An analysis of the exterior Neumann problem for the Poisson equation in connection with a numerical procedure

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This paper is concerned with transformation of the following exterior problem:

\[ \begin{align*}
\Delta u &= f & \text{on } \Omega^c, \\
\frac{\partial u}{\partial n} |_{\Gamma} &= g,
\end{align*} \]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded region, \( \Omega^c = \text{int}(\mathbb{R}^2 \setminus \Omega) \), into a problem represented by two equations. The first one is posed on a bounded domain and the second one is posed on the outer part of the boundary of the domain. This new problem is suitable for a numerical method based on the coupling of the finite and boundary element.

1. Introduction. In physics we often face the problem of finding a potential of a certain physical quantity. For example, a velocity potential in aerodynamics or an electric field potential in electrostatics. This leads to the exterior Dirichlet or Neumann problem for the Laplace or Poisson equation. The main difficulty in finding a numerical solution in this case is the unboundedness of the domain on which the problem is posed.

In the case of the Laplace equation we can transform the basic problem into an integral equation on the boundary of the domain by using the single or double layer representation of the solution to obtain a Fredholm equation of the second kind.

Another approach to the exterior Dirichlet problem for the Laplace equation in \( \mathbb{R}^3 \) was presented by J.C. Nedelec and J. Planchard in [8] where a Fredholm equation of the first kind (the solution is represented by a single layer potential) yielded a certain elliptic variational problem, with unknown being the jump of the normal derivative of the solution across the boundary.
A similar method was applied by M. N. LeRoux in [6] for the plane Dirichlet problem together with a detailed analysis of convergence of numerical solutions to the exact solution.

A method of solving the exterior Neumann problem for the Laplace equation was presented by J. Giroire and J. C. Nedelec in [3]. The authors transformed the above problem to an elliptic variational problem on the boundary, the unknown of which was the jump of the solution across the boundary.

A method for numerical solution of the exterior boundary problem for the Poisson equation was presented by C. Johnson and J. C. Nedelec in [4] for the Dirichlet condition. The basic problem in this case was transformed to a problem represented by two equations. The first was a Poisson equation on a bounded domain, the second was an integral equation on the outer part of the boundary of the domain. The integral equation was obtained by the Green formula for harmonic functions under the assumption that the solution of the basic problem was harmonic in the exterior of the domain.

In this paper we deal with the transformation of the exterior nonhomogeneous Neumann problem for the Poisson equation into a problem posed on a bounded domain and on the outer part of the boundary of the domain using the methods described in [4].

The existence and uniqueness of the solution to the basic problem is proved in the space of potentials introduced by M. N. LeRoux in [5]. It is proved that the associated Dirichlet form is elliptic in the quotient space of potentials modulo the constant functions. The proof is based on Lemma (1–1-1) of [5]. Moreover, a general form of the Green formula from [1] is recalled to prove the equivalence between the basic problem and its variational form (see Th. 2.1). This formula is applied extensively throughout the paper. A similar form of the Green formula in $\mathbb{R}^3$ for potential spaces can be found in the paper by J. Giroire and J. C. Nedelec [3] (see Proposition I–2). In the present paper attention is paid to the exact formulation of the conditions (the choice of proper spaces for the data and the solution, the assumptions concerning the boundary) for the equivalence of the basic problem and the transformed one. The complete exterior problem, i.e. under the assumption that the right-hand side of the Poisson equation can have an unbounded support (see (2.1)), is considered first. Then a problem with the restricted right-hand side is considered (see (3.2)) and the difference between the solution of the basic and the restricted problem is estimated (see Th. 3.1). Finally, the equivalence between the transformed problem (3.10) and the restricted problem (3.2) is established.

According to the ideas from [3] the solution is sought in the quotient spaces modulo the constant functions because the solution of the exterior Neumann problem in the plane is defined up to a constant.
The existence and uniqueness of the solution to the transformed problem is proved by means of the above mentioned equivalence (see Corollary 3.1), in contrast with [4], where the transformed problem was reduced to a Fredholm equation of the second kind (see 2.18 of [4]) and the uniqueness was proved for the latter equation.

The proof of regularity of the solution (see Th. 4.1) is based on the above mentioned equivalence, which leads to a simpler bilinear form (compare (6.4), (6.5) of [4] and the beginning of the proof of Th. 4.1 from the present paper). However, it should be noted that in [4] more general data for the transformed problem was considered, requiring the use of a more complicated bilinear form.

2. The basic problem. We are looking for a solution of the following problem:

\[
\begin{align*}
\begin{cases}
-\Psi^2 \Delta u &= f_1 \quad \text{on } \Omega^c, \\
\frac{\partial u}{\partial n}|r &= g,
\end{cases}
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^2\) is an open bounded simply connected set with boundary \(\Gamma\), \(\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}\) and \(\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}_+\) is the weight function, defined by \(\Psi(x) = \sqrt{1 + r^2(1 + \log \sqrt{1 + r^2})}\), \(r^2 = x_1^2 + x_2^2\), \(x = (x_1, x_2)\) (the role of this function will be explained later on).

Assume that \(\Gamma\) is regular, i.e. it can be described by a finite number of \(\mathcal{C}^1\) functions. In the classical approach we are looking for a solution \(u \in \mathcal{C}^2(\Omega^c) \cap \mathcal{C}^1_\nu(\overline{\Omega}^c)\), where \(\mathcal{C}^1_\nu(\overline{\Omega}^c)\) is defined in [2] (Tome 1, Chap. II, Par. 1, 3b). Roughly speaking \(\mathcal{C}^1_\nu(\overline{\Omega}^c)\) consists of functions in \(\mathcal{C}^1(\overline{\Omega}^c) \cap \mathcal{C}^0(\overline{\Omega}^c)\) for which it is possible to define an outer normal derivative in \(\mathcal{C}^0(\Gamma)\) on \(\Gamma\).

The formulation of sufficient conditions on the functions \(f = f_1/\Psi^2 : \Omega^c \rightarrow \mathbb{R}\), \(g : \Gamma \rightarrow \mathbb{R}\), on the boundary \(\Gamma\) and on the solution \(u \in \mathcal{C}^2(\Omega^c) \cap \mathcal{C}^1_\nu(\overline{\Omega}^c)\) to obtain existence and uniqueness modulo constants can be found in [2] (see Tome 1, Chap. II, Par. 2, Sec. 3, Par. 3, Sec. 2, 3, 4, Par. 4. Sec. 4, 5).

For example, assume that \(f \in \mathcal{C}^0(\overline{\Omega}^c)\) satisfies a local Hölder condition and \(|f(x)| \leq C/|x|^{2+\varepsilon}\) for \(|x| > R\), \(R, \varepsilon > 0\) (\(|\cdot|\) denotes the Euclidean norm in \(\mathbb{R}^n, n \in \mathbb{N}\)), \(g \in \mathcal{C}^0(\Gamma)\). We also assume the following solvability condition:

\[
\int_{\Omega^c} f \, dx + \int_{\Gamma} g \, d\sigma = 0,
\]

where \(\Gamma\) is described locally by a finite number of functions with first derivatives satisfying the Hölder condition. Then there exists a classical solution \(u \in \mathcal{C}^2(\Omega^c) \cap \mathcal{C}^1_\nu(\overline{\Omega}^c)\) satisfying \(|\text{grad } u| \rightarrow 0\) as \(|x| \rightarrow \infty\), which is unique up to an additive constant.
In this paper we discuss a more general situation where \( f_1 \in W^0(\Omega^c) \), \( g \in H^{-1/2}(\Gamma) \), the boundary \( \Gamma \) is Lipschitz continuous, and we look for a solution \( u \) belonging to the space \( W^1(\Omega^c, \Delta) \). The above mentioned spaces can be considered as subspaces of spaces of distributions, and they are defined below. The derivatives in the formulation of the problem are to be understood in the distributional sense.

Let

\[
W^0(\Omega^c) = \{ v \in \mathcal{D}'(\Omega^c) : \Psi^{-1} v \in L^2(\Omega^c) \}.
\]

The norm in \( W^0 \) is denoted by \( \| \cdot \|_{0, \Psi, \Omega^c} \), and defined by

\[
\| v \|_{0, \Psi, \Omega^c} = \| \Psi^{-1} v \|_{0, \Omega^c}, \quad \forall v \in W^0(\Omega^c),
\]

where \( \| \cdot \|_{0, \Omega^c} \) denotes the norm in \( L^2(\Omega^c) \) and \( \mathcal{D}'(\Omega^c) \) is the space of distributions defined on \( \Omega^c \).

Let

\[
W^1(\Omega^c) = \{ v \in W^0(\Omega^c) : \frac{\partial v}{\partial x_i} \in L^2(\Omega^c), \ i = 1, 2 \}.
\]

The norm in \( W^1 \) is denoted by \( \| \cdot \|_{1, \Psi, \Omega^c} \), and defined by

\[
\| v \|_{1, \Psi, \Omega^c} = \| v \|_{0, \Psi, \Omega^c} + \left( \left\| \frac{\partial v}{\partial x_1} \right\|_{0, \Omega^c}^2 + \left\| \frac{\partial v}{\partial x_2} \right\|_{0, \Omega^c}^2 \right)^{1/2}, \quad \forall v \in W^1(\Omega^c).
\]

Let

\[
W^1(\Omega^c, \Delta) = \{ v \in W^1(\Omega^c) : \Psi^2 \Delta v \in W^0(\Omega^c) \}.
\]

The norm in \( W^1(\Omega^c, \Delta) \) is denoted by \( \| \cdot \|_{1, \Psi, \Omega^c, \Delta} \), and defined by

\[
\| v \|_{1, \Psi, \Omega^c, \Delta} = \| v \|_{1, \Psi, \Omega^c} + \| \Psi^2 \Delta v \|_{0, \Psi, \Omega^c}, \quad \forall v \in W^1(\Omega^c, \Delta).
\]

The space \( H^{-1/2}(\Gamma) \) is the dual of \( H^{1/2}(\Gamma) \), and the latter is the space of traces of functions from the space \( H^1(\Omega) \). A definition of this last space can be found in [7].

Notice that for any \( v \in W^1(\Omega^c) \) the restriction of \( v \) to \( \Omega_1 = \tilde{\Omega} \setminus \Omega \), where \( \Omega \subseteq \tilde{\Omega} \) and \( \tilde{\Omega} \) is a bounded domain, belongs to \( H^1(\Omega_1) \) and \( \| v \|_{1, \Omega_1} \leq C \| v \|_{1, \Psi, \Omega^c} \), where \( \| \cdot \|_{1, \Omega_1} \) denotes the standard norm of \( H^1(\Omega_1) \). For \( v \in H^1(\Omega_1) \) we have \( \text{tr} v \big|_{\Gamma} \in H^{1/2}(\Gamma) \), which means \( \| \text{tr} v \|_{1/2, \Gamma} \leq C \| v \|_{1, \Omega_1} \), where \( \| \cdot \|_{1/2, \Gamma} \) denotes the standard norm of \( H^{1/2}(\Gamma) \). This property together with a definition of \( H^{1/2}(\Gamma) \) can be found in [7]. The above implies that any \( v \in W^1(\Omega^c) \) has \( \text{tr} v \big|_{\Gamma} \in H^{1/2}(\Gamma) : \| \text{tr} v \|_{1/2, \Gamma} \leq C \| v \|_{1, \Psi, \Omega^c} \).

A generalized solution of (2.1) is defined as a solution of the following variational problem: find \( u \in W^1(\Omega^c) \) such that

\[
a(u, v) = \langle f_1, v \rangle \Psi + \langle g, \text{tr} v \rangle, \quad \forall v \in W^1(\Omega^c),
\]
where

$$a(v, w) = \sum_{i=1}^{2} \int_{\Omega^c} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} dx, \quad \forall v, w \in W^1(\Omega^c),$$

and $(\cdot, \cdot)_\psi$ denotes the scalar product in $W^0(\Omega^c)$:

$$(v, w)_\psi = \int_{\Omega^c} \frac{v \cdot w}{\psi^2} dx, \quad \forall v, w \in W^0(\Omega^c).$$

From now on we make the convention that round brackets $(\cdot, \cdot)$ denote an inner product, and angle brackets $(\cdot, \cdot)$ denote a dual pair. In the latter case the functional is always in the first place. If there is any doubt as to what spaces we have in mind they will be written explicitly as subscripts.

**Theorem 2.1** Problems (2.1) and (2.2) are equivalent, i.e. if $u \in W^1(\Omega^c, \Delta)$ is a solution to (2.1), then it is a solution to (2.2). Conversely, if $u \in W^1(\Omega^c)$ is a solution to (2.2), then it belongs to $W^1(\Omega^c, \Delta)$ and it is a solution to (2.1).

**Proof.** First, we check that the kernel of the trace operator $tr : W^1(\Omega^c) \to H^{1/2}(\Gamma)$ is dense in $W^0(\Omega^c)$. Obviously, the $D(\Omega^c)$ functions belong to the kernel. Take any $\varepsilon > 0$ and $v \in W^0(\Omega^c)$. We can always find a domain $\tilde{\Omega} \subset \Omega^c$ such that the function $v_r : \Omega^c \to \mathbb{R}$ defined by

$$v_r(x) = \begin{cases} v(x), & x \in \tilde{\Omega}, \\ 0, & x \in \Omega^c \setminus \tilde{\Omega} \end{cases}$$

satisfies $\|v_r - v\|_{0, \psi, \Omega^c} \leq \varepsilon/2$. The restriction of $v_r$ to $\tilde{\Omega}$ is in $L^2(\tilde{\Omega})$. Since $D(\tilde{\Omega})$ is dense in $L^2(\tilde{\Omega})$, we can find $\varphi \in D(\Omega^c)$ such that $\|v_r - \varphi\|_{0, \psi, \Omega^c} \leq \varepsilon/2$. Hence $\|\varphi - v\|_{0, \psi, \Omega^c} \leq \varepsilon$, and the required density follows.

Due to the above density argument we can apply the generalized Green formula to be found in [1]:

$$a(v, w) = (-\psi^2 \Delta v, w)_\psi + \langle \lambda v, tr w \rangle, \quad \forall v \in W^1(\Omega^c, \Delta),$$

$$\forall w \in W^1(\Omega^c),$$

where the operator $\lambda : W^1(\Omega^c, \Delta) \to H^{-1/2}(\Gamma)$ is an extension of the operator $\frac{\partial}{\partial n}|_{\Gamma}$ (the interior normal derivative on $\Gamma$).

If $u \in W^1(\Omega^c, \Delta)$ is a solution to (2.1), then by (2.3) it is also a solution to (2.2). Conversely, if $u \in W^1(\Omega^c)$ is a solution to (2.2), then by the definition of distributional derivatives it also belongs to $W^1(\Omega^c, \Delta)$. Formula (2.3) can be applied once again to show that $u \in W^1(\Omega^c, \Delta)$ is a solution to (2.1). 

Let us now consider the existence and uniqueness of a solution to problem (2.2).
Theorem 2.2. Assume the following solvability condition:

\[(f_1, 1)\psi + (g, 1)_{H^{-1/2}_x H^{1/2}_x} = 0.\]

Then the solution to problem (2.2) exists and is unique up to a constant.

Proof. First, we prove that the form \(a\) from (2.2) is elliptic in \(W^1(\Omega^c) / P_0\), where \(P_0\) is the space of polynomials of degree zero. Take two smooth functions \(\varphi, \psi : \mathbb{R}^2 \rightarrow \mathbb{R}\) such that

\[
\begin{align*}
\varphi(x) + \psi(x) &= 1, \\
0 &
\leq \varphi(x) \leq 1, \\
\varphi(x) &= 0,
\quad |x| \geq R_1, \\
\varphi(x) &= 1, \\
\varphi(x) &= |x| R < R_1,
\end{align*}
\]

where \(R\) is so large that \(\Omega \subset B(0, R), B(0, R)\) is the ball with centre 0 and radius \(R\).

Take \(v \in W^1(\Omega^c).\) We have \(\psi v \in W^1_0(\Omega^c),\) where

\[W^1_0(\Omega^c) = \{v \in W^1(\Omega^c) : \text{tr} v|_\Gamma = 0\}.\]

By Proposition 1–1–1 of [5],

\[\|\psi v\|_{1, \psi, \Omega^c} \leq C\|\psi v\|_{1, \Omega^c},\]

where

\[|v|_{1, \Omega^c}^2 = \left\| \frac{\partial v}{\partial x_1} \right\|_{0, \Omega^c}^2 + \left\| \frac{\partial v}{\partial x_2} \right\|_{0, \Omega^c}^2. \quad \forall v \in W^1(\Omega^c).\]

Together with the definition of \(\psi\) this yields

\[\|\psi v\|_{1, \psi, \Omega^c} \leq C\|v\|_{0, T} + |v|_{1, \Omega^c},\]

where \(T = T(0, R, R_1)\) denotes the annulus with radii \(R\) and \(R_1\) and centre 0. Moreover (see [7]),

\[\|v\|_{0, T} \leq C\left\{ |v|_{1, T} + \int_T v\,dx \right\}.\]

Hence

\[\|\psi v\|_{1, \psi, \Omega^c} \leq C\left\{ |v|_{1, \Omega^c} + \int_T v\,dx \right\}.\]

Similarly,

\[\|\varphi v\|_{1, \psi, \Omega^c} \leq C\left\{ |v|_{1, \Omega^c} + \int_T v\,dx \right\}.\]

Therefore

\[\|v\|_{1, \psi, \Omega^c} \leq C\left\{ |v|_{1, \Omega^c} + \int_T v\,dx \right\}.\]

By choosing an appropriate constant \(c\) we get

\[\|v + c\|_{1, \psi, \Omega^c} \leq C\|v\|_{1, \Omega^c}, \quad \forall v \in W^1(\Omega^c).\]
From the above we infer
\[
\|\tilde{v}\|_{1,\Psi,\Omega^c/P_0} \leq C\|v\|_{1,\Omega^c}, \quad \forall v \in W^1(\Omega^c),
\]
where \(\|\cdot\|_{1,\Psi,\Omega^c/P_0}\) denotes the standard norm in \(W^1(\Omega^c)/P_0\). This implies
\[
a(v, v) = |v|^2_{1,\Omega^c} \geq C^2\|\tilde{v}\|_{1,\Psi,\Omega^c/P_0}^2, \quad \forall v \in W^1(\Omega^c).
\]
Hence the ellipticity of the form \(a\) in \(W^1(\Omega^c)/P_0\), Condition (2.4) implies that the right-hand side of (2.2) can be viewed as a functional in \((W^1(\Omega^c)/P_0)'\), and the theorem is a consequence of the Lax-Milgram theorem. ■

3. The transformed problem. Constructing a finite approximation of problem (2.1) is troublesome due to the unboundedness of \(\Omega^c\). Thus our aim is to construct a problem posed on a bounded domain and approximating problem (2.1). To this end, let \(\widetilde{\Omega}\) be an open bounded simply connected set such that \(\Omega \subset \widetilde{\Omega}\). We define \(\Omega_1 = \widetilde{\Omega}\setminus \Omega\), \(\Omega_3 = \mathbb{R}^2\setminus \widetilde{\Omega}\). Moreover, define \(f_{1r}: \Omega^c \to \mathbb{R}^2\) by
\[
\begin{cases}
f_{1r}(x) = f_1(x) + c & \text{for } x \in \Omega_1, \\
f_{1r}(x) = 0 & \text{for } x \in \Omega_2,
\end{cases}
\]
where the constant \(c\) is chosen in such a way that the solvability condition is satisfied:
\[
(f_{1r}, 1)_\psi + \langle g, 1 \rangle_{H^{-1/2} \times H^{1/2}} = 0.
\]
Then the problem to be solved reads: find \(u_r \in W^1(\Omega^c, \Delta)\) such that
\[
\begin{cases}
-\psi^2 \Delta u_r = f_{1r} & \text{on } \Omega^c, \\
\frac{\partial u_r}{\partial n} \big|_\Gamma = g.
\end{cases}
\]
The above problem approximates problem (2.1) as shown by the following theorem:

**Theorem 3.1.** If \(u\) is a solution to (2.1) and \(u_r\) is a solution to (3.2), then
\[
\|u - u_r\|_{1,\Psi,\Omega^c/P_0} \leq C\left(\int_{\Omega_2} f_{1r}^2/\psi^2 \, dx\right)^{1/2}.
\]

**Proof.** First notice that problem (3.2) has the following variational formulation: find \(u_r \in H^1(\Omega^c)\) such that
\[
a(u_r, v) = (f_{1r}, v)_\psi + \langle g, \text{tr} v \rangle, \quad \forall v \in W^1(\Omega^c).
\]
This problem can be derived in exactly the same way as (2.2). The ellipticity of \(a\) in \(W^1(\Omega^c)/P_0\), the formulation of problems (2.2) and (3.2), and the
solvability conditions (2.4) and (3.1) imply
\[ C\|\tilde{u} - \tilde{u}_r\|_{1,\psi,\Omega^c/P_0} \leq \|f_1 - f_{1,r}\|_{0,\psi,\Omega^c} \|\tilde{u} - \tilde{u}_r\|_{0,\psi,\Omega^c/P_0}, \]
where \(\|\cdot\|_{0,\psi,\Omega^c/P_0}\) is the standard norm in \(W^0(\Omega^c)/P_0\). Hence
\[ \|\tilde{u} - \tilde{u}_r\|_{1,\psi,\Omega^c/P_0} \leq C\|f_1 - f_{1,r}\|_{0,\psi,\Omega^c}. \]
Moreover,
\[ \|f_1 - f_{1,r}\|_{0,\psi,\Omega^c} \leq \left( \int_{\Omega_1} c^2/\psi^2 \, dx \right)^{1/2} + \left( \int_{\Omega_2} f_1^2/\psi^2 \, dx \right)^{1/2} = I_1 + I_2, \]
where the constant \(c\) is from the definition of \(f_{1,r}\). In view of (3.1) and (2.4) this constant satisfies
\[ \int_{\Omega_1} c/\psi^2 \, dx = \int_{\Omega_2} f_1/\psi^2 \, dx. \]
This leads to
\[ I_1^2 \leq \frac{\int_{\Omega_2} 1/\psi^2 \, dx}{\int_{\Omega_1} 1/\psi^2 \, dx} \int_{\Omega_2} f_1^2/\psi^2 \, dx. \]
Now (3.4)–(3.6) yield the assertion. ■

The problem announced at the beginning of this section can be derived from problem (3.2) by decomposition: find \(u_1 \in H^1(\Omega_1, \Delta), u_2 \in W^1(\Omega_2, \Delta)\) such that
\[
\begin{align*}
a) & \quad -\Delta u_1 = f \quad \text{on } \Omega_1, \\
b) & \quad -\Delta u_2 = 0 \quad \text{on } \Omega_2, \\
c) & \quad \text{tr } u_1 = \text{tr } u_2 \quad \text{on } \Gamma_1, \\
d) & \quad \frac{\partial u_1}{\partial n} |_{\Gamma_1} = \frac{\partial u_2}{\partial n} |_{\Gamma_1} \quad \text{on } \Gamma_1, \\
e) & \quad \frac{\partial u_1}{\partial n} |_{\Gamma} = g \quad \text{on } \Gamma,
\end{align*}
\]
where \(\Gamma_1\) is the boundary of \(\Omega_2\), \(f \in L^2(\Omega_1)\), \(f(x) = f_{1,r}(x)/\psi^2\), \(\forall x \in \Omega_1\), \(u_i = u_{1r}|_{\Omega_i}\), \(i = 1, 2\) and \(\frac{\partial}{\partial n}\) denotes the interior normal derivative on \(\Gamma_1\) or \(\Gamma\). The space \(H^1(\Omega_1, \Delta)\) is defined as follows:
\[ H^1(\Omega_1, \Delta) = \{ v \in H^1(\Omega_1) : \Delta v \in L^2(\Omega_1) \}, \]
with the norm
\[ \|v\|^2_{1,\Omega_1,\Delta} = \|v\|^2_{1,\Omega_1} + \|\Delta v\|^2_{0,\Omega_1}, \quad \forall v \in H^1(\Omega_1, \Delta). \]
The space \(W^1(\Omega_2, \Delta)\) is defined analogously to \(H^1(\Omega^c, \Delta)\).

Problem (3.7) can be transformed to a variational problem posed on \(\Omega_1\) and \(\Gamma_1\) since \(u_2 = u_{1r}|_{\Omega_2}\) is harmonic in \(\Omega_2\).

Once again we recall the generalized Green formula from [1] for the spaces \(H^1(\Omega_1), L^2(\Omega_1), H^{1/2}(\Gamma)\) and the trace operator \(\text{tr} : H^1(\Omega_1) \rightarrow H^{1/2}(\partial \Omega_1)\).
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This formula and equations (3.7a, e) imply (3.10a), where \( \lambda_1 = \frac{\partial u_1}{\partial n} |_{\Gamma_1} \) denotes the interior normal derivative. Since \( u_2 \) is harmonic in \( \Omega_2 \) (equation (3.7b)) we can apply the Green formula for harmonic functions (see [2]) to obtain a relation between \( \text{tr} u_1 |_{\Gamma_1} \) and \( \lambda_1 \):

\[
(3.8) \quad \frac{1}{2} u_1(x) = \frac{1}{2\pi} \int_{\Gamma_1} \frac{\partial}{\partial n_y} \log |x - y| u_1(y) \, d\sigma_y - \frac{1}{2\pi} \int_{\Gamma_1} \log |x - y| \lambda_1 \, d\sigma_y + a_0,
\]

where \( a_0 \) is a constant. The Green formula (2.3) for the spaces \( W^1(\Omega_2, \Delta) \), \( W^0(\Omega_2) \), \( H^{1/2}(\Gamma_1) \) and equation (3.7b) yield

\[
a_2(u_2, v) = (\lambda_1, \text{tr} v)_{\Gamma_1}, \quad \forall v \in W^1(\Omega_2),
\]

where

\[
a_2(w, v) = \sum_{i=1}^{2} \int_{\Omega_2} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx, \quad \forall w, v \in W^1(\Omega_2).
\]

Hence \( (\lambda_1, 1)_{\Gamma_1} = 0 \). If \( u_2 \in W^1(\Omega_2, \Delta) \), then \( \lambda_1 \in \dot{H}^{-1/2}(\Gamma_1) \), where \( \dot{H}^{-1/2}(\Gamma_1) \) is defined in (3.11) below. In case \( \lambda_1 \in L^2(\Gamma_1) \), we have

\[
(3.9) \quad \int_{\Gamma_1} \lambda_1 \, d\sigma = 0.
\]

Multiplying (3.8) by a function \( \mu \in L^2(\Gamma_1) \) satisfying (3.9) and integrating along \( \Gamma_1 \) we arrive at (3.10b). The form \( b_1 \) from this equation is defined for regular functions, i.e. those in \( L^2(\Gamma_1) \). However (see [5], [6]), it can be extended continuously onto \( \dot{H}^{-1/2}(\Gamma_1) \times \dot{H}^{-1/2}(\Gamma_1) \). Thus the problem to be solved reads: find \( \tilde{u} = (u_1, \lambda_1) \in H^1(\Omega_1) \times \dot{H}^{-1/2}(\Gamma_1) \) such that

\[
(3.10) \quad \left\{ \begin{array}{l}
\text{a)} \quad a_1(u_1, v) + (\lambda_1, \text{tr} v)_{\Gamma_1} = (f, v)_{L^2} + (g, \text{tr} v)_{\Gamma_1}, \quad \forall v \in H^1(\Omega_1), \\
\text{b)} \quad 2b_1(\lambda_1, \mu) - \langle \mu, \text{tr} u_1 \rangle_{\Gamma_1} + 2 \langle \mu, \frac{\partial}{\partial n} \text{tr} u_1 \rangle_{\Gamma_1} = 0, \quad \forall \mu \in \dot{H}^{-1/2}(\Gamma_1),
\end{array} \right.
\]

where

\[
(3.11) \quad \dot{H}^{-1/2}(\Gamma_1) = \{ \mu \in H^{-1/2}(\Gamma_1) : \langle \mu, 1 \rangle_{H^{-1/2} \times H^{1/2}} = 0 \},
\]

and \( H^{-1/2}(\Gamma_1) \) is defined analogously to \( H^{-1/2}(\Gamma) \). The forms and operators in (3.10) are defined as follows:

\[
a_1(v, w) = \sum_{i=1}^{2} \int_{\Omega_1} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \, dx, \quad \forall v, w \in H^1(\Omega_1),
\]
\[ b_1(\theta, \mu) = -\frac{1}{2\pi} \int_{\Gamma_1} \int_{\Gamma_1} \log |x-y| \theta(y) \mu(x) \, d\sigma_y \, d\sigma_x, \]

\( \forall \theta, \mu \in L^2(\Gamma_1) \cap \dot{H}^{-1/2}(\Gamma_1), \)

\[(\overline{C}_n v)(x) = \frac{1}{2\pi} \int_{\Gamma_1} \frac{\partial}{\partial n_y} \log |x-y| v(y) \, d\sigma_y, \quad \forall v \in H^{1/2}(\Gamma_1).\]

**Theorem 3.2.** If \( u_r \) is a solution to (3.2), then the pair \((u_1, \lambda_1)\), where \( u_1 = u_r|_{\Omega_1}, \lambda_1 = \frac{\partial u_r}{\partial n}|_{\Gamma_1} \), is a solution to (3.10). Conversely, if \((u_1, \lambda_1)\) is a solution to (3.10), then there exists a harmonic extension \( u_r \in W^1(\Omega^c) \) of \( u_1 \) to the whole \( \Omega^c \) and this extension is a solution to (3.2).

**Proof.** The first part of the theorem comes from the derivation of problem (3.10).

Now assume that we have a solution \((u_1, \lambda_1)\) of (3.10). Then there exists a solution \( u_2 : \Omega_2 \to \mathbb{R} \) to the following Dirichlet problem: find \( u_2 \in W^1(\Omega_2) \) such that

\[
\begin{cases}
-\Delta u_2 = 0 & \text{on } \Omega_2, \\
\text{tr } u_2|_{\Gamma_1} = \text{tr } u_1|_{\Gamma_1} & \text{on } \Gamma_1.
\end{cases}
\]

For this function we can derive an equation similar to (3.10b):

\[(3.12) \quad 2b_1(\lambda_2, \mu) - \langle \mu, \text{tr } u_2 \rangle + 2\langle \mu, \overline{C}_n \text{tr } u_2 \rangle = 0, \quad \forall \mu \in \dot{H}^{-1/2}(\Gamma_1),\]

where \( \lambda_2 = \frac{\partial u_2}{\partial n}|_{\Gamma_1} \in \dot{H}^{-1/2}(\Gamma_1) \). If we subtract (3.12) from (3.10b), then, bearing in mind \( \text{tr } u_1|_{\Gamma_1} = \text{tr } u_2|_{\Gamma_1} \), we get

\[b_1(\lambda_2 - \lambda_1, \mu) = 0, \forall \mu \in \dot{H}^{-1/2}(\Gamma_1).\]

The form \( b_1 \) is elliptic in \( \dot{H}^{-1/2}(\Gamma_1) \) (see [5], [6]). Hence \( \lambda_2 = \lambda_1 \). Let \( u_r : \Omega^c \to \mathbb{R} \) be such that \( u_r|_{\Omega_i} = u_i, i = 1,2 \). We know that

\[
\begin{cases}
-\Delta u_r = f & \text{in } \Omega_1, \\
-\Delta u_r = 0 & \text{in } \Omega_2.
\end{cases}
\]

We want to check that

\[(3.13) \quad \langle -\Delta u_r, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} = (f|_{\Omega_1}, \varphi)|_{\Psi}, \quad \forall \varphi \in \mathcal{D}(\Omega^c).\]

The definition of distributional derivatives implies

\[\langle -\Delta u_r, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} = a_1(u_1, \varphi) + a_2(u_2, \varphi), \quad \forall \varphi \in \mathcal{D}(\Omega^c).\]

The Green formula applied to the forms \( a_1 \) and \( a_2 \) yields

\[
a_1(u_1, \varphi) = (f, \varphi)_{L^2} - \langle \lambda_1, \text{tr } \varphi \rangle_{\Gamma_1},
\]

\[
a_2(u_2, \varphi) = \langle \lambda_2, \text{tr } \varphi \rangle_{\Gamma_1}, \quad \forall \varphi \in \mathcal{D}(\Omega^c).\]
Adding the above equalities and taking into account \( \lambda_1 = \lambda_2 \) and \( f(x) = f_{1r}(x)/\Psi^2, \forall x \in \Omega_1 \) we get (3.13). Hence \(-\Psi^2 \Delta u_r = f_{1r} \) in \( \Omega^c \). Obviously \( \frac{\partial u_r}{\partial n}\big|_\Gamma = g \), by (3.10a). This shows the second part of the theorem. ■

Remark 3.1. A transformation of this type was first applied in [4] to the Dirichlet problem for the Poisson equation.

**Corollary 3.1.** Assume that the solvability condition (3.1) is satisfied. Then there exists a solution \((u_1, \lambda_1)\) of (3.10) and it is unique up to a constant in the first component of the solution.

**Proof.** Assume that we have two solutions \((u_1^1, \lambda_1^1), (u_1^2, \lambda_1^2)\) of (3.10). By Theorem 3.2 the harmonic extensions \(u_1^1, u_1^2\) of \(u_1^1, u_1^2\) satisfy (3.2). The solvability condition (3.1) and Theorem 2.2 imply \(u_1^1 = u_1^2 + c\), where \(c\) is a constant. The first equation of (3.10) yields \(\lambda_1^1 = \frac{\partial u_1^1}{\partial n}\big|_{\Gamma_1}, \lambda_1^2 = \frac{\partial u_1^2}{\partial n}\big|_{\Gamma_1}\). Hence \(\lambda_1^1 = \lambda_1^2\). The existence of a solution comes from the solvability condition (3.1) and Theorem 2.2. ■

4. A regularity result. In this section we consider the regularity of a solution \(\hat{u} = (u_1, \lambda_1)\) to (3.10). This is important for the numerical analysis of this problem.

**Theorem 4.1.** Assume that \(f \in H^m(\Omega_1), g \in H^{m+1/2}(\Gamma_1), m \in \mathbb{N}\), the boundary \(\partial \Omega_1\) is \(C^{m+1,1}\), i.e. it can be described locally by functions with Lipschitz continuous \(m+1\) derivatives, and there exists a solution \(\hat{u} = (u_1, \lambda_1) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma_1)\) to (3.10). Then \(\hat{u} \in H^{m+2}(\Omega_1) \times H^{m+1/2}(\Gamma_1)\).

**Proof.** The proof is by induction. First we establish regularity of \(u_1\) near the boundary \(\Gamma_1\) in the direction tangent to the boundary. Then we prove full regularity near the boundary by making use of the relation between partial derivatives given by equation (3.10a) of the problem. Finally, we establish global regularity of \(\hat{u}\) and, consequently, the regularity of \(\lambda_1\).

From Theorem 3.2 we obtain
\[
a(u_r, \varphi) = (f_{1r}, \varphi)_{\Psi}, \quad \forall \varphi \in \mathcal{D}(\Omega^c),
\]
where \(u_r\) is the harmonic extension of \(u_1\). In general, \(u_r\) cannot be regular near \(\Gamma_1\) in the normal direction due to the definition of \(f_{1r}\). Thus we check the regularity in the tangent direction. Assume that the theorem holds for \(m-1\), i.e.
\[
f \in H^{m-1}(\Omega_1), g \in H^{m-1/2}(\Gamma_1) \Rightarrow u_1 \in H^{m+1}(\Omega_1), \lambda_1 \in H^{m-1/2}(\Gamma_1).
\]
We prove that it holds for \(m\), i.e.
\[
f \in H^{m}(\Omega_1), g \in H^{m+1/2}(\Gamma_1) \Rightarrow u_1 \in H^{m+2}(\Omega_1), \lambda_1 \in H^{m+1/2}(\Gamma_1).
\]
For \(m = 0\) the proof is the same as for the induction step.
Let then \( f \in H^m(\Omega_1) \), \( g \in H^{m+1/2}(\Gamma) \). Let \( x_0 \) be a certain point on the boundary \( \Gamma_1 \) and \( C \) a neighbourhood of \( x_0 \) in \( \mathbb{R}^2 \). Take a diffeomorphism \( \theta \), with \( \theta^{-1} \) being \( C^{m+1,1} \), transforming \( C \) onto a disc \( D \) in \( \mathbb{R}^2 \). We assume that the images of \( C_0 = C \cap \Gamma_1 \), \( C_- = C \cap \Omega_1 \) and \( C_+ = C \cap \Omega_2 \) under \( \theta \) are \( D_0 = D \cap \{(y_1, y_2) : y_2 = 0\} \), \( D_- = D \cap \{(y_1, y_2) : y_2 < 0\} \) and \( D_+ = D \cap \{(y_1, y_2) : y_2 > 0\} \) respectively.

Let \( \varphi_0 \) be a function from the space \( C^\infty(\mathbb{R}^2) \) with \( \varphi_0 \equiv 1 \) in the neighbourhood of \( x_0 \in \Gamma_1 \) and \( \text{supp} \varphi_0 \subset C \). In view of

\[
 a(x, \varphi_0 v) = (f, \varphi_0 v)_{L^2(\Omega_1)}, \quad \forall v \in W^1(\Omega^c),
\]

we arrive at

\[
(4.1) \quad a(\varphi_0 u_r, v) = (\varphi_0 f, v)_{L^2(\Omega_1)} + \sum_{i=1}^2 \int_{\Omega^c} \left( \frac{\partial \varphi_0}{\partial x_i} u_r \frac{\partial v}{\partial x_i} - \frac{\partial \varphi_0}{\partial x_i} \frac{\partial u_r}{\partial x_i} v \right) dx,
\]

\( \forall v \in W^1(\Omega^c) \).

Define the form \( d : (H^1_0(D))^2 \rightarrow \mathbb{R} \) by

\[
d(w, v) = a(\overline{w \circ \theta}, \overline{v \circ \theta}), \quad \forall w, v \in H^1_0(D),
\]

where the bar denotes the extension by zero of a function from \( H^1_0(C) \) onto the whole plane \( \mathbb{R}^2 \) and \( \circ \) denotes the composition of two functions. Then by the Friedrichs inequality,

\[
(4.2) \quad \|w\|_{1, D} \leq C \sup_{v \in H^1_0(D)} \frac{d(w, v)}{\|v\|_{1, D}}, \quad \forall w \in H^1_0(D).
\]

Let \( \overline{u} = (\varphi_0 u_r) \circ \theta^{-1} \). Then \( \overline{u} \in H^1_0(D) \). By (4.1) and the definition of \( \overline{u} \),

\[
(4.3) \quad d(\overline{u}, w) = a(\varphi_0 \overline{u_r}, \overline{w \circ \theta}) = (\varphi_0 f, \overline{w \circ \theta}) + \sum_{i=1}^2 \int_{\Omega^c} \left( \frac{\partial \varphi_0}{\partial x_i} u_r \frac{\partial w \circ \theta}{\partial x_i} - \frac{\partial \varphi_0}{\partial x_i} \frac{\partial u_r}{\partial x_i} \frac{w \circ \theta}{\partial x_i} \right) dx, \quad \forall w \in H^1_0(D).
\]

The next step is to prove that \( \frac{\partial^{m+1} \overline{u}}{\partial y_i^{m+1}} \in H^1_0(D) \). To this end let us introduce the finite difference:

\[
\Delta_h w(y) = (\overline{w}(y + h) - \overline{w}(y))/h, \quad y \in D, \quad w \in H^1_0(D),
\]

where \( h = (h_1, 0) \) and \( \overline{w} \) denotes the extension of \( w \) by zero onto \( \mathbb{R}^2 \). Notice that if \( \text{supp} w \subset D \), then \( \Delta_h w(y) \in H^1_0(D) \) for \( h \) sufficiently small. Furthermore,

\[
(4.4) \quad \sum_{i=1}^2 \int_D \frac{\partial}{\partial x_i} (\Delta_h w) \circ \theta \frac{\partial}{\partial x_i} v \circ \theta J(\theta^{-1}) dy
\]
Exterior Neumann problem

\[ -\sum_{i=1}^{2} \int_{\mathcal{D}} \frac{\partial}{\partial x_{i}} w \circ \theta \frac{\partial}{\partial x_{i}} (\Delta_{h} v) \circ \theta J(\theta^{-1}) \, dy + I_{1}(w, v), \]

where \(\text{supp} \, w \subset \mathcal{D}, w, v \in H^{1}(\mathcal{D}),\) and

\[ (4.5) \quad -\sum_{i=1}^{2} \int_{\mathcal{D}} \frac{\partial}{\partial x_{i}} \left( \frac{\partial}{\partial y_{1}} w \right) \circ \theta \frac{\partial}{\partial x_{i}} v \circ \theta J(\theta^{-1}) \, dy \]

\[ = -\sum_{i=1}^{2} \int_{\mathcal{D}} \frac{\partial}{\partial x_{i}} w \circ \theta \frac{\partial}{\partial y_{1}} \left( \frac{\partial}{\partial y_{1}} v \right) \circ \theta J(\theta^{-1}) \, dy + I_{2}(w, v), \]

where \(w \in H_{0}^{2}(\mathcal{D}), v \in H^{2}(\mathcal{D}),\) and

\[ I_{i} \leq C \|w\|_{1,\mathcal{D}} \|v\|_{1,\mathcal{D}}, \quad i = 1, 2. \]

Formulae (4.4), (4.5) can be rewritten in the following way:

\[ (4.6) \quad d(\Delta_{h} w, v) = -d(w, \Delta_{h} v) + I_{1}(w, v), \]

\[ (4.7) \quad d\left( \frac{\partial w}{\partial y_{1}}, v \right) = -d\left( w, \frac{\partial v}{\partial y_{1}} \right) + I_{2}(w, v). \]

We use formula (4.7) \(m\) times and formula (4.6) once to obtain

\[ (4.8) \quad d\left( \Delta_{h} \frac{\partial^{m} \bar{u}}{\partial y_{1}^{m}}, v \right) = (-1)^{m+1} d\left( \bar{u}, \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} \right) + I_{3}(\bar{u}, v). \]

Expression \(I_{3}\) can be estimated as follows:

\[ |I_{3}(\bar{u}, v)| \leq C \|\bar{u}\|_{m+1,\mathcal{D}} \|v\|_{1,\mathcal{D}}. \]

Now we deal with the first term of the right-hand side of (4.8). Rewrite formula (4.3) as

\[ (4.9) \quad d\left( \bar{u}, \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} \right) = (\varphi_{0} f, \left( \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} \right) \circ \theta) \]

\[ + \sum_{i=1}^{2} \int_{\Omega^{c}} \left( \frac{\partial \varphi_{0}}{\partial x_{i}} \bar{u} \frac{\partial}{\partial x_{i}} \left( \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} \right) \circ \theta - \frac{\partial \varphi_{0}}{\partial x_{i}} \frac{\partial \bar{u}}{\partial x_{i}} \left( \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} \circ \theta \right) \right) \, dx \]

\[ = I_{4} + I_{5}. \]

Change of variables yields

\[ I_{4} = \int_{\mathcal{C}^{-}} \varphi_{0} f \left( \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} \right) \circ \theta \, dx = \int_{\mathcal{P}^{-}} \varphi_{0} \circ \theta^{-1} f \circ \theta^{-1} \Delta_{h} \frac{\partial^{m} v}{\partial y_{1}^{m}} J(\theta^{-1}) \, dy. \]

By the Gauss formula

\[ I_{4} = (-1)^{m} \int_{\mathcal{P}^{-}} \frac{\partial^{m}}{\partial y_{1}^{m}} (\varphi_{0} \circ \theta^{-1} f \circ \theta^{-1} J(\theta^{-1})) \Delta_{h} v \, dy. \]
Hence
\[ |I_4| \leq C\|f\|_{m,c^-}\|v\|_{1,D}.\]
It is easy to see that the form \( I_5 \) can be estimated as follows:
\[ |I_5| \leq C\|u_r\|_{m+1,c}\|v\|_{1,D}.\]
Estimate (4.2) and equalities (4.8), (4.9) yield
\[
\left\| \Delta_h \frac{\partial^m u_r}{\partial y_1^m} \right\|_{1,D} \leq C\left\{ \|u_r\|_{m+1,c} + \|f\|_{m,c^-} \right\}.
\]
The last estimate implies \( \frac{\partial^{m+1} u}{\partial y_1^{m+1}} \in H^1_0(D), \) which in turn yields
\[ (4.10) \quad \frac{\partial^{m+2} \bar{u}}{\partial y_1^{m+2}} \in L^2(D), \quad \frac{\partial^{m+2} \bar{u}}{\partial y_2 \partial y_1^{m+1}} \in L^2(D).\]
We cannot expect that \( \bar{u} \in H^{m+2}_0(D) \) for the reasons explained above, but we can prove that \( \bar{u}|_{\partial D} \in H^{m+2}(\partial D) \) using the relation between partial derivatives given by the equation itself. By virtue of the formula for the derivative of the product of two functions we arrive at
\[
\Delta(\varphi_0 u_r) = \varphi_0 \Delta u_r + u_r \Delta \varphi_0 + 2 \sum_{i=1}^2 \frac{\partial \varphi_0}{\partial x_i} \frac{\partial u_r}{\partial x_i}.
\]
Since by assumption \( f|_{\Omega_1} \in H^m(\Omega_1) \) and \( u \in H^{m+1}(\Omega_1) \), we get
\[
\frac{\partial^2(\varphi_0 u_r)}{\partial x_1^2} + \frac{\partial^2(\varphi_0 u_r)}{\partial x_2^2} = \eta,
\]
where \( \eta \in H^m(\partial \Omega_1) \). In view of \( \varphi_0 u_r = \bar{u} \circ \theta \), by writing out the expressions \( \frac{\partial(\bar{u} \circ \theta)}{\partial x_i} \), \( i = 1, 2 \), we arrive at
\[ (4.11) \quad \alpha \frac{\partial^2 \bar{u}}{\partial y_2^2} + \beta \frac{\partial^2 \bar{u}}{\partial y_1 \partial y_2} + \gamma \frac{\partial^2 \bar{u}}{\partial y_1^2} + \delta = \eta,
\]
where \( \alpha, \beta, \gamma, \delta \) are sufficiently smooth functions and \( \alpha > 0 \). Let us differentiate both sides of (4.11) \( m \) times: \( m-l \) times with respect to \( y_1 \) and \( l \) times with respect to \( y_2 \), \( l = 0, 1, \ldots, m \). As a consequence, by (4.10), we see that all the derivatives of \( \bar{u} \) of order \( m+2 \) are in \( L^2(D_-) \), hence \( \bar{u} \in H^{m+2}(\partial D_-) \), which implies \( \varphi_0 u_r \in H^{m+2}(\partial \Omega_1) \). The fact that \( u_1 \in H^{m+2}(\Omega_1) \) is a simple consequence of the last remark and general results concerning solutions of elliptic problems (see [7]). These results give us the local regularity and the regularity near the boundary \( \Gamma \). Since \( \lambda_1 = \frac{\partial u_1}{\partial n}|_{r_1} \), it is then easy to see that \( \lambda_1 \in H^{m+1/2}(I_1) \).

In the next paper we will show how to find an approximate solution to problem (2.1) using the theorems of this paper.
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References