Substantial Learning Environment “Tiles”

Abstract: Work with pupils within a Substantial Learning Environment (SLE) “Tiles” in the frames of NaDiMa project is described in this paper. The idea of this SLE came to being on the basis of Swoboda’s researches and concerned building the ideas of isometric geometrical transformations. We showed that SLE “Tiles” is didactically rich for both the youngest children and older primary class pupils. It seems to build intuitions that help crossing the line between visual understanding of geometric transformations and dynamic thinking in geometry.

1 Introduction

In 2009 a new national curriculum in the primary schools was introduced. According to this curriculum “pupils finishing the first grade [...] would perceive symmetry (for example in a picture of a butterfly); notice that a specific figure is a larger or smaller version of another figure; continue a regular pattern (for example, a band)”\(^2\). Pupils finishing the third grade “would draw the symmetrical half of a figure; a larger of smaller figure; continue a simple regular motive (band, rosette)”\(^3\). The expected geometrical skills of a child has clearly been broadened by including those that depend on perceiving regularities and symmetry in bands, patterns and tessellation, as well as in nature. Pointing out the skills of recognition of the figures’ symmetry means that at this stage symmetry should function as a procedure rather than geometrical objects (Marchini, Vighi 2009, pp. 169).

As E. Swoboda states (2009), “geometrical regularities in ornaments are still not sufficiently exploited a field which aims at introducing children in

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\(^2\) Edukacja matematyczna i techniczna 6, p. 25; www.reformaprogramowa.men.gov.pl

\(^3\) Ibidem p. 26
the world of geometry”. They make a rich environment for creating didactical situations connected with mathematics. Building regularities by children gives a chance of discovering many intuitions, which are the basis of more difficult geometrical concepts.

As a math teacher, I cooperate with Swoboda in an international project NaDiMa - Motivation via Natural Differentiation in Mathematics. On the basis of my own observations, I noticed that the work of some teachers is restricted to proposals taken from teachers’ guides. Teachers rarely reach for literature from the field of didactic of mathematics that presents new ideas. The cooperation with Swoboda in the framework of NaDiMa project gave me as a teacher the possibility of using the results of her researches. On this basis, my project and research tool (tiles) were designed. Our common idea is to build educational environments involving geometrical regularities on the plane.

The NaDiMa project, which started in the school year 2008/09⁴ meets changes that occurred in the mathematical education. Introducing more geometric content to mathematics at schools is in line with research conducted in Poland (Swoboda, 2006; Jagoda, 2004, 2008; Rożek, 2009) and all over the world (Vopenka, 1989; Callingham 2004; Clements, Swaminathan, Hannibal. Sarama, 1999; Marchini, Vighi, 2009). Conclusions drawn from that research state that geometry should be acquired by the child not only by means of basic shapes (circle, square, and triangle) and measurements, but also through relations between these shapes (similarities, symmetries). Examining these shapes and geometric phenomena may proceed dynamically – not only by perception but also by creating regularities. Research shows that by making tessalations or drawing bands children are able not only to guess what will follow, but also transform one band into another, or depict a regularity in various ways (Mulligan & Michelmore, 2009; Ginsburg 2002). Talking with children about the perceived relations encourage to pay more attention to specific (definitive) properties of the relations and to express them in an increasingly precise way. Therefore geometry classes conducted in an didactic environment based on regularity prepare for the sort of behaviour and thinking that is the basis of intellectual activity in mathematics classes.

That sort of proceeding comes into accord with European trends which inspire the changes in education policy. The European trends on educational policy stress the importance of formation of pupils’ competences as: competence to learn, to communicate, to solve problems, to make conjectures, etc. This process should start on the primary school level supported mainly by the

⁴Project “Motivation via Natural Differentiation in Mathematics”, no. 142453-2008-LLP-Poland-Comenius; www.nadima.eu. Participants from Poland: Mathematical Institute of the Rzeszów University and Zespół Szkół Społecznych Nr 1 in Rzeszów.
In our project we presumed the following:
On the general level the project aims at the development of primary school pupils:

- to support the development of their learning competences.
- to support the consciousness of the meaning of mathematics as a part of human culture.
- to encourage pupils’ motivation to learn mathematics,
- to realize pupils’ individual (cognitive) potentials,
- to create the possibility for students to experience success in the process of problem solving.

These aims should be achieved by means of support and enhancement of teachers directed on:

- getting to know examples for substantial learning environments (SLE) and by that experience the nature of mathematics, and recognize and use the potential for teaching and learning,
- strengthening teacher’s mathematical content knowledge in relation to and requested for these specific learning environments,
- realizing the heterogeneity and natural differentiation in mathematics classrooms,
- supporting the change of beliefs on the substance and importance of mathematics for primary school level.

The idea of SLE is widely encouraged among Germans (Wittmann 1995, 2001; Scherer, 2007), but also among Czechs (Hejný, Jirotkova 2009). Wittmann describes SLE as a teaching/learning unit with the following properties (Wittmann, 1995, 365-366):

1. It represents central objectives, contents and principles of teaching mathematics at a certain level.
2. It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.
3. It is flexible and can be adapted to the special conditions of a classroom.
4. It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.
Stressing the importance of SLE for mathematics didactics, Wittmann (2001) states:

The concept of SLE is a very powerful one. It can be used to tackle successfully one of the big issues of mathematics education which has become more and more urgent and which is of crucial importance for the future of mathematics education as a discipline: the issue of theory and practice. [...] The design of substantial learning environments around long-term curricular strands should be placed at the very centre of mathematics education. Research, development, and teacher education should be consciously related to them in a systematic way.

[Wittmann, 2001, pp. 3-4]

![Diagram](image)

**Figure 1.** The design of substantial learning environment (Wittmann, 2001 p. 5).

In addition, he states:

It is not by chance that development projects based on SLEs have been successful in changing mathematics teaching as well as in changing teachers’ attitudes: in these projects fundamental systemic conditions have been taken into account.

[Wittmann, 2001, p. 5]

## 2 Mathematical, psychological and pedagogical aspects of SLE: “Tiles”

Activities in geometry for the youngest children – as E.Swoboda states (Swoboda, 2006, pp. 228-230) – cannot be headed on achieving a readymade product in the form of a notion or geometrical ability. They are to give a
chance for gaining experience, which will be the basis for constructing mathematical notions in further education. Through properly organized activities, one can activate intuitions taken from children’s behavior. These intuitions are the basis for further creation of more complex mathematical notions. Analyzing relations on the plane or creating such relations by children has a big developmental importance for their thinking processes.

The stage to which we relate constructing our SLE “Tiles” is more connected to the creation of a “concept image” related to geometrical transformations, rather than a “concept definition” (Tall, Viener 1987). It is to build mental pictures-representations and help discovering its related properties. Additionally, we intended to build such intuitions that would help crossing the line between visual understanding of geometrical transformations and dynamic thinking in the domain of geometry. In this case, it would be a transition from static visual representation – a relation between the figures (knowing what) – to the understanding of dynamic relations between the objects: shift, placing one object over the other etc. (knowing how).

There is a problem with creating an intuition for mathematical concepts: it is that it requires considering relations between theories of learning and those of intuition. It is an issue that has a long and rich research tradition in philosophy, psychology and as well as didactics of mathematics.

In the constructive approach to learning mathematics, stemming from Piaget, the basic concept is schema. Piaget (1980) himself claimed that “the schema of action consists of those aspects, which are repeatable, transposable and generalisable”. Schemata develop by two mechanisms: assimilation and accommodation. Rumelhart and Norman (1985) describe schemata as “data structures for representing the generic concepts stored in memory. […] Schemata in some sense represent the stereotypes of these concepts”. According to Hershkowitz (2009), both concepts – schema and intuition can be put together. Fischbein (1999), while referring to teaching mathematics, stated that schema is “a kind of condensed, simplified representation of a class of objects or events” or “adaptive behavior of an organism achieved by assimilation and accommodation”. It is also “a program which enables the individual to: (a) record, process, control, and mentally integrate information, (b) react meaningfully and efficiently to environmental stimuli” (Hershkovitz 2009, pp. 31-33). He contrasted “intuitive understanding” (insight), “formal understanding” and “procedural understanding”. In addition, he distinguished between two kinds of intuitions: “affirmatory intuitions” and “anticipatory intuitions”. Affirmatory intuition is direct and self-evident cognition without the need for checking and proving. He claimed that this type of intuition is the “effect of
compression if a *structural schema lies behind this cognition*\(^5\) The statement “effect of compression” points out the necessity for existence of other schemata. Anticipatory intuition help:

- grasping the problem,
- distinguishing between the given information and the question,
- searching for strategies,
- finding a schema for solving.

In her conclusion, Hershkovitz states that the relation between intuition and constructive assumptions of teaching mathematics is as follows:

> We can look at intuition and schemata as a closed circuit. The more schemata a person acquires, the more intuition he or she has. The educational challenge is to enable children to develop rich mathematical schemata leading to more intuitions for solving mathematical problems.

*(Hershkovitz 2009, p. 40)*

The above approach towards intuition was the psychological basis for creating SLE “Tiles”. Utilizing Fischbein’s terminology, we create schemata that would be connected to the affirmatory intuition directed to geometrical transformations.

\(^5\)David G. Myers describes intuition in a similar way: *Intuitive thinking is processing information by the right hemisphere. [...] Intuition is no other than recognition* (p. 19). These views connect intuition with functioning of the right hemisphere. From the research that Myers elaborates (on, among others: M. Gazzaniga, 1967, *The Split Brain in Man*. “Scientific American”, August; M. Gazzaniga: 1988, Organization of the Human Brain. “Science” 245: R. Sperry: 1964, Problems Outstanding in the Evolution of Brain Function. James Arthur Lecture, Americal Museum of Natural History, New York.; D. Schacter: 1992, Understanding Implicit Memory. A Cognitive Neuroscience Approach. “American Psychologists” 47 L. R. Squire 1987: Memory and Brain, Oxford University Press, New York) conducted in various sites in USA using various techniques and methods we might conclude that the right hemisphere gathers images, non-verbal information which we cannot follow or verbalize, procedures of action that we are not able to repeat in a conscious way. The major part of everyday thinking, feeling and activity is done unconsciously, but traces of these actions remain and are gathered in the right hemisphere. This knowledge reveals itself in certain unconscious procedures, actions and conclusions (Chlewinski, 1999, p. 126). We know more than we can express. This knowledge is gained through experience. Often it is called tacit knowledge. This sort of knowledge is usually inaccessible to consciousness. It is procedural type of knowledge. Contrary to open knowledge or knowing “what”, tacit knowledge is knowing “how”
3 The Project’s implementation

The work within SLE “Tiles” was conducted in several stages of education. Initially, the project was introduced to kindergarten children (6-year-olds), then first and third grade of primary school, and grades IV, V and VI of elementary school (children aged 5–12). Some of the classes were conducted with secondary school pupils (13-year-olds). The SLE “Tiles” developed along. Each stage of the project was the effect of experience acquired on the previous stage and became the basis for the next stage.

The idea assumed that this would provide an opportunity to create various scenarios of classes adjusted to various age groups. This allowed continuing different scenarios depending on the children’s needs and solutions suggested by them. At the same time, common analysis of observations gathered on various levels of education provided a better basis for interpreting the observed phenomena. In this paper I will only describe the observations concerning grades IV, V, VI (students aged 10–12). The observations were made on the basis of detailed analysis of classes that were video recorded.

4 The Project’s progress

4.1 Initial stage

- Introducing teachers to the project’s ideas; choice of activities realised within the project:

The teachers had to enter the world of geometric regularities (learn the idea of opening the world of geometry in the environment of regularities) as well as become aware of the conception of SLE and work with tools that would allow to realize the conception in class. We wanted these tools to be easy to use and modify. According to these theoretical assumptions, a research tool was developed: tiles with a motif. There were two variants; one was a mirror image of the other. Additionally, this motif was printed on a square and rectangular cards. Therefore, the basic set consisted of four kinds of tiles (fig. 2).

![Figure 2. Variants of tiles.](image)
If the pattern on the tile is not axially symmetrical a mirror reflection is required for creating a symmetrical composition, thus putting the tile the other way around. With a tile printed only on one side it is impossible. That is why we need to have two kinds of tiles: “left” and “right”. Other possibilities arise when we use a rectangular tile. Using the same pattern on a square and rectangular tile causes some motifs to be near to one another and others to be more distant. These facts were to enable discovering various relations between the figures in a tessellation created from the tiles. The pattern that was printed on a tile also enabled various degrees of precision – for some reasons existing dots could have been omitted and all the attention could be focused only on the arch, in another case the dot at the end of the arch could have been a vital element.

- Making the children acquainted with the project and the research tool (the tiles);

This was a time of the children’s spontaneous activity, free creativity and fun⁶. For the teachers that observed the stage it was an opportunity to verify the results of didactical research. The usefulness of the tool (the tiles) was also tested with regard to mathematical concepts and procedures possible to achieve during classes, as well as the attractiveness of the tool on various educational stages. We observed spontaneous behaviour of the children – their ways of using the tool. We wanted the next class to be conducted in a more direct way with regards to mathematical goals, but at the same time we wanted to minimize the teachers’ activities and instructions focusing on the strategies presented by the children.

4.2 One-dimensional tesselations (bands)

In the second stage of our project we decided to focus the children’s attention on a one-dimensional pattern, namely the band. The classes focused on building these bands and “destroying” them.

According to mathematical assumptions the goal was to differentiate between the mirror image (changing the orientation of the plane) from shifting and rotating (which keep the orientation of the surface). “Destroying the band” was a backward engineering task to building the band and thus provided other possibilities to learn about the features of mirror images and confront this knowledge with the features of shift and rotation.

⁶In E.Swoboda’s research children were to create a beautiful floor. In our project in classes IV-VI, the pupils were to check what can be built of tiles. In earlier classes, the teacher started all activities from telling a story.
4.3 Coding and decoding the band

A schematic drawing reflected the structure of the band. The pupils were presented with the task in the form of information that the band was built by children from another school who have sent us only the scheme (fig. 3). Two schemes were prepared – one for the younger pupils, the other one for the older ones from grades IV – VI.

![Diagram](image)

**Figure 3.** Schematic drawings of a band: a) younger children, b) older children from grades IV-VI.

The purpose of these activities was to provoke the children to describe relations of geometrical bands and use words for describing properties. We assumed that the activities will provoke argumentation and discussion – vexed in the field of geometric phenomena.

4.4 Topical tessellations

Another stage of the project were the so called topical tessellations. With specially prepared music the pupils created tessellations with a specific topic. The aim was to observe the children’s natural associations with the topic, which was enhanced by music. Music functions in a natural way, it speaks of a sort of transition and changes from moment A to moment B. In the assumptions we referred to building of dynamic associations with visual representations created by the child. We tried to choose the themes of the tessellations in such a way that they would refer to or could be associated with geometrical transformations.

Suggested topics were: (a) rapid river – to direct to the use of translation, (b) merry-go-round – use of rotation, (c) mirror – use of axial symmetry.

This type of didactic activity was caused by the lack of effects that have been anticipated earlier – the reflection on transformations while regarding placing of the tiles. The children still did not pay any attention to the sorts of shifts done by placing the tiles; they were only interested in the end result.
4.5 Reconstructing of the tiled floor

This task was to some extent similar to the task with fixing the broken band. The pupils were given three two-dimensional mosaics constructed according to various rules (fig. 4) The construction of tile floors was inspired by S. Turnau’s comments on the construction of two-dimensional ornaments (Turnau, 1988, pp. 230-234).

The floors were made of square and rectangular tiles, which was to make the pupils aware of the distance between the elements.

![Floors for reconstruction.](image)

5 Participants

A group of children, the work of which is described in the paper, counted 46 pupils (17 girls and 29 boys). At the start of the project activities one of the fourth graders had not yet gone through mirror reflection in their mathematics lessons. Other pupils knew mirror symmetry: they would draw and recognize figures being each other’s mirror images; they pointed the symmetry axis and recognized the axially symmetric figures.\(^7\) The range of the experiences regarding symmetry varied depending on the age. The number of experiences regarding intuitive feedback of the axial symmetry concept would increase along the age.

6 Observation results

Each stage of the project proved interesting for the children – at every stage of education the pupils participated in the activities with great interest. For the teachers, the classes also proved to be beyond routine not only due to their form. Often the reactions and solutions that the pupils displayed were surprising. This also motivated searching explanations in didactic literature and interpret the phenomena with regard to these theories. At this stage, the supervision of the work by Swoboda was necessary.

\(^7\) The selected classes implemented the curriculum *Matematyka 2001.*
6.1 Introductory stage: acquainting the children with the project and research tool

We observed various behaviour of the children after we took the tiles out of the envelopes. Some of the children grouped the tiles; we noticed that they perceived the difference between squares and rectangles. Other children immediately started to put the tiles together looking for a proper arrangement.

The pattern on the tiles provoked the children to continue the line. The children did not feel any boundaries in moving across the surface. Their works were not restricted to a linear pattern; on the contrary, compositions were made that organised the surface. Older children often very quickly introduced rules to their tessellations.

In the observed groups differences between children quickly appeared. The group of children that was not able to create the tessellation with specific rules or create a regular pattern or band was the minority (11%). The rest of the children introduced strong rules from the start and created rich geometrical patterns.

![Examples of work by 10-12-year-olds.](image)

We observed to what extent tessellation patterns involved axial symmetry, rotation, and shift. In the group that did not have the idea of symmetry introduced during mathematics classes, 40% of works involved axial symmetry. In other works, axial symmetry was used locally. In the second group, 58% had one axis of symmetry and 11% two axes. In several works symmetry appeared on a local level (8%). Only in one work (2.8%) rotational symmetry and in one translations were found. This proves that the dominating arrangement was axial symmetry.
Some of the children, dived deep in the analysis of the tool and the whole tool along with all the details, were subdued to the idea of symmetry. Other children treated the tool globally, ignoring the details. Each one of them realised the idea of symmetry but on various levels. Their actions were autonomous, not suggested by the teacher. The children realised the rule all by themselves, but only in the familiar range.

In the works of other children the most often present sort was mirror symmetry – a symmetrical continuation of lines, often with regard to asymmetric arrangement of dots (fig. 5). Also symmetric creations were made – for example the picture of an ant (fig. 5) – in which the child worked on symmetry of the whole image including the dots. Probably due to the level of complication of the pattern, in parts of it, dots on the left hand side do not match the dots on the right hand side.

When the children saw the tessellation based on translation only they were astonished by its simplicity. The creator was very pleased. Few works contained rotational symmetry. Creating such a tessellation proved very difficult for the children. The work in fig. 5 is an example of central symmetry. One of the child began the work from circles placed in the opposite corners of the sheet (though the placement of the circles was incorrect here) and gradually moved diagonally toward the middle. At the end, the elements on the edges were placed. The tiles with longer edges were placed according to the idea of central symmetry, but the difficulty level made the creator use the second type of tiles instead of the first type, which lead to the symmetry being distorted. However, the placement of the elements shows that the child had central symmetry in mind all the time (for the child it probably was rotation by 180°). The last work in fig. 5) is an example of perfect realization of rotational symmetry: the elements have different orientation (the central part clockwise, the corners counter clockwise). The child had rotational symmetry in mind as a symmetry axis is not visible here. Additionally, it is worth mentioning that rotational symmetry was used by gifted children and those having broader geometric experience rather than arithmetic one.

**General conclusion:** The pupils who could present regularity clearly did not lose spontaneous sensitivity to regularities. Such regular patterns seem to be culturally closer for them. The tools (tiles) produced a fondness for searching for geometric phenomena, and that sort of action is close to the theoretic basis for creating geometric ideas described by Hejný and Vopěnka. In the second group one could notice lack of consistent work. The children either did not have proper experiences or the range of these experiences was limited. On the second stage of education it becomes vivid in mathematics classes.
We have also noticed that this sort of classes motivate the pupils who are hyperactive. Dyslectic children also have difficulty with the jigsaw puzzles.

6.2 One-dimensional tessellations (bands)

In the second stage of the project we tried to focus the children’s attention on a one-dimensional pattern, namely the band. We assumed that a simpler structure would prove easier for the children.

This project stage has shown that:

1. Work with tiles requires familiarizing with the tool. This stage in particular directs geometrical phenomena that become important to the child by themselves through examination of the tool and direct the child’s further work on establishing mathematical intuition.

2. The children’s attention was focused on the pattern of the tiles not on the material (the children would put one tile over the other if continuing the line required it; they covered parts of the rectangular tiles). From the mathematical point of view, the rectangular shape of the tiles was to suggest the distance between the elements and how it would influence building of a pattern; it was important in terms of transformations. But the children ignored the rectangular shape of the tiles. This can be considered a confirmation of the fact that shift is not an easy transformation for the children.

3. The children did not limit their work to the sheet of paper, nor did they try to fill the sheet; they went out of the surface if the pattern required it. They functioned in “rich structures” (according to Hans Freudenthal).

4. The patterns varied and were filled with ideas.

5. Some of the children approached the task on a pre-mathematical level. creating tessellations that had little regularities. Nevertheless, the children participated in the classes, were active and satisfied with their work. From a didactic point of view, their activity was also important, as it could have been treated as a prolonged stage of gaining experience in interacting in the world of regularities. This allowed them to gain better knowledge of the tool.

6. Variation appeared in the forms created by the children, but also in the mathematical precision; for example, during the creation of a symmetrical tessellation, some children ignored the dots and others subdued the dot pattern to the idea of symmetry.
During the construction process, the children noticed the following facts: there are two sorts of tiles, “left” and “right” ones. By shifting and rotating the tiles one sort could not become the other, so the tiles are different.

The children were presented with the following games: creating the band, destroying it, coding and decoding the band. These activities were conducted in separate classes. Below are the observations and reflections from this stage.

6.2.1 Creating and destroying the band

The classes focused on creating bands and destroying them. One of the children created a pattern on the blackboard. We chose a person to repair the pattern and a person to destroy it. The first person would leave the room, not to see the destruction process, and then come back to repair the band. The children were told to keep the pattern one-dimensional and continuous. The band had to be a continuous sequence of signs.

The classes proved very exciting for the children, although they were not easy. The child had to memorize the structure of the pattern visually as a whole or in a sequence, recording the order and placement of the tiles. The second way is harder; but observing the work of the children showed that they tended to work that way. While testing the regularities, children usually analysed the tiles on the blackboard checking the sequence. Therefore, their work depended on transforming the global reception of information in the pattern into a sequence of activities. Additionally, it was clear that the children’s activities are not a passive copying of the pattern, but an activity dependent on following a specific rule that they had processed in their memory.

During the creation of bands that were to be destroyed (fig. 6) we observed a significant difference between the older and the younger children’s behaviour. The younger ones created a band that was based on two tiles. The older ones created longer combination of tiles, harder to memorize. In the first example there were no repeating elements. During its creation the following conversation went on:

*T: Grześ, can you build a band?*
*G: Any band? Can it be simple?*
*T: Yes, any.*

Grześ puts: 🎵🎵🎵

The children laugh.
*T: Longer, we have many tiles.*

Grześ begins to build a band (fig. 6):
This situation can be interpreted in various ways, but regardless of the interpretation it became an opportunity for mathematically important class activities. It also became the basis of the teachers' reflection.

The teacher wanted that a band was build – that means repetition of a motif. The pupil was encouraged to continue the pattern. The encouragement was read as putting more tiles. The pupil misunderstood the message. It turned out that this pattern is impossible to recreate as we are not able to memorize the whole motif.

On the educational level, the band meant for the children a long sequence of signs; the motif could be repetitive or not. However, never before was there any need for verbalization of this rule. In the task presented to older children, the pupils noticed that it would be hard to repair when regularities were not visible. Thus, they built long and complicated motifs (fig. 7) that usually did not reappear on the blackboard, as there was not enough room to repeat them. That is why there was no possibility to state if what is on the blackboard is a whole motif and what is the rule for its transformation.

Such a situation provoked a discussion about the band directed to clarifying its definition. The pupils mentioned “sort of like repeating frames”. They noted that the pattern should not take the whole blackboard, as we do not have control over such a band. We agreed that the bands will contain at least one repetition, which will enable us to verify the correctness of the band’s reconstruction.

Equally interesting were the observations regarding the children’s interpretation of destroying the pattern. Initially the pupil only turned the tiles around ($90^0$, $180^0$ or $270^0$); this caused the pattern to brake, but fixing it was not difficult. Then they began to turn the tiles and switch their places. In the process of fixing one needed to remember the original pattern and recreate it
tile after tile. If at some point a tile was turned that did not fit the pattern, the child would look for another tile in the next part of the band. By substitution and rotation of the tiles the pattern could be rebuilt. Searching for the right tile required of the children to know what they were looking for, what sort of tile was needed.

The pupils deduced themselves that it was possible to switch a tile from the blackboard with another one that had not been used. The teacher had to react at this point and took over the part of the person destroying the pattern. Some of the tiles were moved and rotated, but usually the “left” type was exchanged for the “right” type. At first the pupils were surprised with the teacher’s actions, but soon they realised what sort of difficulty will the person recreating the pattern encounter and anticipated the moment when this person would state that the pattern cannot be reconstructed any further because none of the tiles fit. Sometimes the children showed that the odd tile should be turned to the other side, but it proved impossible as the other side was blank. Through this we managed to stress the difference between the “left” and the “right” tiles and also inspire the children to notice the rotation and place switching (substituting parallel shifting) that do not place “left” tiles over the “right” tiles.

**General reflection.** In the first stage of the project we observed the children’s spontaneous creation of regularities. In the task set in this stage the children reacted differently. I believe that in this case the cognitive network referring to regularities has been reconstructed.

### 6.2.2 Coding and decoding the band

The second part of the classes was dedicated to decoding schematically written information. The schematic drawing reflected the band that was said to have been created by children from another school in another country. We only got the scheme with no additional information (fig. 8).

![Figure 8. The scheme of a band for grades IV-V.](image)

Each child tried to decode the band according to their own idea. It turned out that different bands were created. This situation lead to a discussion over the interpretation possibilities of a given code. The children stated that the arrow tips could (but do not have to) indicate the dots in the pattern and star-
ted to spontaneously draw their ideas on the blackboard. Each child wanted to draw their own band on the blackboard and share their ideas for decoding the information.

The first two ideas for decoding appeared on the blackboard (fig. 9a, 9b). The children explained how they interpret the relations between the arrows and the motifs on the tiles (the location of the dot and the line). They also contemplated two tiles at a time with different dots layout ("right" and "left" tile). The discussion revealed that the children had a very precise knowledge of the pattern on the "left" and "right" tile and remembered that the line starts closer to the corner of the tile and the dot is near the middle of the edge. The bands on the blackboard were considered different ones. The third band (fig. 9c) was also considered to be different. The children noticed that it contained parts of the second and first band (fig. 9b and 9a). The three types of bands often appeared in the class but another solution also was found (fig. 9d). Its creator explained that he had not taken the arrow tips into account, but only the direction and rate of obliquity of the lines. After drawing the fourth possibility, the teacher asked if there were any others. The pupils immediately stated that there were. They commented: "I just turned 180°", "I switched it – in the place where there is the beginning of the arrow, there is a dot and at the end there is nothing". Such statements turned the attention to the problem of the figure’s positioning (represented by the band) in the space as well as initiated the discussion: was a figure represented both in vertical and horizontal position the same figure?

![Diagram](image)

**Figure 9.** Ways of decoding the band.

In the whole process of decoding the band it was important to notice the structure and different interpretation possibilities of the given information. It was also crucial to provoke a discussion over the band’s structure, to search for different solutions and discuss their correctness and acceptability. The children
created bands responding to the code, compared their solutions and looked for similarities and differences between them.

At this stage the pupils’ attention was focused on the differences between the two tiles. These differences were made vivid by discussing the code (the arrows). The arrow code did not stress these differences, it seemed immune to an element’s orientation in the space – it did not indicate if there should be a “right” or a “left” tile in a given place. The children would rather read the code as: there are supposed to be four tiles in a two-to-two inner relationship.

Some of the pupils wondered how many ways are there to create a band reflecting this pattern (elements of speculative thinking).

**General reflection**

During the classes with the bands there appeared certain significant mathematical problems:

- regarding understanding of the band:
  - The band as a whole (motif) and as a sequence of repeating motifs
  - The band as a whole (reflected on the blackboard) and as a fragment of something that can be continued

- regarding the interpretation of the code:
  - Variety of interpretations
  - The inner relations between the elements (tiles)
  - Differences between the elements (tiles)

It turned out that in the second stage of education classes with bands are creative and can be broadened in many ways, directing pupils to various mathematical problems, like for example:

- Band made of one kind of tile (“left” tiles or “right” tiles), discussing the differences between the tiles
- Transferring the structure of a band (building bands with the same structure using different objects like geometric figures)
- Coding of a band
- Decoding of a band: what bands represent the code; how many solutions such a task has; What solutions can be considered equal?
6.3 Topical tessellations

Another stage of the project were the so called topical tessellations. With specially prepared music, the children were to create tessellations on a given topic. The assumption was that these tessellations were to have semantic basis. In order to create mathematical associations we wanted to use the meaning of words from the common language. Additionally, we wanted these associations to be dynamic, that is why the children would hear music during performing the task. Thus, during the class we were building relations between the word (its meaning), music, and visual representation created as a relation between specific objects. From the mathematical point of view we wanted to give proper meanings to geometric relations, but each child created this meaning based on his/her own experience. By this we wanted to connect the world of the children’s natural imagination with the world of mathematics. We tried to choose the themes of the tessellations in such a way that they would represent or could be associated with geometric transformations. We wanted to observe to what extent rapid river will be represented by the use of translation, merry-go-round by rotation; mirror by mirror symmetry.

Figure 10. Tessellations for 'Rapid river'.

The “Rapid river” (fig. 10) was often represented with a curved line. This topic in general evoked associations with translation, but some works contained a visible repeating motif. In the repetitions children used translation (they created a sort of band). The process of creating the pattern was connected with making premeditated calculations – rotations and shifts of specific tiles.

The children’s tessellations on “Merry-go-round” topic (fig. 11) contained mainly rotation. Placing several tiles required many full turns of single elements. Observing the process proved very interesting. Some of the works started with distinguishing the central element and then other elements were arranged around it – rotational arrangement. During these tasks the child obse-
erved the pattern in rotation, which enhanced the understanding of “left” and “right” tiles’ differences. A two-dimensional pattern with rotational symmetry was formed. Some children added tiles with regard to the rhythm of the music. Maintaining only rotational symmetry was difficult, as it can be seen in the children’s works. There were also works that seemed closer to mirror symmetry rather than rotational, and those where mirror symmetry gradually transformed into rotational symmetry. Observing the child’s work did not pose any doubts that they try to match the constructed arrangement with rotation – the pupil would draw an oval line with their finger, trying to see if the tiles go round one after another. While doing this, the pupil would adjust the tiles and rearrange them in such a way that the dot pattern would represent rotation and not mirror symmetry. Finally, a tessellation with a rotational pattern would appear.

![Figure 11. "Merry-go-round" puzzles.](image)

Tessellations on “Mirror” were varied. Some students built a physical object – a mirror. The glass was represented by blank tiles, and the ones with lines were used to build the frame. Almost every work contained regularities, often connected with mirror symmetry. The process of creation was different than on “merry-go-round”. The children used both movement of the right and the left hand, moving two tiles at the same time and placing them symmetrically. The tessellation was made by adjusting the left hand side to the right hand side.
In general, the tessellations in this stage of the project displayed an increasing level of regularity. Even children, who in the earlier stages conducted their work with the possibility to observe certain geometric rules, now used regularities or sequences of tiles.

**Observations:**

- The closest to matching our expectations were the works for “Merry-go-round”. In the first stage patterns with rotation were rare.
- It can be assumed that the term “Mirror” gave a slightly too vast field of interpretation. The term “mirror symmetry” is used during mathematics classes – the children’s spontaneous compositions were organised in a way that a mathematician interprets according to the idea of axial symmetry. The children use a mirror to recognize and draw mirror images as well as to observe the “second half” of an axially symmetric figure. Tessellations for the “Mirror” topic evoked various associations, not always connected with mirror symmetry. The term “mirror” contains a “static” content. It is difficult to combine mirror symmetry with proper movement because, as the only transformation on the surface, it is not a continuous transformation. The movement which corresponds to it is realized in the space. The semantic association with a mirror (or reflection) is also static and refers to the relation between two objects (the true object and its image).
- The most difficult topic was “Rapid river” with its parallel shift. In the stages realized so far, including this one, we were unable to expose that sort of tessellation.
General reflection. The classes with „topical tessellations” have shown how hard it is to associate certain own ideas with the pupils’ ideas as well as how hard it is to find the right semantic basis for the subsequent types of isometric transformations on the plane. During the discussion and analysis of the received results, there has been a suggestion that instead of terms such as “rapid river”, “mirror”, “merry-go-round”, dynamic terms could be used. such as “riding the merry-go-round”8. It is undoubtedly a suggestion worth trying out.

6.4 Reconstructing the tiled floor

While observing the pupils’ activities we were able to clearly perceive their expectations as to the challenging problems they had to solve using their knowledge. Three versions of tiled floors were prepared for reconstruction. In old buildings one can see tiled floors that have tiles missing and the pattern is partially broken. It is possible to reconstruct such floors. Such was also the case with the sheets prepared for the pupils – some tiles were missing, some patterns were vivid others were not.

This task proved to be a challenge for the pupils. They struggled with it in various ways. Some discovered the rule in the pattern quite fast saying: “This is easy, because each floor has a pattern”. Some did not know how to recreate the whole floor just having a fragment. The children were convinced by the teacher to help the ones who had no idea how to recreate the pattern – this forced them to verbalize their observations and explain how they had come up with their idea for solving the problem. Various ideas were presented. Each child explained their idea. Below are examples of argumentations:

Karolina:
I saw a pattern (here she shows the tiles along the left edge of the sheet) and then I moved it up by one.

Beatka:
I chose an element (she shows the rectangle from the right side of the sheet that has missing tile above) and looked for an identical one (she shows the same motif on the left side of the sheet). So you need to paste the same one (rectangle).

Asia:
If there was a rectangle here than I tried to find it somewhere close to the identical looking squares below and above. So I looked for a rectangle which

\footnote{I would like to thank professor Maria Korcz from UAM in Poznań for this advice.}
has an identical square under it. Here were two identical squares so there has
to be a rectangle between them.
Maryia:
Each floor has a pattern. This has a diagonal pattern. Identical tiles are put
in diagonal lines.
Kinga:
There are lines here: down, up, left, right... Like a long
pattern.
Maciek:
This pattern is diagonal, but there is also a pattern right
and left.
Kasia:
I know! This goes with that, only backwards (she makes a gesture with her
hand). I think I see something: this goes this way and that goes like this (she
makes gestures representing the tile’s position on the sheet and its position
after it has been rotated). When you turn this one (she rotates it 90\(^0\)) you will
get the same as this one.
Bartek:
This fragment (lower right corner) was the hardest to do, because it did not
have any pattern started. You had to look at the square that wasn’t pasted and
find this block on the sheet (he shows four printed squares) and do the same.
We need only one type of squares to finish the tessellation.
Kasia:
I have noticed that there is an identical pattern everywhere, only it goes diffe-
rent ways (she shows two tiles forming a motif). In this line (diagonal) there
is the pattern and in that line it’s the same (another diagonal line) but goes
another way. As if someone turned it 90\(^0\). There is one fragment here that is
everything.

The activities proved to be a challenge for the pupils, they provoked to
discuss and argument. The children’s argumentation points to deep analysis
of the pattern’s structure. Although the children did not know the concept
of angle, they were able to say how many degrees is a piece to be turned
in order to make it look like another one. They were able to recognize the
same tile in rotation. They first looked on the whole piece locally looking for
local relations. Then these local relations were used on the whole tessellation.
The most difficult part for the children was the rectangles’ positioning in the
tiled floor on fig. 13. Some of the rectangles had squares pasted over them.
So again, the idea that children would notice the distances between elements
functioning in a parallel shift and accept the fact as a differentiating factor
among the transformation of this group failed. The parallel shift seems to be the most difficult issue to teach.

**General reflection:** Speaking about geometry comes spontaneously when the children feel the lack of information. The missing pieces of information are generated through logical argumentation (reasoning). Language is a tool that helps to recreate what is not visible. Thus, describing what is visible may not have much sense for the children. This might be the reason why the children functioning on a “visual level” (according to van Hiele), do not use language.

7 Conclusion

Tiles, as it is stated in research results by Swoboda, are a good tool for both the youngest children and older primary-school pupils. They allow the children to work on their own level, with their own creativity. Even if the task is directed (for example building a pattern or a tiled floor), the children create their own work by themselves. Such activities stimulate a pupil to discover regularities and to function according to the rule. This happens regardless of the pupil’s age. This displays the diversity of the children’s perception and use of regularities in their creations, in “geometric perception”. Swoboda states that those 6-year-olds who use regularities “can do much better in school education than those who do not apply regularities while organizing space”. Our observation of grade-IV-VI pupils proves this true.

While designing tasks and activities for the children we were not aware of such broad possibilities of using these tasks. The children positively answered the given problems. Looking from the perspective of year’s work it is necessary to say that the classes significantly developed the children’s perception. A proof for that is solving more and more complicated tessellations by the children. Children who were initially unable to form a pattern began using regularities in their tessellations. Often they were not yet present in a long sequence of repetitions; however, the children’s works began displaying a sort of order and clarity. The dominating mirror symmetry is gradually supplemented by rotational symmetry and the rotation itself is identified by the children.

Although it is difficult to compare activities of 6-7 year old children with activities of children from older primary school classes, one can state that the observed domination of mirror symmetry in children’s work is similar to the research results of Swoboda. We noticed that the observed children seldom use shifts or rhythmical continuation of the pattern in their work. This may be caused by the lack of a topic of the task – a floor may be naturally associated with the appearance of a continuous pattern. The lack of shift in tessellations
may have also been caused by the type of the applied tiles. In Swoboda’s researches, tiles are symmetrical. Such construction of tiles did not differentiate whether a child repeated the pattern or thought about the symmetry. The visual effect of the work did not reveal these differences. Tiles used in my research were not symmetrical. Repeating the same pattern may have been treated by the children as a way of impoverishing the environment. They purposely used “left” and “right” tiles and they felt the difference between the two. Therefore, the application of two types of tiles (“left” and “right”) develops the children’s mathematical abilities.

According to the assumptions concerning SLE, the created environment must be directed towards important mathematical content as only then can the children act on various levels according to the natural differences between them. On the other hand, a rich mathematical environment reveals the natural diversity of a given society and becomes a motivating factor for every individual. The teacher is part of the SLE and also has to be reinforced. The teacher directs a given society, but the society also has an influence on the teacher, although that is not sufficient. The teacher cannot alone understand the natural diversity and the possibilities of using it as a source of motivation. The support of researchers is necessary. Here it is the construction of the Substantial Learning Environment.

As a teacher, I believe that SLE “Tiles” fulfils the assumptions of SLE. On the other hand, I also see that understanding of the world of geometry formed in the children’s minds still requires extensive research. Often the observed behaviour was a surprise and their interpretation proved difficult even with the information from contemporary literature. Being sensitive for the children’s natural behaviour should be a natural feature of all teachers, but the next stages: interpreting of this behaviour and building theories that explain this behaviour require close collaboration among the teachers researchers.

References


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Wrocław.


Bogate Środowisko Edukacyjne „Kafelki”

Streszczenie


1. Reprezentuje ono podstawowe cele, treści i pryncepty dotyczące nauczenia matematyki na określonym poziomie edukacyjnym.

2. Dotyczy istotnych matematycznych zagadnień, procesów i procedur wykraczających poza ten poziom i jest bogatym źródłem aktywności matematycznych.

3. Jest elastyczne i może być z łatwością adaptowane do specyficznych warunków klasowych.

4. Integruje matematyczne, pedagogiczne i psychologiczne aspekty nauczenia matematyki, a przez to tworzy bogate pole empirycznych badań w dydaktyce matematyki.

Podkreślając wagę SLE dla dydaktyki matematyki, Wittmann (2001) stwierdza, że pojęcie SLE ma dużą moc. Może być użyte dla pomyslnego rozwiązania jednego z najważniejszych zagadnień dydaktyki matematyki, które staje się coraz bardziej naślad, a które ma istotne znaczenie dla dydaktyki matematyki jako dyscypliny naukowej: związku teorii z praktyką.

Projekt SLE „Kafelki” dotyczył budowania intuicji izometrycznych przekształceń geometrycznych: przesunięcia równoległego, obrotu, symetrii osiowej. W wyniku działań w tym środowisku dziecko miałoby nabyć takich wyobrażeń i umiejętności, które w przyszłości pomogłyby mu w rozwiązywaniu matematycznych problemów związanych z tymi pojęciami, funkcjonującymi na poziomie formalnym (Hejny 1997). Poziom, do którego odnosimy się konstruując SLE „Kafelki” związany jest raczej z tworzeniem przez dziecko jego „obrazu pojęcia” związewanego z przekształceniami geometrycznymi, niż „definicji pojęcia” (Tall, Vinner, 1987), czyli budowania myślowych reprezentacji oraz odkrywania własności z nimi związanych. Dodatkowo zastanawialiśmy się nad budowaniem takich intuicji, które pomogą w przekroczeniu poziomu wizualnego rozumienia przekształceń i ukierunkują na dynamiczne rozumowania w obrębie geometrii. W tym przypadku byłoby to przejście od statycznej, wizualnej reprezentacji relacji między figurami („knowing that”) w kierunku
rozumienia dynamicznych związków między obiektami i ruchem nakładającym jeden obiekt na drugi („knowing how”).

Tworzenie propozycji SLE przebiegalo etapami. Każda faza projektu była efektem działań na etapie poprzednim oraz stanowiła podstawę dla powstania fazy następnej. W tym artykule opisane zostały jedynie obserwacje dotyczące uczniów klas IV, V, VI (dzieci 10-12 letnich).

Przebieg projektu

I. Faza wstępna

- Zapoznanie nauczycieli z ideą projektu, wybór zagadnień realizowanych w ramach projektu,
- Zapoznanie uczniów z projektem i z narzędziem badawczym (kafelekami).

Był to czas spontanicznej aktywności uczniów, swobodnej twórczości, zabawy. Dla nauczycieli obserwujących ten etap była to możliwość zweryfikowania wyników badań dydaktycznych. Testowano również przydatność narzędzi – kafelek, zarówno pod kątem pojęć i procedur matematycznych, możliwych do osiągnięcia na zajęciach, jak i atrakcyjności narzędzi na różnych poziomach edukacyjnych.

II. Układanki jednowymiarowe (szlaczki)

W drugiej fazie projektu postanowiliśmy skupić uwagę uczniów na wzorze jednowymiarowym, czyli szlaczku. Zajęcia przeprowadzone w klasach dotyczyły układania szlaczków oraz burzenia ich.

W założeniach matematycznym celem tego typu zajęć było odróżnienie odbicia lustrzanego (zmieniającego uporządkowanie płaszczyzny) od przesunięcia i obrotu (które zachowują uporządkowanie płaszczyzny). „Psucie szlaczka” było zadaniem odwrotnym do „układania”.

III. Kodowanie i dekodowanie szlaczka

Schematyczny rysunek obrazował strukturę szlaczka. Zadaniem uczniów było odkodowanie informacji.

IV. Układanki tematyczne

Kolejną fazą projektu były tzw. układanki tematyczne. Do odpowiednio przygotowanej muzyki uczniowie tworzyli układankę o zadanym temacie. Zasugerowane tematy to: (a) rwała rzeka - mająca ukierunkowywać na wykorzystanie translacji, (b) karuzela – wykorzystanie obrotów, (c) lustro – wykorzystanie symetrii osiowej.
V. Rekonstrukcja posadzki

To zadanie było w pewnym sensie dualne do zadania polegającego na naprawianiu zepsutego szklacza. Uczniowie dostali do uzupełnienia 3 mozaiki dwuwymiarowe, konstruowane według różnych reguł.

Grupa dzieci, których praca opisana jest w artykule, liczyła 46 uczniów (w tym 17 dziewcząt i 29 chłopców). W momencie rozpoczynania działań związanych z projektem jedna z klas czwartych nie realizowała jeszcze w ramach lekcji matematyki zagadnień związanych z odbiciem lustrzanym. Pozostali uczniowie poznań symetrię osiową; rysowali i rozpoznawali figury będące swoimi lustrzannymi odbiciami, wskazywali oś symetrii, rozpoznawali figury osiowo-symetryczne. Zakres doświadczeń związanych z symetrią był różny w zależności od wieku.

Wnioski

„Kafelki” są dobrym materiałem dydaktycznym dla dzieci najmłodszych. a także dla uczniów starszych klas szkoły podstawowej. Pozwalają one dzieciom pracować na swoim własnym poziomie, uruchomić własną twórczość. Nawet jeżeli zadanie jest ukierunkowane (np. budujemy wzorek, posadzkę), dziecko samo tworzy swe dzieło. Takie działania stymulują je w kierunku zauważania regularności i funkcjonowania zgodnie z regulą. Dzieje się to niezależnie od wieku dzieci. Uwidocznia się duże zróżnicowanie między dziećmi w dostrzeganiu i stosowaniu regularności w swoich wytworach, w ”geometrycznym patrzeniu”. Swoboda stwierdza, że sześcioletki, które stosują regularności, znacząco lepiej radzą sobie w edukacji szkolnej od tych, które organizując przestrzeni takich regularności nie stosowały. Nasza obserwacja uczniów klas IV – VI potwierdza tę tezę.

Patrząc z perspektywy roku pracy stwierdzam, że zajęcia w znacznym stopniu rozwinięły dziecięcą percepcję. Dowodem na to jest układanie coraz bardziej skomplikowanych układanek przez uczniów. Te dzieci, które początkowo nie były w stanie ułożyć wzoru, w swoich układankach potem stosowały już regularności. Często nie były one jeszcze zachowane w długim ciągu powtórzeń, jednak w pracach dzieci zapanował pewnego rodzaju ład i przejrzystość. Dominująca symetria lustrzana stopniowo uzupełniana jest o symetrię obrotową, a same obroty są przez dzieci identyfikowane.

Mimo iż trudno jest porównywać działania dzieci 6-7 letnich z uczniami starszych klas szkoły podstawowej, można stwierdzić, że zaobserwowana dominacja symetrii lustrzanej w pracach dzieci stanowi podobieństwo do wyników badań Swobody. Wśród obserwowanych uczniów uzużyliśmy natomiast bardzo rzadkie stosowanie przesunięcia, czy też rytmiczne kontynuowanie wzoru


Jako nauczyciel sadzę, że zaproponowane środowisko edukacyjne „Każełki” spełnia założenia SLE, zarysowane na początku tej pracy. Z drugiej strony widzę, że rozumienie geometrycznego świata kształtowanego w umysłach dzieci wymaga jeszcze wielu badań. Bardzo często obserwowane zachowania były zaskoczeniem, a ich interpretacja była bardzo trudna nawet w świetle tego, co znajduje się we współczesnej literaturze. Wyczuwanie na naturalne dziecie zachowania powinno być naturalną cechą nauczyciela, ale dalsze etapy: interpretacja tych zachowań, budowanie teorii tłumaczącej te zachowania – wymagają ścisłego kontaktu z badaczem.