On the approximate derivative of superposition

In this paper a theorem on the approximate derivative of superposition is proved. This theorem has an application in the proofs of theorems on substitution in the Denjoy integrals. As a corollary, a generalization of Tolstoff's well-known theorem (see [5]), concerning also the approximate derivative of superposition, is obtained.

**Theorem.** Let $F$ be a finite function, approximately derivable almost everywhere on an interval $(a, b)$ and let $F$ fulfil condition $(N)^{(1)}$ on this interval. If a function $\varphi$ is approximately derivable almost everywhere on an interval $(c, d)$ and is such that $\varphi((c, d)) \subset (a, b)$, then

$$G'_{ap}(t) = F'_{ap}(\varphi(t))q'_{ap}(t)$$

almost everywhere on the set $T$ of all points at which the finite approximate derivative of the function $G = F(\varphi)$ exists. (We assume that the right side of (1) is equal to zero if at least one of the factors is equal to zero, whether the other is defined or not).

The proof will require the following lemmas.

**Lemma 1.** Let $Z$ be any set of the real line. Then the sets of points which are respectively points of outer right-hand (left-hand) density and outer right-hand (left-hand) dispersion for the set $Z$ are $F_{\sigma\delta}$ sets.

**Lemma 2.** For every finite function, the set of all points of approximate derivability is measurable in the Lebesgue sense.

Lemmas 1 and 2 follow from Lemmas 1 and 4 in [2].

**Lemma 3.** If a finite function $F$ has the approximate derivative at every point of a set $A$ and if $|F(A)| = 0$, then $F'_{ap}(x) = 0$ almost everywhere on $A$.

**Proof.** Let $B$ be the set of all points $x \in A$ such that $F'_{ap}(x) > 0$. We shall prove that $B$ is of the Lebesgue measure zero. For this purpose let us observe that there exist a sequence of sets $\{B_n\}_{n=1}^\infty$ and a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$ such that (a) $B = \bigcup_{n=1}^\infty B_n$ and

(1) A function $F$ is said to fulfil condition $(N)$ on an interval $(a, b)$ if $|F(Z)| = 0$ for every $Z \subset (a, b)$ of the Lebesgue measure zero.
(b) \( F(x_2) - F(x_1) \geq \epsilon_n (x_2 - x_1) \), if \( x_1 < x_2 \) and \( x_1, x_2 \in B_n \) (see the proof of Theorem (10.8), p. 237 in [3]). Hence, it is sufficient to show that \( |B_n| = 0 \) for each positive integer. For this purpose let \( \epsilon \) be any positive number. Since \( |F(B_n)| = 0 \), there exists a sequence of intervals 
\[^{\text{1}}\] \( \{I_{n,s}\}_{n,s=1,2,...} \) such that \( F(B_n) \subseteq \bigcup_{s=1}^{\infty} I_{n,s} \) and \( \sum_{s=1}^{\infty} |I_{n,s}| < \epsilon \epsilon_n \) for each positive integer \( n \). We may assume that \( I_{n,s} \cap F(B_n) \neq \emptyset \) for \( s, n = 1, 2, \ldots \) Now let \( P_{n,s} \), for \( s, n = 1, 2, \ldots \), denote smallest closed interval containing the set \( B_n \cap F^{-1}(I_{n,s} \cap F(B_n)) \). (Some of \( P_{n,s} \) may reduce to single points). In view of (b), we obtain the inequality \( |I_{n,s}| \geq \epsilon_n |P_{n,s}| \) for \( s, n = 1, 2, \ldots \) Hence, it follows that \( \sum_{s=1}^{\infty} |P_{n,s}| < \epsilon \) for \( n = 1, 2, \ldots \) In such a way we have proved that \( B \) is of the Lebesgue measure zero.

Since, by symmetry, the same is true for the set of all points \( x \in A \) at which \( F' \) is negative, the proof of the lemma is finished.

A finite function defined on a set \( E \) will be termed LG on that set if \( E \) is expressible as the sum of a finite or countable sequence of sets on each of which this function fulfills the Lipschitz condition. The following lemma is known (see Theorem (10.14), p. 239 in [3]).

**Lemma 4.** If at every point \( x \) of a set \( E \), except perhaps at the points of a countable subset, a function \( F \) fulfills at least one of the following conditions

\[ -\infty < F_{\text{ap}}^{+}(x) \leq F_{\text{ap}}^{-}(x) < +\infty, \quad -\infty < F_{\text{ap}}^{-}(x) \leq F_{\text{ap}}^{+}(x) < +\infty, \]

then \( F \) is LG on the set \( E \).

The proof of the following lemma is simple and may be omitted.

**Lemma 5.** If a function \( F \) is LG on a set \( E \) and \( \varphi \) is a function VBG\(^{(2)}\) on a set \( S \) and is such that \( \varphi(S) \subseteq E \), then the function \( G = F(\varphi) \) is VBG on \( S \).

We now proceed to the proof of the theorem. In view of Lemma 4, there exists a sequence \( \{E_n\}_{n=1,2,...} \) of sets contained in \( \langle a, b \rangle \) such that

\[ |\langle a, b \rangle - \bigcup_{n=1}^{\infty} E_n| = 0 \]

and that \( F \) fulfills the Lipschitz condition on each \( E_n \).

Since \( F \) is approximately derivable almost everywhere on \( \langle a, b \rangle \), \( F \) is measurable in the Lebesgue sense on \( \langle a, b \rangle \). Hence, in view of the fact that every set measurable in the Lebesgue sense differs from a Borel set by a set of the Lebesgue measure zero, we may assume that each \( E_n \) is the Borel set. There exists a function \( F_n \), for \( n = 1, 2, \ldots \), which fulfills the Lipschitz condition on \( \langle a, b \rangle \) and is such that \( F_n \) coincides

\(^{(2)}\) For the definition of function VBG, see p. 221 in [3].
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with $F$ on $E_n$. Now let $A_n$, for $n = 1, 2, \ldots$, denote the set of all points of ordinary derivability of $F_n$. For every set $Z$, let $Z^{(o)}$ denote the set of all points of outer derivability for $Z$ which belong to $Z$. Since $A_n$ is Borel, therefore in view of Lemma 1 the set $B_n = (A_n \cap E_n)^{(o)}$, for $n = 1, 2, \ldots$, is also Borel. Further, since $F_n$ is derivable almost everywhere and almost every point of a set is a point of its outer density, the set $E_n - B_n$ is of the Lebesgue measure zero. We have also $B_n^{(o)} = B_n$ and

$$F'_{ap}(x) = F'_{B_n}(x)$$

for every $x \in B_n$, where $F'_{A}(x)$ denotes the derivative of a function $F$ at a point $x$ with respect to a set $A$. Since $\varphi$ is measurable in the Lebesgue sense on $\langle c, d \rangle$, we infer, in view of Lemma 2, that each set $T_n = T \cap \varphi^{-1}(B_n)$ is measurable in the Lebesgue sense. By Lemma 4, for each positive integer $n$ there exists a sequence $\{T_{n,k}\}_{k=1}^{\infty}$ of sets contained in $T_n$ such that $|T_n - \bigcup_{k=1}^{\infty} T_{n,k}| = 0$ and that $\varphi$ fulfills the Lipschitz condition on each $T_{n,k}$. In view of the argument used at the beginning of the proof, we may assume that $T_{n,k}$ is Borel, $T_{n,k}^{(o)} = T_{n,k}$ and $\varphi$ is derivable with respect to $T$ at every point belonging to $T_{n,k}$, for $n, k = 1, 2, \ldots$. Then

$$\varphi'_{ap}(t) = \varphi'_{T_{n,k}}(t)$$

at every point $t \in T_{n,k}$. By the ordinary formula on the derivative of superposition, we get

$$G'_{T_{n,k}}(t) = F'_{B_n}(\varphi(t))\varphi'_{T_{n,k}}(t),$$

for $t \in T_{n,k}$, $n, k = 1, 2, \ldots$. Since $T_{n,k}^{(o)} = T_{n,k}$ in view of (2), (3), (4) we infer that (1) is true at every point $t \in T_{n,k}$ and therefore almost everywhere on each $T_n$. Let us now put $B = \langle a, b \rangle - \bigcup_{n=1}^{\infty} B_n$. Since $B$ is of the Lebesgue measure zero, we have $|\varphi(A)| = 0$, where $A = T \cap \varphi^{-1}(B)$. Hence, in view of Lemma 3, $\varphi'_{ap}(t) = 0$ almost everywhere on $A$. Since $F$ fulfills condition (N), we get $|G(A)| = 0$, and hence, in view of Lemma 3, it follows that $G'_{ap}(t) = 0$ almost everywhere on $A$. It is easy to see that this completes the proof.

Remark 1. Let us observe that no inclusion between the sets $T$ and $T_\ast = \{ t : |\varphi'_{ap}(t)| < +\infty, |F'_{ap}(\varphi(t))| < +\infty \}$ is true in the above theorem. One can easily give an example of functions $F$, $\varphi$ which satisfy the hypotheses of the theorem and are such that $|T - T_\ast| > 0$, $|T_\ast - T| > 0$.

Corollary 1. Let $F$ be a finite function, measurable in the Borel sense and $LG$ on an interval $\langle a, b \rangle$. If a function $\varphi$ is approximately derivable almost everywhere on an interval $\langle c, d \rangle$ and is such that $\varphi(\langle c, d \rangle) \subset \langle a, b \rangle$,
then the function \( G = F(p) \) is approximately derivable almost everywhere and (1) is satisfied almost everywhere on \( (c, d) \).

**Proof.** In view of Lemma 4, we have \( (c, d) = T_1 \cup T_2 \), where \( |T_1| = 0 \) and \( p \) is LG and therefore also VBG on \( T_2 \). Since \( G \) is evidently measurable in the Lebesgue sense on \( T_2 \) by the theorem of Denjoy-Khintchine (see Theorem (4.3), p. 222 in [3]) \( G \) is approximately derivable almost everywhere on \( T_2 \) and therefore almost everywhere also on \( (c, d) \).

By the above mentioned theorem, \( F \) is also approximately derivable almost everywhere on \( (a, b) \). Since \( F \) fulfils condition (N) on \( (a, b) \) (see Theorem (6.1), p. 225 in [3]), the corollary follows directly from the theorem proved above.

**Remark 2.** The hypothesis of measurability in the Borel sense of \( F \) in the above corollary may be weakened. It is sufficient to assume that \( F \) is absolutely measurable on \( (a, b) \). One can show that this changed hypothesis is essential if one requires the statement of Corollary 1 to be true for such function \( p \) which is approximately derivable almost everywhere on \( (c, d) \). The author does not know whether the remaining hypothesis concerning \( F \) is essential in this sense.

The following corollary is a generalisation of Tolstoff's well-known theorem on the approximate derivative of superposition (see p. 325 in [5]).

**Corollary 2.** Let a function \( F \) satisfy the hypotheses of Lemma 4, where the set \( E \) is equal to an interval \( (a, b) \). Then, for every function \( p \) which is approximately derivable almost everywhere on an interval \( (c, d) \) and is such that \( p((c, d)) \subseteq (a, b) \), the statement of Corollary 1 is satisfied.

**Proof.** In view of Lemma 4 and Corollary 1 it is sufficient to show that \( F \) is measurable in the Borel sense on \( (a, b) \). For this purpose let us observe that there exists at most a countable set \( K_1 \subseteq (a, b) \) such that for \( x \in (a, b) - K_1 \) at least one of the following equalities is satisfied:

\[
F(x) = \lim_{t \to x^+} F(t), \quad F(x) = \lim_{t \to x^-} F(t).
\]

Let us now put

\[
K_2 = \{x: \lim_{t \to x^+} F(t) < \lim_{t \to x^-} F(t)\} \cup \{x: \lim_{t \to x^-} F(t) < \lim_{t \to x^+} F(t)\}.
\]

The set \( K_2 \) is at most countable (see p. 6 in [1]). Now let \( \alpha \) be any finite number. By (5) we get

\[
\{x: F(x) \leq \alpha, x \in S\} = \{x: \lim_{t \to x^+} F(t) \leq \alpha, x \in S\} \cup \{x: \lim_{t \to x^-} F(t) \leq \alpha, x \in S\},
\]

(\(^2\) For the definition of absolutely measurable functions, see [4].)
where $S = (a, b)$, $(K_1 \cup K_2)$. We shall now prove that the sets on the right side of the above equality are $F_{\alpha}$ with respect to $S$. For this purpose it is sufficient to show that the sets

$$\{x: \lim_{t \to x^+} F(t) > a\}, \quad \{x: \lim_{t \to x^-} F(t) > a\}$$

are $G_{\delta}$. Let $Z_n$, for $n = 1, 2, \ldots$, denote the set of all points $x$ at which the outer right-hand upper density of the set $\{t: F(t) > a + 1/n\}$ is positive. From the definition of the approximate right-hand upper limit it follows that

$$\{x: \lim_{t \to x^+} F(t) > a\} = \bigcup_{n=1}^{\infty} Z_n.$$

Hence, in view of Lemma 1, we infer $\{x: \lim_{t \to x^+} F(t) > a\}$ is a $G_{\delta}$-set.

In the same way we can show that $\{x: \lim_{t \to x^-} F(t) > a\}$ is also a $G_{\delta}$-set.

This completes the proof of the corollary.

In the same way one can show that $\{x: F(x) < a, x \in S\}$ is also an $F_{\alpha}$-set with respect to $S$. Hence, it follows that $F$ is of Baire class 2 with respect to $S$ and therefore also with respect to $\langle a, b \rangle$. In a different way one can prove that there exists at most a countable set $P$ such that $F$ is of Baire class 1 with respect to $\langle a, b \rangle - P$.

References