On the approximation of conjugate functions from $L^p$ by some special matrix means of conjugate Fourier series

Abstract. The results corresponding to some theorems of W. Łenski and B. Szal are shown. From the presented pointwise results the estimates on norm approximation are derived. Some special cases as corollaries are also formulated.

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1. Introduction. Let $L^p$ ($1 \leq p < \infty$) be the class of all $2\pi$–periodic real–valued functions integrable in the Lebesgue sense with $p$–th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \left(\int_Q |f(t)|^p \, dt\right)^{1/p}.$$ 

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x),$$

and the conjugate one

$$\tilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x),$$

with the partial sums $\tilde{S}_k f$. We know that if $f \in L^1$, then

$$\tilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_\epsilon(t) \frac{1}{2} \cotg \frac{t}{2} dt = \lim_{\epsilon \to 0^+} \tilde{f}(x, \epsilon),$$
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where

$$\tilde{f}(x, \epsilon) := -\frac{1}{\pi} \int_{\epsilon}^{\pi} \psi_x(t) \frac{1}{2} \cotg \frac{t}{2} dt$$

with

$$\psi_x(t) = f(x + t) - f(x - t),$$

exists for almost all $x$ and if $\tilde{f} \in L^1$, then $\tilde{S}f(x) = \tilde{S}\tilde{f}(x)$ [3].

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

(1) 

$$a_{n,k} \geq 0 \quad \text{and} \quad b_{n,k} \geq 0 \quad \text{when} \quad k = 0, 1, 2, ... n,$$

$$a_{n,k} = 0 \quad \text{and} \quad b_{n,k} = 0 \quad \text{when} \quad k > n,$$

(2) 

$$\sum_{k=0}^{n} a_{n,k} = 1 \quad \text{and} \quad \sum_{k=0}^{n} b_{n,k} = 1, \quad \text{where} \quad n = 0, 1, 2, ... .$$

Let the $AB$–transformation of $(\tilde{S}_k f)$ be given by

$$\tilde{T}_{n,A,B} f(x) := \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{S}_k f(x) \quad (n = 0, 1, 2, ... ).$$

As a measure of approximation of function $\tilde{f}$ and $\tilde{f} \left( \cdot, \frac{\pi}{n+1} \right)$ by $\tilde{T}_{n,A,B} f$ in the space $L^p(1 \leq p < \infty)$ we will use the pointwise modulus of continuity of $f$ defined, for $\beta > 0$, by the formula

$$\tilde{w}_x f(\delta)_{p,\beta} := \left\{ \frac{1}{\delta} \int_{0}^{\delta} \left[ |\psi_x(t)| \sin^{\beta} \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}},$$

and

$$\tilde{w}_{L^p} f(\delta)_{p,\beta} := \sup_{0 < t \leq \delta} \left\| \psi_x(t) \sin^{\beta} \frac{t}{2} \right\|_{L^p}.$$
then
\[
|\tilde{T}_{n,A,B}f(x) - \tilde{f}(x, \frac{\pi}{n+1})| \ll \sum_{r=0}^{n} \left( a_{n,r} + \sum_{k=1}^{r} a_{n,k} \right) \left[ \frac{1}{r+1} \sum_{k=0}^{r} \tilde{\omega}_x f \left( \frac{\pi}{k+1} \right)_{1,0} \right] \\
+ \frac{1}{n+1} \sum_{k=0}^{n} \tilde{\omega}_x f \left( \frac{\pi}{k+1} \right)_{1,0}
\]

and under the additional condition
\[
\frac{1}{\pi} \int_{0}^{\pi/(n+1)} \frac{|\psi_x(t)|}{t} dt \ll \tilde{\omega}_x f \left( \frac{\pi}{n+1} \right)_{1,0},
\]

\[
|\tilde{T}_{n,A,B}f(x) - \tilde{f}(x)| \ll \sum_{r=0}^{n} \left( a_{n,r} + \sum_{k=1}^{r} a_{n,k} \right) \left[ \frac{1}{r+1} \sum_{k=0}^{r} \tilde{\omega}_x f \left( \frac{\pi}{k+1} \right)_{1,0} \right] \\
+ \frac{1}{n+1} \sum_{k=0}^{n} \tilde{\omega}_x f \left( \frac{\pi}{k+1} \right)_{1,0}
\]

for every natural \( n \) and all considered real \( x \).

In this paper we shall consider the pointwise deviations \( \tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot, \frac{\pi}{n+1}) \) and \( \tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot) \). In the theorems we formulate the general and precise conditions for the matrices \((a_{n,r})_{r=0}^{n}\) and \((b_{r,k})_{k=0}^{n}\) and for the modulus of continuity type. Finally, we also give some results on the norm approximation.

We shall write \( K_1 \ll K_2 \), if there exists a positive constant \( C \), sometimes depending on some parameters, such that \( K_1 \ll CK_2 \).

2. Statement of the results. Now we formulate our main results.

**Theorem 2.1** Let \( f \in L^p \) (\( 1 \leq p < \infty \)). If the entries of matrices \((a_{n,r})_{r=0}^{n}\) and \((b_{r,k})_{k=0}^{n}\) satisfy the conditions (1), (2), and

\[
\sum_{r=0}^{n} \sum_{k=0}^{n-r} |a_{n-r,k} - b_{n-r,k+1}| \ll \frac{1}{n+1},
\]

then, for \( 0 \leq \beta \leq 1 \) when \( p = 1 \) and \( 0 \leq \beta < 2 - \frac{1}{p} \) when \( p > 1 \),

\[
|\tilde{T}_{n,A,B}f(x) - \tilde{f}(x, \frac{\pi}{n+1})| \ll (n+1)^{\beta} \left\{ \tilde{\omega}_x f \left( \frac{\pi}{n+1} \right)_{p,\beta} + \frac{1}{n+1} \sum_{k=0}^{n} \tilde{\omega}_x f \left( \frac{\pi}{k+1} \right)_{1,\beta} \right\}
\]

holds for almost all considered \( x \).
Let us consider the classes
\[ L^p(\tilde{\omega}_x) = \left\{ f \in L^p : \tilde{\omega}_x f(\delta) \leq \tilde{\omega}(\delta) \right\}, \]
and
\[ L^p(\tilde{\omega}) = \left\{ f \in L^p : \tilde{\omega}_{L^p} f(\delta) \leq \tilde{\omega}(\delta) \right\}, \]
with \( \beta \geq 0 \) and \( 1 \leq p < \infty \), where \( \tilde{\omega}_x \) and \( \tilde{\omega} \) are the functions of modulus of continuity type.

**Theorem 2.2** Let \( f \in L^p(\tilde{\omega}_x) \) (\( 1 < p < \infty \)). If the entries of matrices \( (a_{n,r})_{r=0}^n \) and \( (b_{r,k})_{k=0}^n \) satisfy the conditions (1), (2), (3) and \( \tilde{\omega}_x \) satisfies
\[ f \in L^p(\tilde{\omega}_x) \]
 holds for almost all considered \( x \) such that \( \tilde{\omega}(x) \) exists.

**Theorem 2.3** Let \( f \in L^p \) (\( 1 \leq p < \infty \)). If the entries of matrices \( (a_{n,r})_{r=0}^n \) and \( (b_{r,k})_{k=0}^n \) satisfy the conditions (1), (2), (3) and \( \tilde{\omega}_x \) satisfies
\[ f \in L^p(\tilde{\omega}_x) \]
 holds for almost all considered \( x \) such that \( \tilde{\omega}(x) \) exists and for \( 0 \leq \beta \leq 1 \) when \( p = 1 \), and \( 0 \leq \beta < 2 - \frac{1}{p} \) when \( p > 1 \).

**Theorem 2.4** Let \( f \in L^p(\tilde{\omega}) \) (\( 1 < p < \infty \)). If the entries of matrices \( (a_{n,r})_{r=0}^n \) and \( (b_{r,k})_{k=0}^n \) satisfy the conditions (1), (2), (3) and \( \tilde{\omega} \) satisfies
\[ f \in L^p(\tilde{\omega}) \]
 holds for almost all considered \( x \) such that \( \tilde{\omega}(x) \) exists and for \( 0 \leq \beta \leq 1 \) when \( p = 1 \), and \( 0 \leq \beta < 2 - \frac{1}{p} \) when \( p > 1 \).
3. Corollaries.

**Corollary 3.1** Under the assumptions of Theorem 2.1 we have

\[
\left| \tilde{T}_{n,A,B} f(x) - \tilde{f}(x, \frac{\pi}{n+1}) \right| \ll (n+1)^{\beta - \frac{1}{p}} \left\{ \sum_{k=0}^{n} \tilde{w}_x f \left( \frac{\pi}{k+1}, p, \beta \right) \right\}^\frac{1}{p}.
\]

**Proof** By Lemma 4.3

\[
\left| \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) \right| \ll (n+1)^{\beta} \left\{ \sum_{k=0}^{n} \tilde{w}_x f \left( \frac{\pi}{k+1}, p, \beta \right) \right\}^\frac{1}{p}.
\]

This completes the proof of Corollary 3.1.

**Corollary 3.2** Under the assumptions of Theorem 2.1, Corollary 3.1 and Lemma 4.3 we obtain another estimation for norm approximation

\[
\left\| \tilde{T}_{n,A,B} f (\cdot) - \tilde{f} \left( \cdot, \frac{\pi}{n+1} \right) \right\|_{L^p} \ll (n+1)^{\beta - \frac{1}{p}} \left\{ \sum_{k=0}^{n} \tilde{w}_x f \left( \frac{\pi}{k+1}, p, \beta \right) \right\}^\frac{1}{p}.
\]

If matrix \(A\) is defined by \(a_{n,r} = \frac{1}{(r+1)\ln(n+1)}\) when \(r = 0, 1, 2, \ldots, n\) and \(a_{n,r} = 0\) when \(r > n\), and matrix \(B\) is defined by \(b_{r,k} = \frac{1}{r+1}\) when \(k = 0, 1, 2, \ldots, r\) and \(b_{r,k} = 0\) when \(k > r\), then from Theorem 2.1 we obtain

**Corollary 3.3** Let \(f \in L^p (1 \leq p < \infty)\), then

\[
\left| \tilde{T}_{n,A,B} f (x) - \tilde{f} (x) \right| = \frac{1}{\ln(n+1)} \sum_{r=0}^{n} \frac{1}{(r+1)^2} \sum_{k=0}^{r} \tilde{S}_k f (x) - \tilde{f} (x) \]

\[
= O_x (n+1)^{\beta} \left\{ \tilde{w}_x f \left( \frac{\pi}{n+1}, p, \beta \right) + \frac{1}{n+1} \sum_{k=0}^{n} \tilde{w}_x f \left( \frac{\pi}{k+1}, 1, \beta \right) \right\}^\frac{1}{p},
\]

for almost all considered \(x\) and \(0 \leq \beta < 1 - \frac{1}{p}\), when \(p > 1\), and \(\beta = 0\), when \(p = 1\).

**Proof** For the proof we will show that \((a_{n,r})\) and \((b_{r,k})\) satisfy the assumptions of Lemma 4.2. The conditions (1) and (2) are satisfied evidently. Since

\[
\sum_{r=0}^{n} \frac{1}{(r+1)\ln(n+1)} \left( \sum_{k=0}^{n-r-1} \left| \frac{1}{n-r+1} - \frac{1}{n-r+1} \right| + \frac{1}{n-r+1} \right)
\]
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\[ \frac{1}{\ln(n+1)} \sum_{r=0}^{n} \frac{1}{r+1} \frac{1}{n-r+1} = 1 \ln(n+1) \]
\[ = \frac{1}{\ln(n+1)} \sum_{r=0}^{n} \left( \frac{1}{r+1} + \frac{1}{n-r+1} \right) \]
\[ \leq \frac{1}{\ln(n+1)} n + 2 \ln(n+1) \leq \frac{1}{n+1} \]

the proof of Corollary 3.3 is complete.

4. Auxiliary results. We begin this section by some notations following A. Zygmund [3, Section 5 of Chapter II].

It is clear that

\[ \tilde{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \tilde{D}_k(t) \, dt \]

and

\[ \tilde{T}_{n,A,B} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}_k(t) \, dt, \]

where

\[ \tilde{D}_k(t) = \sum_{\nu=0}^{k} \sin \nu t = \frac{\cos \frac{\nu}{2} - \cos(k + \frac{1}{2})t}{2 \sin \frac{\nu}{2}}. \]

Hence

\[ \tilde{T}_{n,A,B} f(x) - \tilde{f}(x, \frac{\pi}{n+1}) = -\frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}_k(t) \, dt + \frac{1}{\pi} \int_{\pi}^{\pi} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}^2_k(t) \, dt, \]

and

\[ \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) = \frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}_k^2(t) \, dt, \]

where

\[ \tilde{D}^2_k(t) = \frac{\cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}. \]

Now, we formulate some estimates for the conjugate Dirichlet kernels.

Lemma 4.1 (see [3]) If $0 < |t| \leq \pi/2$, then

\[ |\tilde{D}_k^2(t)| \leq \frac{\pi}{2|t|} \text{ and } |\tilde{D}_k(t)| \leq \frac{\pi}{|t|}, \]

and for any real $t$ we have

\[ |\tilde{D}_k(t)| \leq \frac{1}{2} k(k+1)|t| \text{ and } |\tilde{D}_k(t)| \leq k+1. \]
Lemma 4.2 If \( (a_{n,r})_{n,r=0}^n \) and \( (b_{r,k})_{r,k=0}^r \) satisfy (1), (2) and (3), then

\[
\left| \sum_{r=0}^n a_{n,n-r} \sum_{k=0}^r b_{r,k} \tilde{D}_k^r(t) \right| \ll \frac{\tau^2}{n+1}
\]

for \( n = 0, 1, 2, 3, \ldots \), where \( \tau = \left\lceil \frac{\pi}{t} \right\rceil \leq n/2 \).

Proof Since

\[
K_n(t) := \sum_{r=0}^n a_{n,n-r} \sum_{k=0}^r b_{r,k} \tilde{D}_k^r(t) = \sum_{r=0}^n a_{n,r} \sum_{k=0}^n b_{n-r,k} \tilde{D}_k^r(t)
\]

and using Abel’s transformation and from (7) we get

\[
= \sum_{r=0}^n \frac{a_{n,r}}{2\sin \frac{t}{2}} \left[ \sum_{k=0}^{n-r-1} (b_{n-r,k} - b_{n-r,k+1}) \sum_{l=0}^k \cos \left( \frac{2(l+1)\pi}{2} \right) \right. \\
+ b_{n-r,n-r} \sum_{l=0}^{n-r} \cos \left( \frac{2(l+1)\pi}{2} \right) \left. \right]
\]

\[
= \frac{1}{2\sin \frac{t}{2}} \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} (b_{n-r,k} - b_{n-r,k+1}) \left[ \sum_{l=0}^k \cos \left( \frac{2(l+1)\pi}{2} \right) \right].
\]

Then

\[
|K_n(t)| \leq \frac{1}{\sin \frac{t}{2}} \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} |b_{n-r,k} - b_{n-r,k+1}| \left| \sum_{l=0}^k \cos \left( \frac{2(l+1)\pi}{2} \right) \right|.
\]

A simple calculation for the last sum gives

\[
\left| \sum_{l=0}^k \cos \left( \frac{2(l+1)\pi}{2} \right) \right| \leq \frac{1}{\sin \frac{t}{2}}.
\]

So

\[
|K_n(t)| \ll \frac{\pi^2}{t^2} \sum_{r=0}^n a_{n,r} \sum_{k=0}^{n-r} |b_{n-r,k} - b_{n-r,k+1}|
\]

and from (3) we obtain

\[
|K_n(t)| \ll \frac{\tau^2}{n+1}.
\]

The desired estimate is now evident.

Lemma 4.3 Let \( f \in L^p \) (\( 1 \leq p < \infty \)) and \( \beta \geq 0 \), then

\[
\tilde{w}_x f \left( \frac{\pi}{n+1} \right)_{p,\beta} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^n \left[ \tilde{w}_x f \left( \frac{\pi}{k+1} \right)_{p,\beta} \right]^p \right\}^{\frac{1}{p}}
\]

holds for every natural \( n \) and real \( x \).
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**Proof** By our assumption we get

$$\tilde{w}_x f\left(\frac{\pi}{n+1}\right)_{p,\beta} = \left\{ \frac{n+1}{\pi} \right\} \int_0^{\pi/\pi} \left| \psi_x(u) \sin^\beta \frac{u}{2} \right|^p du \sum_{k=0}^{n} \frac{2 (k+1)}{(n+1) (n+2)} \right\}^{\frac{1}{p}}$$

$$= \left\{ \frac{2}{n+1} \sum_{k=0}^{n} \frac{(k+1) (n+1)}{\pi (n+2)} \int_0^{\pi/\pi} \left| \psi_x(u) \sin^\beta \frac{u}{2} \right|^p du \right\}^{\frac{1}{p}}$$

$$\ll \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \frac{(k+1)}{\pi} \int_0^{\pi/\pi} \left| \psi_x(u) \sin^\beta \frac{u}{2} \right|^p du \right\}^{\frac{1}{p}}$$

$$= \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \tilde{w}_x f\left(\frac{\pi}{k+1}\right)_{p,\beta} \right\}^{\frac{1}{p}}$$

and this completes the proof. ■

4.1. **Proof of Theorem 2.1.** We start with

$$\tilde{T}_{n,A,B} f (x) - \bar{f} \left( x, \frac{\pi}{n+1} \right) = -\frac{1}{\pi} \int_0^{\pi/\pi} \psi_x (t) \sum_{r=0}^n \sum_{k=0}^n a_{n,n-r,b_{r,k}} \bar{D}_k (t) dt$$

$$+ \frac{1}{\pi} \int_0^{\pi/\pi} \psi_x (t) \sum_{r=0}^n \sum_{k=0}^n a_{n,n-r,b_{r,k}} \bar{D}_k^2 (t) dt$$

$$= \tilde{I}_1 (x) + \tilde{I}_2^2 (x)$$

and

$$|\tilde{T}_{n,A,B} f (x) - \bar{f} \left( x, \frac{\pi}{n+1} \right)| \leq |\tilde{I}_1 (x)| + |\tilde{I}_2^2 (x)|$$

In case $p = 1$ and $0 \leq \beta \leq 1$ from Lemma 4.1 and (9), we obtain

$$|\tilde{I}_1 (x)| \leq \frac{1}{2\pi} \sum_{r=0}^n \sum_{k=0}^n a_{n,n-r,b_{r,k}} (k+1) \int_0^{\pi/\pi} |\psi_x (t)| t dt,$$

and using the inequality $\sin \frac{t}{2} \geq \frac{t}{2} (0 \leq t \leq \pi)$, we get

$$|\tilde{I}_1 (x)| \leq \frac{n(n+1)}{\pi} \int_0^{\pi/\pi} |\psi_x (t)| \left( \sin^\beta \frac{t}{2} \right) t^{1-\beta} dt$$

$$= n \left( \frac{\pi}{n+1} \right)^{1-\beta} \tilde{w}_x f\left(\frac{\pi}{n+1}\right)_{1,\beta} \ll (n+1)^{\beta} \tilde{w}_x f\left(\frac{\pi}{n+1}\right)_{1,\beta}.$$

In case $p > 1$ and $0 \leq \beta < 2 - \frac{1}{p}$, the Hölder inequality for integrals, with $\frac{1}{p} + \frac{1}{q} = 1$, Lemma 4.1 and (9) give

$$|\tilde{I}_1 (x)| \leq \frac{n(n+1)}{2\pi} \int_0^{\pi/\pi} |\psi_x (t)| t dt.$$
Now we estimate the term $\tilde{I}_2$. Using Lemma 4.2 and the inequality $\sin t \leq t$ for $0 \leq t \leq \pi$, we obtain

$$\left| \tilde{I}_2 (x) \right| \leq \frac{1}{\pi} \int_{\pi/\pi}^{\pi} |\psi_x (t)| \left| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r,b_{r,k}} D_k^r (t) \right| dt \leq \frac{\pi}{n+1} \int_{\pi/\pi}^{\pi} \frac{|\psi_x (t)|}{t^2} dt,$$

for $\beta \geq 0$. Integrating by parts we get

$$\left| \tilde{I}_2 (x) \right| \ll (n + 1)^{\beta - 1} \left\{ \frac{1}{\pi^2} \int_{0}^{\pi} |\psi_x (s)| \sin^{\beta} \frac{s}{2} ds \right\}^{\frac{\pi}{\pi^2}} + 2 \int_{\pi/\pi}^{\pi} \frac{1}{t^2} \left[ \int_{0}^{t} |\psi_x (s)| \sin^{\beta} \frac{s}{2} ds \right] dt,$$

$$\ll (n + 1)^{\beta - 1} \left\{ \frac{1}{\pi^2} \int_{0}^{\pi} |\psi_x (s)| \sin^{\beta} \frac{s}{2} ds + 2 \int_{\pi/\pi}^{\pi} \frac{1}{t^2} \tilde{w}_x f (t)_{1,\beta} dt \right\} \ll (n + 1)^{\beta - 1} \left\{ \tilde{w}_x f (\pi)_{1,\beta} + \int_{1}^{n+1} \tilde{w}_x f \left( \frac{\pi}{u} \right)_{1,\beta} du \right\} \ll (n + 1)^{\beta - 1} \left\{ \tilde{w}_x f (\pi)_{1,\beta} + \sum_{m=1}^{n+1} \tilde{w}_x f \left( \frac{\pi}{m} \right)_{1,\beta} du \right\} \ll (n + 1)^{\beta - 1} \left\{ \tilde{w}_x f (\pi)_{1,\beta} + \sum_{m=0}^{n} \tilde{w}_x f \left( \frac{\pi}{m+1} \right)_{1,\beta} \right\} \ll (n + 1)^{\beta - 1} \sum_{m=0}^{n} \tilde{w}_x f \left( \frac{\pi}{m+1} \right)_{1,\beta}. $$
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4.2. Proof of Theorem 2.2. Let us usually 

$$\tilde{T}_{n,A,B}f(x) - \tilde{f}(x) = -\frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}_k(t) dt$$

$$+ \frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}_k(t) dt$$

$$= \tilde{I}_1(x) + \tilde{I}_2(x)$$

and

$$|\tilde{T}_{n,A,B}f(x) - \tilde{f}(x)| \leq |\tilde{I}_1(x)| + |\tilde{I}_2(x)|.$$ 

From Lemma 4.1 and (8), using the Hölder inequality for integrals, with $\frac{1}{p} + \frac{1}{q} = 1$, and from (4) and (5) we obtain

$$|\tilde{I}_1(x)| \leq \frac{1}{\pi} \int_{0}^{\pi} |\psi_x(t)| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} |\tilde{D}_k(t)| dt$$

$$\ll \left\{ \int_{0}^{\pi} \frac{|\tilde{w}_x(t)|}{t} dt \right\}^{\frac{1}{p}} \left\{ \int_{0}^{\pi} \left( \frac{\tilde{w}_x(t)}{t \sin^{\beta} t} \right)^q dt \right\}^{\frac{1}{q}}$$

$$\ll (n+1)^{-\frac{1}{p}} (n+1)^{\frac{\beta+1}{2}} \tilde{w}_x \left( \frac{\pi}{n+1} \right) = (n+1)^{\beta} \tilde{w}_x \left( \frac{\pi}{n+1} \right)$$

$$\ll (n+1)^{\beta-1} \sum_{k=0}^{n} \tilde{w}_x \left( \frac{\pi}{n+1} \right)$$

for $1 < p < \infty$. Using Lemma 4.2 for $\tilde{I}_2$ we have

$$|\tilde{I}_2(x)| \leq \frac{1}{\pi} \int_{0}^{\pi} |\psi_x(t)| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,n-r} b_{r,k} \tilde{D}_k(t) \frac{1}{t^2} dt \leq \frac{\pi^2}{n+1} \int_{0}^{\pi} \frac{|\psi_x(t)|}{t^2} dt.$$ 

Analogously to the second part of the proof of Theorem 2.1, we obtain

$$|\tilde{I}_2(x)| \ll (n+1)^{\beta-1} \sum_{m=0}^{n} \tilde{w}_x f \left( \frac{\pi}{m+1} \right)^{1,\beta}$$

and

$$|\tilde{I}_2(x)| \ll (n+1)^{\beta-1} \sum_{m=0}^{n} \tilde{w}_x \left( \frac{\pi}{m+1} \right).$$
Thus, our proof is complete.

4.3. Proof of Theorem 2.3. If \( f \in L^p \), then by monotonicity of the norm as a functional

\[
\left\| \tilde{T}_{n,A,B} f (\cdot) - \tilde{f} \left( \cdot, \frac{\pi}{n+1} \right) \right\|_{L^p} \leq (n+1)^\beta \left\{ \left\| \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right\|_{L^p} + \frac{1}{(n+1)} \sum_{k=0}^{n} \left\| \tilde{\omega} \left( \frac{\pi}{k+1} \right) \right\|_{L^p} \right\}.
\]

Using monotonicity of \( \tilde{\omega}_x f \), with respect to \( p \), we obtain

\[
\left\| \tilde{\omega} \left( \frac{\pi}{k+1} \right) \right\|_{L^p} \leq \left\| \tilde{\omega} \left( \frac{\pi}{k+1} \right) \right\|_{L^p} \leq \left\{ \frac{1}{2\pi} \int_{0}^{\pi} \left| \psi_x (u) \sin^\beta \frac{u}{2} \right| du \right\} \frac{1}{p} \]

\[
= \left\{ \frac{k+1}{\pi} \int_{0}^{\pi} \left| \psi_x (u) \sin^\beta \frac{u}{2} \right| du \right\} \frac{1}{p} \leq \left\{ \sup_{0 < u \leq \frac{\pi}{k+1}} \left| \psi_x (u) \sin^\beta \frac{u}{2} \right| \right\} \frac{1}{p} \]

\[
= \tilde{\omega}_{L^p} f \left( \frac{\pi}{k+1} \right) \beta
\]

for \( k = 0, 1, 2, 3, \ldots \). Using monotonicity of \( \tilde{\omega}_{L^p} f (\delta)_\beta \), in view of \( \delta \), we have

\[
\left\| \tilde{T}_{n,A,B} f (\cdot) - \tilde{f} \left( \cdot, \frac{\pi}{n+1} \right) \right\|_{L^p} \leq (n+1)^{\beta-1} \sum_{k=0}^{n} \tilde{\omega}_{L^p} f \left( \frac{\pi}{k+1} \right)_\beta
\]

and the desired result follows.

4.4. Proof of Theorem 2.4. Analogously to the proof of Theorem 2.2 we have

\[
\left| \tilde{T}_{n,A,B} f (x) - \tilde{f} (x) \right| \leq \left\{ \int_{0}^{\frac{\pi}{\delta}} \left[ \left| \tilde{\psi}_x (t) \right| \right]^p \frac{t^{\beta p}}{2} dt \right\} \frac{1}{p} \left\{ \int_{0}^{\frac{\pi}{\delta}} \left[ \tilde{\omega} (t) \right]^q \frac{t^{\beta p}}{2} dt \right\} \frac{1}{q} \]

\[
+ (n+1)^{\beta-1} \sum_{m=0}^{n} \tilde{\omega}_x f \left( \frac{\pi}{m+1} \right)_\beta.
\]
Using (6) we obtain
\[
\|\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot)\|_{L^p} \ll \left\{ \frac{\int_0^{\pi n+\frac{\beta}{2}} \left[ \frac{\tilde{\omega}(t)}{t\sin^\beta t} \right]^q dt}{\int_0^{\pi n+\frac{\beta}{2}} \left[ \frac{\psi_0(t)}{\omega(t)} \right]^p \sin^{\beta p} \frac{t}{2} dt} \right\}^{\frac{1}{q}}
\]
\[
\quad + (n+1)^{\beta-1} \sum_{m=0}^{n} \left\| \tilde{\omega} f \left( \frac{\pi}{m+1} \right) \right\|_{L^p}^{\frac{1}{p}}
\]
\[
\ll (n+1)^{\beta+\frac{1}{p}} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \left\{ \frac{\int_0^{\pi n+\frac{\beta}{2}} \left[ \frac{\psi_0(t)}{\omega(t)} \right]^p \sin^{\beta p} \frac{t}{2} dt}{\int_0^{\pi n+\frac{\beta}{2}} \left[ \frac{\psi_0(t)}{\omega(t)} \right]^p \sin^{\beta p} \frac{t}{2} dt} \right\}^{\frac{1}{q}}
\]
\[
\quad + (n+1)^{\beta-1} \sum_{m=0}^{n} \left\| \tilde{\omega} f \left( \frac{\pi}{m+1} \right) \right\|_{L^p}^{\frac{1}{p}}
\]
\[
\ll (n+1)^{\beta+\frac{1}{p}} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \left( \frac{\pi}{n+1} \right)^{\frac{1}{p}} + (n+1)^{\beta-1} \sum_{m=0}^{n} \tilde{\omega} \left( \frac{\pi}{m+1} \right)
\]
\[
\ll (n+1)^{\beta-1} \sum_{m=0}^{n} \tilde{\omega} \left( \frac{\pi}{m+1} \right).
\]

Thus, our proof is complete. \[\blacksquare\]

References

