to \infty$.

Proof. By (7) and from Lemma 1, we have

$$k_t(R(a_t x + b_t))^{l_t} \to V(x) \text{ as } t \to \infty \text{ for } x \in C_V.$$

Hence, for every natural $\nu \geq 2$, we get

$$k_{\tau_{\nu}} (R(a_{\tau_{\nu}} x + b_{\tau_{\nu}}))^{l_{\tau_{\nu}}} \to V(x) \text{ as } t \to \infty \text{ for } x \in C_V.$$

Now (8) follows from the transformation

$$k_{\tau_{\nu}} (R(a_{\tau_{\nu}} x + b_{\tau_{\nu}}))^{l_{\tau_{\nu}}} = k_{\tau_{\nu}} [(R(a_{\tau_{\nu}} x + b_{\tau_{\nu}}))^{l_{\nu}}]^{t_{\nu}/l_{\nu}}$$

$$= k_{\tau_{\nu}} k_t^{-t_{\nu}/l_{\nu}} [k_t(R(a_{\tau_{\nu}} x + b_{\tau_{\nu}}))^{l_{\nu}}]^{t_{\nu}/l_{\nu}}.$$

(8) is a necessary condition for the nondegenerate reliability function $R(x)$ given by (2) to be a limit reliability function of the system. We will translate this condition into a functional equation with unknown function $V(x)$. From this equation we shall determine the class of possible nondegenerate functions $V(x)$. To this end, we prove two auxiliary theorems. We introduce the following notation: $x(t) \sim y(t)$ means that $x(t)/y(t) \to 1$ as $t \to \infty$.

Lemma 5. If

$$k_t = 1, \ l_t = c \log t, \ \text{where} \ c > 0, \ t \in (0, \infty),$$

and the nondegenerate reliability function $R(x)$ given by (2) is a limit reliability function of a regular homogeneous series-parallel system, then for every natural $\nu \geq 2$ there exist numbers $\alpha_{\nu} > 0$ and $\beta_{\nu} \in (-\infty, \infty)$ such that

$$[V(\alpha_{\nu} x + \beta_{\nu})]^{\nu} = V(x) \text{ for } x \in (-\infty, \infty).$$
Proof. By Lemma 4, (8) holds for any $\tau_\nu(t)$, which tends to $\infty$ as $t$ tends to $\infty$. Take $k_t = t$, $\tau_\nu = t^\nu$ for $\nu \geq 2$, $t \in (0, \infty)$. Then for every natural $\nu \leq 2$ and all $t \in (0, \infty)$,

\begin{equation}
(10) \quad k_t \tau_\nu k_t^{-1} \tau_\nu / t = 1
\end{equation}

and

\begin{equation}
(11) \quad \ell_\nu / t = \nu.
\end{equation}

Hence $[k_t(R(a_{\tau_\nu}x + b_{\tau_\nu}))^\nu]$ converges to a nondegenerate function $V_0(x)$, i.e.

\begin{equation}
(12) \quad (R(a_{\tau_\nu}x + b_{\tau_\nu}))^\nu \rightarrow V_0(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for} \quad x \in C_{V_0}.
\end{equation}

Moreover, by Lemma 1,

\begin{equation}
(12) \quad k_t(R(a_{\tau_\nu}x + b_{\tau_\nu}))^\nu \rightarrow V(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for} \quad x \in C_V.
\end{equation}

By (12) and Lemma 3, we conclude that for every natural $\nu \geq 2$ there exist numbers $\alpha_\nu > 0$ and $\beta_\nu \in (-\infty, \infty)$ such that $V_0(x) = V(\alpha_\nu x + \beta_\nu)$ for all $x \in (-\infty, \infty)$. From (10)–(12) and (8), we now get (9). 

**Lemma 6.** Every nondegenerate solution of the equation (9) is one of the following forms:

\begin{align}
V_1(x) &= \begin{cases} 
\infty, & x < x_0, \\
w_1(x), & 0 < w_1(x) < 1, \quad x > x_0, \quad \text{where} \quad x_0 = \frac{\beta_\nu}{1 - \alpha_\nu},
\end{cases} \\
V_2(x) &= \begin{cases} 
w_2(x), & 1 < w_2(x) < \infty, \quad x < x_0, \\
0, & x > x_0, \quad \text{where} \quad x_0 = \frac{\beta_\nu}{1 - \alpha_\nu},
\end{cases} \\
V_3(x) &= \begin{cases} 
w_3(x), & 1 < w_3(x) < \infty, \quad x < x_0, \\
w_3(x), & 0 < w_3(x) < 1, \quad x < x_0, \quad \text{where} \quad x_0 = \frac{\beta_\nu}{1 - \alpha_\nu},
\end{cases} \\
V_4(x) &= \begin{cases} 
\infty, & x < x_1, \\
1, & x_1 < x < x_2, \\
0, & x > x_2, \quad \text{where} \quad x_1 < x_2,
\end{cases}
\end{align}

where $w_1(x), w_2(x), w_3(x), w_3(x)$ are nonincreasing and right-continuous. Moreover, $V_1(x), V_2(x)$ and $V_3(x)$ are solutions of (9) if and only if $\alpha_\nu < 1$ for every $\nu \geq 2$, and $V_4(x)$ is a solution of (9) if and only if $\alpha_\nu = 1$ and $\beta_\nu = 0$ for every $\nu \geq 2$.

**Proof.** Case 1: $\alpha_\nu > 1$ for some $\nu \geq 2$. Then $\alpha_\nu x + \beta_\nu > x$ for $x > x_0$. By monotonicity $V(\alpha_\nu x + \beta_\nu) \leq V(x)$ for $x > x_0$. Now (9) gives $V(x) \leq [V(x)]^\nu$ for $x > x_0$. The properties given in Corollary 2 now imply that

\begin{equation}
(17) \quad V(x) = 0 \quad \text{for} \quad x > x_0.
\end{equation}
Since \( \alpha_\nu > 1 \), we have \( \alpha_\nu x + \beta_\nu < x \) for \( x < x_0 \), and so \( V(\alpha_\nu x + \beta_\nu) \geq V(x) \) for \( x < x_0 \). By (9) \( V(x) \geq [V(x)]^\nu \) for \( x < x_0 \). The properties given in Corollary 2 now yield

\[
V(x) = \infty \quad \text{for} \quad x < x_0.
\]

Combining (17) and (18), we conclude that if \( \alpha_\nu > 1 \), then (9) is only satisfied by a degenerate function.

\textbf{Case 2}: \( \alpha_\nu < 1 \) for some \( \nu \geq 2 \). Then \( \alpha_\nu x + \beta_\nu > x \) for \( x < x_0 \), and hence \( V(\alpha_\nu x + \beta_\nu) \leq V(x) \) for \( x < x_0 \). This gives either

\[
V(x) = \infty \quad \text{for} \quad x < x_0.
\]

or

\[
1 < V(x) < \infty \quad \text{for} \quad x < x_0.
\]

Since \( \alpha_\nu < 1 \), we have \( \alpha_\nu x + \beta_\nu < x \) for \( x > x_0 \) and so \( V(\alpha_\nu x + \beta_\nu) \geq V(x) \) for \( x > x_0 \). Hence \( V(x) \geq [V(x)]^\nu \) for \( x > x_0 \). This leads to

\[
0 < V(x) < 1 \quad \text{for} \quad x > x_0
\]

or

\[
V(x) = 0 \quad \text{for} \quad x > x_0.
\]

Combining (19)–(20) we conclude that in this case, (9) can only be satisfied functions of the form \( V_1(x) \), \( V_2(x) \), \( V_3(x) \) or by a degenerate function.

\textbf{Case 3}: \( \alpha_\nu = 1 \) for some \( \nu \geq 2 \). Then (9) takes the form

\[
[V(x + \beta_\nu)]^\nu = V(x) \quad \text{for} \quad x \in (-\infty, \infty).
\]

If \( \beta_\nu = 0 \), then the only solution with the properties given in Corollary 2 is

\[
V(x) = \begin{cases} 
\infty & \text{for} \quad x < x_1, \\
1 & \text{for} \quad x_1 < x < x_2, \\
0 & \text{for} \quad x > x_2, \quad \text{where} \quad x_1 < x_2.
\end{cases}
\]

If \( \beta_\nu \neq 0 \), then no such solution exists. Therefore, in this case, (9) can only be satisfied by functions of the form \( V_4(x) \).

In Cases 1, 2 and 3 of the proof we showed that if \( \alpha_\nu < 1 \) for every \( \nu \geq 2 \), then every nondegenerate function satisfying (9) is of the form \( V_1(x) \), \( V_2(x) \) or \( V_3(x) \), and if \( \alpha_\nu = 1 \) and \( \beta_\nu = 0 \) for every \( \nu \geq 2 \), then a nondegenerate function satisfying (9) is of the form \( V_4(x) \). It remains to check whether there exist any nondegenerate functions of other types that satisfy (9) in the case when \( \alpha_\nu < 1 \) for some \( \nu \) and \( \alpha_\nu = 1 \) for other \( \nu \). Suppose that this is true. Let \( \alpha_{\nu_1} < 1 \) for some \( \nu_1 \geq 2 \). Then, by the previous considerations, (9) is only satisfied by nondegenerate functions of the form \( V_1(x) \), \( V_2(x) \) or \( V_3(x) \). Suppose that \( \alpha_{\nu_2} = 1 \) for some \( \nu_2 \geq 2 \), \( \nu_2 \neq \nu_1 \). In this case (9) is only satisfied by a nondegenerate function of the form \( V_4(x) \). Since \( V_1(x) \neq 0 \) for
all \( x \in (-\infty, \infty) \) and there exists \( x \in (-\infty, \infty) \) such that \( V_4(x) = 0 \) and \( V_2(x) \neq \infty \) for all \( x \in (-\infty, \infty) \) and there exists \( x \in (-\infty, \infty) \) such that \( V_4(x) = \infty \) and besides \( V_3(x) \neq 0 \) and \( V_3 \neq \infty \) for all \( x \in (-\infty, \infty) \), it follows that there is no nondegenerate function \( V(x) \) that satisfies (9) in the case when \( \alpha_{\nu} < 1 \) for some \( \nu \) and \( \alpha_{\nu} = 1 \) for some \( \nu \).

From the above discussion it follows that if \( \alpha_{\nu} < 1 \) for some \( \nu \geq 2 \), then (9) is only satisfied by a nondegenerate function \( V(x) \) of the form \( V_1(x) \), \( V_2(x) \) or \( V_3(x) \) and, at the same time, \( \alpha_{\nu} < 1 \) for every \( \nu \geq 2 \). On the other hand, if \( \alpha_{\nu} = 1 \) for some \( \nu \geq 2 \), then (9) is only satisfied by a nondegenerate function \( V(x) \) of type \( V_4(x) \) and at the same time, \( \alpha_{\nu} = 1 \) for every \( \nu \geq 2 \). This means that the only nondegenerate functions that satisfy (9) are functions of the form \( V_1(x) \), \( V_2(x) \), \( V_3(x) \) or \( V_4(x) \). Moreover, if \( V(x) \) is of the form \( V_1(x) \), \( V_2(x) \) or \( V_3(x) \), then \( x_0 = \frac{\beta_{\nu}}{1-\alpha_{\nu}} \) is constant for every \( \nu \geq 2 \). ■

Remark 1. In the proof of Lemma 6, it can be noticed ([6]) that apart from the functions of the forms (13)–(16), the only nonincreasing nondegenerate functions which satisfy (9) are

\[
V(x) = 0 \quad \text{for } x \in (-\infty, \infty),
\]

\[
V(x) = 1 \quad \text{for } x \in (-\infty, \infty)
\]

and

\[
V(x) = \infty \quad \text{for } x \in (-\infty, \infty)
\]

in all cases discussed there, and moreover,

\[
V(x) = \begin{cases} 
\infty & \text{for } x < x_0, \\
w(x), 1 < w(x) < \infty & \text{for } x > x_0, w(\infty) = 1,
\end{cases}
\]

\[
V(x) = \begin{cases} 
w(x), 0 < w(x) < 1 & \text{for } x < x_0, w(-\infty) = 1, \\
0 & \text{for } x > x_0,
\end{cases}
\]

where \( x_0 = \frac{\beta_{\nu}}{1-\alpha_{\nu}} \),

\[
V(x) = \begin{cases} 
\infty & \text{for } x < x_1, \\
1 & \text{for } x > x_1, \text{ where } x_1 \in (-\infty, \infty),
\end{cases}
\]

and

\[
V(x) = \begin{cases} 
1 & \text{for } x < x_1, \\
0 & \text{for } x > x_1, \text{ where } x_1 \in (-\infty, \infty),
\end{cases}
\]

in the case when \( \alpha_{\nu} > 1 \) and \( \beta_{\nu} = 0 \) for every \( \nu \geq 2 \), and

\[
V(x) = w(x), 1 < w(x) < \infty \quad \text{for } x \in (-\infty, \infty), w(\infty) = 1.
\]

in the case when \( \alpha_{\nu} = 1 \) and \( \beta_{\nu} > 0 \) for every \( \nu \geq 2 \), and

\[
V(x) = w(x), 0 < w(x) < 1 \quad \text{for } x \in (-\infty, \infty), w(-\infty) = 1.
\]
in the case when \( \alpha_\nu = 1 \) and \( \beta_\nu < 0 \) for every \( \nu \geq 2 \), where \( w(x) \) is a nonincreasing and right-continuous function.

**Definition 6** ([3]). A positive function \( u(x) \) defined for \( x \in (0, \infty) \) varies regularly at infinity if there exists \( r \in (-\infty, \infty) \) such that

\[
\frac{u(tx)}{u(t)} = x^r \quad \text{as} \quad t \to \infty \quad \text{for} \quad x \in (0, \infty).
\]

The number \( r \) is called the exponent of regular variation of \( u(x) \).

**Lemma 7.** ([3]) A monotone positive function \( u(x) \) defined for \( x \in (0, \infty) \) varies regularly if and only if there exist sequences \( d_\nu \) and \( e_\nu \) of positive numbers with \( d_{\nu+1}/d_\nu \to 1 \) and \( e_\nu \to \infty \) as \( \nu \to \infty \) such that \( d_\nu u(x_\nu x) = f(x) \) as \( \nu \to \infty \) for \( x \in (0, \infty) \), where \( f(x) \) is positive and finite. Moreover, \( \frac{f(x)}{f(1)} = x^r \) for \( x \in (0, \infty) \), where \( r \in (-\infty, \infty) \) is the exponent of regular variation of \( u(x) \).

**Theorem 1.** If

\[ k_t = t, \quad l_t = c \log t, \quad \text{where} \quad c > 0, \quad t \in (0, \infty), \]

then the only possible nondegenerate limit reliability functions of the regular homogeneous series-parallel system are:

\[
\begin{align*}
\mathcal{R}_1(x) &= \begin{cases} 
1 & \text{for } x < 0, \\
1 - \exp[- \exp[-x^{\alpha}]] & \text{for } x \geq 0, \quad \text{where } \alpha > 0,
\end{cases} \\
\mathcal{R}_2(x) &= \begin{cases} 
1 - \exp[- \exp[-x^{\alpha}]] & \text{for } x < 0, \quad \text{where } \alpha > 0, \\
0 & \text{for } x \geq 0,
\end{cases} \\
\mathcal{R}_3(x) &= \begin{cases} 
1 - \exp[- \exp[\beta(-x)^{\alpha}]] & \text{for } x < 0, \\
1 - \exp[- \exp[-x^{\alpha}]] & \text{for } x \geq 0, \quad \text{where } \alpha > 0, \quad \beta > 0,
\end{cases} \\
\mathcal{R}_4(x) &= \begin{cases} 
1 & \text{for } x < x_1, \\
1 - \exp[-1] & \text{for } x_1 \leq x \leq x_2, \\
0 & \text{for } x \geq x_2, \quad \text{where } x_1 < x_2.
\end{cases}
\]

**Proof.** According to Lemmas 5 and 6, under the assumption (24), if the reliability function

\[
\mathcal{R}(x) = 1 - \exp[-V(x)], \quad x \in (-\infty, \infty),
\]

is a nondegenerate limit reliability function of the regular homogeneous series-parallel system, then the nondegenerate function \( V(x) \) satisfies the following equation

\[
[V(\alpha_\nu + \beta_\nu)]^\nu = V(x)
\]

for \( \nu \geq 2 \) and \( x \in (-\infty, \infty) \). Moreover, there is no such nondegenerate function if \( \alpha_\nu > 1 \) for every \( \nu \geq 2 \), and \( V(x) \) is of the form \( V_1(x) \) given by
or $V_2(x)$ given by (14) or $V_3(x)$ given by (15) if $\alpha_\nu < 1$ for every $\nu \geq 2$, and it is of the form $V_3(x)$ given by (16) if $\alpha_\nu = 1$ for every $\nu \geq 2$.

**Case 1.** If $\alpha_\nu > 1$ for every $\nu \geq 2$, then there is no nondegenerate limit reliability function of the system.

**Case 2.** If $\alpha_\nu < 1$ for every $\nu \geq 2$, then from (30) and (13) it follows that $[w_1(\alpha_\nu x + \beta_\nu)]^\nu = w_1(x)$ for $\nu \geq 2$ and $x > x_0$. Without loss of generality, we assume $\beta_\nu = 0$. Then $x_0 = 0$ and

$$[w_1(\alpha_\nu x)]^\nu = w_1(x)$$

for $\nu \geq 2$ and $x \in (0, \infty)$, and moreover.

$$V_1(x) = \begin{cases} \infty & \text{for } x < 0, \\ w_1(x) & \text{for } x > 0, \end{cases} \text{ where } 0 < w_1(x) < 1.$$  

Taking the logarithm of both sides of the equality (31), we have $\nu [- \log w_1(\alpha_\nu x)] = - \log w_1(x)$ for $\nu \geq 2$ and $x \in (0, \infty)$ and next, substituting $x := \frac{1}{\alpha_\nu} x$, we get

$$\frac{1}{\nu} \left[ - \log w_1 \left( \frac{1}{\alpha_\nu} x \right) \right] = - \log w_1(x)$$

for $\nu \geq 2$ and $x \in (0, \infty)$. Since $- \log w_1(x)$ is nondecreasing, $- \log w_1(x) < \infty$ for $x \in (0, \infty)$, and $- \log w_1(\infty) = \infty$ and $\frac{1}{\nu} \to 0$ as $\nu \to \infty$. (33) it follows that

$$\frac{1}{\alpha_\nu} \to \infty \text{ as } \nu \to \infty.$$  

From (33) and (34), assuming $c_\nu = \frac{1}{\alpha_\nu}$, $d_\nu = \frac{1}{\nu}$, we conclude by Lemma 7 that the function $- \log w_1(x)$ is regularly varying. Therefore, by Definition 6, there exists $r \in (-\infty, \infty)$ such that

$$- \log w_1 \left( \frac{1}{\alpha_\nu} x \right) - x^r \text{ as } \nu \to \infty \text{ for } x \in (0, \infty).$$

Hence

$$x^r = \lim_{\nu \to \infty} \frac{- \log w_1 \left( \frac{1}{\alpha_\nu} x \right)}{- \log w_1 \left( \frac{1}{\alpha_\nu} \right)}$$

$$= \lim_{\nu \to \infty} \frac{1}{\nu} \left[ - \log w_1 \left( \frac{1}{\alpha_\nu} \right) \right] = \frac{- \log w_1(x)}{- \log w_1(1)}.$$
for $x \in (0, \infty)$, i.e. $-\log w_1(x) = ax^\nu$, where $a = -\log w_1(1) > 0$ by (32). Since the function $-\log w_1(x)$ is nondecreasing, then $r = \alpha$, where $\alpha > 0$. Hence $V_1(x)$ is of the type

\begin{equation}
V_1(x) = \begin{cases} 
\infty & \text{for } x < 0, \\
\exp[-x^\alpha] & \text{for } x > 0, \text{ where } \alpha > 0.
\end{cases}
\end{equation}

It follows that $R(x)$ is given by (25).

Similarly, from (30) and (14), we get $[w_2(\alpha, x + \beta)]^\nu = w_2(x)$ for $\nu \geq 2$ and $x < x_0$. Proceeding in an analogous way, we conclude that $V_2(x)$ is of the type

\begin{equation}
V_2(x) = \begin{cases} 
\exp[(x)^\alpha] & \text{for } x < 0, \text{ where } \alpha > 0 \\
0 & \text{for } x > 0.
\end{cases}
\end{equation}

Hence, $R(x)$ is given by (26). Now, taking under consideration (30) and (15), we obtain $[w_3(\alpha, x + \beta)]^\nu = w_3(x)$ for $\nu \geq 2$ and $x < x_0$ and $[w_3(\alpha, x + \beta)]^\nu = w_3(x)$ for $\nu \geq 2$ and $x > x_0$. Without loss of generality, we assume $\beta = 0$. Then $x_0 = 0$ and

\begin{equation}
[w_3(\alpha, x)]^\nu = w_3(x)
\end{equation}

for $\nu \geq 2$ and $x \in (-\infty, 0)$, and

\begin{equation}
[w_3(\alpha, x)]^\nu = w_3(x)
\end{equation}

for $\nu \geq 2$ and $x \in (0, \infty)$, and moreover,

\begin{equation}
V_3(x) = \begin{cases} 
w_3^1(x) & \text{for } x < 0, \text{ where } 1 < w_3^1(x) < \infty \\
w_3^2(x) & \text{for } x > 0, \text{ where } 0 < w_3^2(x) < 1.
\end{cases}
\end{equation}

Proceeding with $V_3(x)$ for $x \in (-\infty, 0)$ in the same way as with $V_2(x)$, we get

\begin{equation}
w_3^1(x) = \exp[c_1(-x)^a_1]
\end{equation}

for $a_1 \in (-\infty, 0)$, where $a_1 > 0$ and $c_1 > 0$; and proceeding with $V_3(x)$ for $x \in (0, \infty)$ in the same way as with $V_1(x)$, we have

\begin{equation}
w_3^2(x) = \exp[-c_2x^{a_2}]
\end{equation}

for $x \in (0, \infty)$, where $a_2 > 0$ and $c_2 > 0$. Since, by (37) and (38), \( \exp[\nu c_1(-x)^{a_1}] = \exp[c_1(-x)^{a_1}] \) for every $\nu \geq 2$ and $x \in (-\infty, 0)$ and $\exp[\nu c_2(\alpha, x)^{a_2}] = \exp[-c_2x^{a_2}]$ for every $\nu \geq 2$ and $x \in (0, \infty)$, we have $\nu a_1^{a_1} = 1$ and $\nu a_2^{a_2} = 1$ for every $\nu \geq 2$. But this is possible only if $a_1 = a_2 = \alpha$, where $\alpha > 0$. We conclude that

\begin{equation}
V_3(x) = \begin{cases} 
\exp[\beta(-x)^\alpha] & \text{for } x < 0 \\
\exp[-x^\alpha] & \text{for } x > 0, \text{ where } \alpha > 0, \beta > 0.
\end{cases}
\end{equation}

Hence, $R(x)$ is given by (27).

Case 3. If $\alpha_\nu = 1$, then from (29) and (16) it immediately follows that $R(x)$ is given by (28). \( \blacksquare \)
Remark 2. In Remark 1 we noticed that the family \( R_t(x) \) can be convergent to some nondegenerate nonincreasing functions which are not reliability functions. These functions will be called singular limits. Some general information about their forms is given in Remark 1. From that remark and from (2) it immediately follows that, under the assumptions of Theorem 1, the types of singular limits of the family \( R_t(x) \) are

\[
\begin{align*}
S_1(x) &= 0 \quad \text{for } x \in (-\infty, \infty), \\
S_2(x) &= 1 \quad \text{for } x \in (-\infty, \infty), \\
S_3(x) &= 1 - \exp[-1] \quad \text{for } x \in (-\infty, \infty), \\
S_4(x) &= \begin{cases} 1 & \text{for } x < 0, \\ 1 - \exp[-1] & \text{for } x \geq 0, \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
S_5(x) &= \begin{cases} 1 - \exp[-1] & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}
\end{align*}
\]

Moreover, it can be shown ([16]) that, in this case, the only remaining types of singular limits of the family \( R_t(x) \) are

\[
\begin{align*}
S_6(x) &= \begin{cases} 1 & \text{for } x \leq 0, \\ 1 - \exp[-\exp[x^{-\alpha}]] & \text{for } x > 0, \text{ where } \alpha > 0, \end{cases} \\
S_7(x) &= \begin{cases} 1 - \exp[-\exp[-(-x)^{-\alpha}]] & \text{for } x < 0, \text{ where } \alpha > 0, \\ 0 & \text{for } x \leq 0, \end{cases} \\
S_8(x) &= 1 - \exp[-\exp[\exp[-x]]] \quad \text{for } x \in (-\infty, \infty) \\
S_9(x) &= 1 - \exp[-\exp[-\exp x]] \quad \text{for } x \in (-\infty, \infty).
\end{align*}
\]

Theorem 2. If

\[
k_t = t, \quad l_t = c \log l + r(l), \quad r(l) \sim s, \quad s \in (-\infty, \infty), \quad c > 0, \quad l \in (0, \infty),
\]

then the only possible nondegenerate limit reliability functions of the regular homogeneous series-parallel system are:

\[
\begin{align*}
\tilde{R}_1(x) &= \begin{cases} 1 & \text{for } x < 0, \\ 1 - \exp\left[ -\exp\left[ -x^\alpha - \frac{s}{c} \right] \right] & \text{for } x \geq 0, \text{ where } \alpha > 0, \end{cases} \\
\tilde{R}_2(x) &= \begin{cases} 1 - \exp\left[ -\exp\left[ (-x)^\alpha - \frac{s}{c} \right] \right] & \text{for } x < 0, \text{ where } \alpha > 0, \\ 0 & \text{for } x \geq 0, \end{cases} \\
\tilde{R}_3(x) &= \begin{cases} 1 - \exp\left[ -\exp\left[ (\theta(-x)^\alpha - \frac{s}{c} \right] \right] & \text{for } x < 0, \\ 1 - \exp\left[ -\exp\left[ -x^\beta - \frac{s}{c} \right] \right] & \text{for } x \geq 0, \text{ where } \alpha > 0, \beta > 0, \end{cases}
\end{align*}
\]
Class of limit reliability functions

\[
\tilde{R}_4(x) = \begin{cases} 
1 & \text{for } x < x_1, \\
1 - \exp \left[ - \exp \left( - \frac{s}{c} \right) \right] & \text{for } x_1 \leq x < x_2, \\
0 & \text{for } x \geq x_2.
\end{cases}
\]

Proof. If \( l_t \) is given by (21), then, according to Theorem 1, the only possible nondegenerate limit reliability functions of the system are \( R_1(x), \ R_2(x), \ R_3(x) \) and \( R_4(x) \). This means that there exist functions \( a_t > 0 \) and \( b_t \in (-\infty, \infty) \) such that

\[
t(R(a_t x + b_t))^c \log t \rightarrow V(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for } x \in C_V,
\]

where \( R(x) \) is the reliability function of a particular component of the system and \( V(x) \) is given by (35), (36), (39) or (16). Since

\[
k_t(R(a_t x + b_t))^c \log t + r(t) = t^{1 - \frac{c \log t + r(t)}{c \log t}} \left[t(R(a_t x + b_t))^c \log t\right]^{\frac{c \log t + r(t)}{c \log t}}
\]

for \( x \in (-\infty, \infty), \ t \in (0, \infty) \),

\[
t^{1 - \frac{c \log t + r(t)}{c \log t}} \rightarrow e^{-\zeta}
\]

and

\[
\frac{c \log t + r(t)}{c \log t} \rightarrow 1 \quad \text{as} \quad t \rightarrow \infty,
\]

by (40) we have

\[
k_t(R(a_t x + b_t))^c \log t + r(t) \rightarrow e^{-\zeta} V(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for } x \in C_V.
\]

Hence, from (35), (36), (39) and (16), by Lemma 1, we deduce that the reliability functions \( \tilde{R}_1(x), \ \tilde{R}_2(x), \ \tilde{R}_3(x) \) and \( \tilde{R}_4(x) \) are limit reliability functions of the system. Now, if for any other norming functions \( \alpha_t > 0 \) and \( \beta_t \in (-\infty, \infty) \) there exist nondegenerate limit reliability functions of the system, then, by Lemma 2, they are of the same type.

4. Examples. Using Lemma 1, it is easy to check the following examples.

Example 1. If the regular homogeneous series-parallel system is such that

\[
R(x) = \begin{cases} 
1, & x < 0, \\
\exp[-A \exp[xB]], & x \geq 0, \ A > 0, \ B > 0, 
\end{cases}
\]

and the pairs \((k_t, l_t)\) and \((a_t, b_t)\) satisfy the conditions

(i) \( k_t \rightarrow \infty, \ A l_t - \log k_t \rightarrow 0 \) as \( t \rightarrow \infty, \)

(ii) \( a_t = \frac{1}{(A l_t)^{1/B}}, \ b_t = 0 \) for \( t \in (0, \infty) \), then

\[
\tilde{R}_1(x) = \begin{cases} 
1, & x < 0, \\
1 - \exp[-\exp[-xB]], & x \geq 0,
\end{cases}
\]

is the limit reliability function of the system.
Example 2. If the regular homogeneous series-parallel system is such that
\[ R(x) = \begin{cases} \exp[-A \exp[-x^2]], & x < 0, A > 0, \\ 0, & x \geq 0, \end{cases} \]
and the pairs \((k_t, l_t)\) and \((a_t, b_t)\) satisfy the conditions
(i) \(k_t = \infty, A l_t = \log k_t = 0\) as \(t \to \infty\),
(ii) \(a_t = \frac{1}{\sqrt{A l_t}}, b_t = 0\) for \(t \in (0, \infty)\), then
\[ \mathbb{R}_2(x) = \begin{cases} 1 - \exp[-\exp(-x^2)], & x < 0, \\ 0, & x \geq 0, \end{cases} \]
is the limit reliability function of the system.

Example 3. If the regular homogeneous series-parallel system is such that
\[ R(x) = \begin{cases} \exp[-A \exp[-x^2]], & x < 0, \\ \exp[-A \exp[x^2]], & x \geq 0, A > 0, \end{cases} \]
and the pairs \((k_t, l_t)\) and \((a_t, b_t)\) satisfy the conditions
(i) \(k_t = \infty, A l_t = \log k_t = 0\) as \(t \to \infty\),
(ii) \(a_t = \frac{1}{\sqrt{A l_t}}, b_t = 0\) for \(t \in (0, \infty)\), then
\[ \mathbb{R}_3(x) = \begin{cases} 1 - \exp[-\exp[-x^2]], & x < 0, \\ 1 - \exp[-\exp[-x^2]], & x \geq 0, \end{cases} \]
is the limit reliability function of the system.

Example 4. If the regular homogeneous series-parallel system is such that
\[ R(x) = \begin{cases} 1, & x < 0, \\ \exp^{-1}, & 0 \leq x < A, \\ 0, & x \geq A, A > 0, \end{cases} \]
and the pairs \((k_t, l_t)\) and \((a_t, b_t)\) satisfy the conditions
(i) \(k_t = \infty\) as \(t \to \infty\), \(l_t = \log k_t\) for \(t \in (0, \infty)\),
(ii) \(a_t = 1, b_t = 0\) for \(t \in (0, \infty)\), then
\[ \mathbb{R}_4(x) = \begin{cases} 1 - \exp[-1], & 0 \leq x < A, \\ 0, & x \geq A, \end{cases} \]
is the limit reliability function of the system.

5. Summary. In the paper a class of limit reliability functions for a regular homogeneous series-parallel system is found. The result is obtained under the assumption that the lifetimes of the particular components are independent random variables with the same distribution function and the pair \((k_t, l_t)\) has the property that \(k_t = t\) and \(l_t = e \ln t + r(t), r(t) \sim s,\)
s ∈ (-∞, ∞), c > 0 where t tends to infinity. The class obtained is four-element and is different from the class of limit distributions of min-max statistics of independent random variables with the same distribution obtained in [2] for the case when k_i = n and l_i = n, n ∈ N. Because of the duality ([1]), the only possible limit reliability functions of the regular homogeneous parallel-series system are \( \overline{R}_i(x) = 1 - \overline{R}_i(-x) \), i = 1, 2, 3, 4.

References


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