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Existence and Topological Properties of Solution Sets for Differential Inclusions with Delay

Abstract. We consider the problem \( \dot{x}(t) \in A(t)x(t) + F(t, \theta_t x) \) a.e. on \([0, b]\), \( x = \kappa \) on \([-d, 0]\) in a Banach space \( E \), where \( \kappa \) belongs to the Banach space, \( \mathcal{C}_E([-d, 0]) \), of all continuous functions from \([-d, 0]\) into \( E \). A multifunction \( F \) from \([0, b] \times \mathcal{C}_E([-d, 0])\) into the set, \( \mathcal{P}_{fc}(E) \), of all nonempty closed convex subsets of \( E \) is weakly sequentially hemi-continuous, \( \theta_t x(s) = x(t + s) \) for all \( s \in [-d, 0] \) and \( \{A(t) : 0 \leq t \leq b\} \) is a family of densely defined closed linear operators generating a continuous evolution operator \( S(t, s) \). Under a generalization of the compactness assumptions, we prove an existence result and give some topological properties of our solution sets that generalizes earlier theorems by Papageorgiou, Rolewicz, Deimling, Frankowska and Cichon.

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1. Introduction. Differential inclusions appear in the study of nonsmooth Hamiltonian systems and nonsmooth optimal control problems as in Clark [7]. In addition, the works of de Korvin-Kleyle [27], Papageorgiou ([26], [27]), Salinetti-Wets ([35], [36]) and Yovits-Foulk-Rosi [38] illustrate the importance of the multifunctions in various applied fields, like optimization ([35], [36]), mathematical economics ([26], [27]) and in the analysis of uncertain information system ([27],[38]). In this paper we will deal with the differential inclusion

\[
(P) \left\{ \begin{array}{ll}
\dot{x}(t) \in A(t)x(t) + F(t, \theta_t x), & t \in [0, b] \\
x = \kappa & \text{on } [-d, 0],
\end{array} \right.
\]

where \( b, d \geq 0 \) and \( \kappa \) belongs to the Banach space, \( \mathcal{C}_E([-d, 0]) \), of all continuous functions from \([-d, 0]\) into \( E \). \( F : [0, b] \times \mathcal{C}_E([-d, 0]) \rightarrow E \) is weakly sequentially hemi-continuous, \( \theta_t x(s) = x(t + s) \) for all \( s \in [-d, 0] \) and \( \{A(t) : 0 \leq t \leq b\} \) is a
family of densely defined closed linear operators generating a continuous evolution operator \( S(t,s) \).

Our purpose is to prove an existence theorem for integral solution of the problem \((P)\). Moreover, we give some topological properties for our solution set, \( S(\mathcal{K}) \), of all integral solutions for \((P)\). The problem \((P)\) was investigated, without delay, by many authors ([10], [11], [30], [34], [6], [13] for instance). In this case when \( A(t) = 0 \), \( S(t,s) = id \) and a mild solution is a Carathéodory one, we have a generalization to the existence theorems of Deimling [10], Ibrahim-Gomaa [18], Kisielewicz [20], Papageorgiou [30] and [32]. As \( A(t) \neq 0 \) our results extend that of [11], [30], [34] and [6]. Also in [14] we give a generalization to recent results, to the Cauchy problem

\[
\begin{aligned}
\dot{x}(t) &= f(t, x(t)), \quad t \in [0, T] \\
x(0) &= x_0,
\end{aligned}
\]

where \( f : [0, T] \times E \to E \) and \( E \) is a Banach space, by using weak and strong measures of noncompactness, and in [12] we study the solution set for the inclusion \( \dot{x}(t) \in F(t, x(t), \dot{x}(t)) \) under three boundary conditions \( x(0) = x_0, \ x(\eta) = x(T) \) where \( 0 < \eta < T \). Moreover much work has been done to study the topological properties of the solution set for the differential inclusions see ([1], [2], [5], [19], [17], and [16] for instance). Papageorgiou, [31], consider the problem \( x(t) \in p(t) + \int_0^t k(t,s) \text{ext} F(s, x(s)) \, ds \) where \( \text{ext} F(t,s) \) denotes the set of extremal points of \( F(s, x(s)) \), this problem arise in the study of control system, also Papageorgiou, [28], study semilinear evolution inclusions and their use in optimal problems. Volterra integral inclusions of the type \( x(t) \in p(t) + \int_0^t k(t,s) \text{ext} F(s, x(s)) \, ds \) has been studied by Papageorgiou, [29], in a separable Banach space \( E \) with \( p(\cdot), x(\cdot) \in C([0, b], E) \), \( K(t,s) \) be a function from \([0, b] \times [0, b] \) into the set of all continuous, linear operators from \( E \) into \( E \) for all \( 0 \leq s \leq t \leq b \), \( \|k(t,s)\| \leq C < \infty \) and \( F \) is a multifunction on \([0, b] \times E \) with nonempty closed valued in \( E \). Under certain conditions on \( F \) and \( K \) existence theorems are proved and continuity properties of the solution set are studied. Moreover, S. Hu - V. Lakshmikantham and N. S. Papageorgiou, [15], study semilinear evolution inclusions in which the linear, closed and densely defined operator which generates the strongly continuous semigroup depends on parameter.

2. Preliminaries. A multivalued function \( F \) on a Banach space \( E \) into the set of nonempty closed subsets, \( P(E) \), of \( E \) is said to be upper semicontinuous if and only if for all closed subset \( A \) of \( E \) \( F^{-}(A) = \{ x \in E : F(x) \cap A \neq \emptyset \} \) is closed in \( E \) and \( F \) is called \( w-w \) sequentially upper semicontinuous if every weakly closed subset \( A \) of \( E \) \( F^{-}(A) \) is weakly sequentially closed. Such a multivalued function is called upper hemi-continuous ( resp. weakly upper hemi-continuous) if and only if for any \( x^{*} \in E^{*} \), \( c \in \mathbb{R} \) \( \{ x \in E : \sup_{x \in F(x)}(x^{*}, x) < c \} \) is open in \( E \) (resp. in \( E_{w}^{*} \), where \( E_{w}^{*} \) is the Banach space \( E \) with the weak topology and \( E^{*} \) is the dual space. Furthermore \( F \) is called weakly sequentially upper hemi-continuous if and only if for any \( x^{*} \in E^{*} \) the function \( h : E_{w}^{*} \to \mathbb{R} \) defined by \( h(x) = \sup_{x \in F(x)}(x^{*}, x) \) is sequentially upper hemi-continuous. For other properties of the multivalued function we refer to [21], [8], and [4] for instance.

The following lemmas is necessary in the proof our main results.
Lemma 2.1 ([24], [22]). If \( \gamma \) is a measure of weak (strong) noncompactness and \( A \subset C_w(I, E) \) is a family of strongly equicontinuous functions, then \( \gamma(A(I)) = \sup \{ \gamma(A(t)) : t \in I \} \).

Lemma 2.2 ([6]) Let \( Y \) and \( E \) be Banach spaces and let \( F : E \to P_{fc}(Y) \) be weakly sequentially upper hemi-continuous. If there exist \( a \in L^1(I, \mathbb{R}), (x_n)_{n \in \mathbb{N}} \subset C(I, E) \) and \( (y_n)_{n \in \mathbb{N} \cup \{0\}} \subset L^1(I, E) \) such that \( \|F(x)\| \leq a(t) \) for all \( x \in C(I, E) \), \( x_n(t) \to x_0(t) \) weakly a.e. on \( I \), \( y_n \to y_0 \) weakly in \( L^1(I, E) \) and \( y_n(t) \in F(x_n(t)) \) a.e. on \( I \), then \( y_0(t) \in F(x_0(t)) \) a.e. on \( I \).

Lemma 2.3 [6] Let \( F : I \times E \to P_{fc}(E) \) and, for each \( t \in I \), let \( F(t, \cdot) \) be weakly sequentially upper hemi-continuous. If there exists \( a \in L^1(I, \mathbb{R}) \) such that, for each \( (t, x) \in I \times E \), \( \|F(t, x)\| \leq a(t) \) and, for each \( x \in E \), \( F(\cdot, x) \) has a measurable selection, then for each \( y \in C(I, E) \) there exists at least one measurable (and integrable) selection \( \sigma(\cdot) \) of \( F(\cdot, y(\cdot)) \).

In this paper we consider \( E \) is a Banach space, \( I = [0, b] \) and \( P_{fc}(E) \) is the family of nonempty closed convex subsets of \( E \) and we shall use some steps that used by Szulfa [37]. For any nonempty bounded subset \( Z \) of \( E \), the Kuratowski measure of noncompactness, \( \alpha \), and the Hausdorff measure of noncompactness, \( \alpha^* \), are defined as:

\[
\alpha(Z) = \inf \{ \varepsilon > 0 : Z \text{ admits a finite number of sets with diameter} \leq \varepsilon \},
\]

\[
\alpha^*(Z) = \inf \{ \varepsilon > 0 : Z \text{ admits a finite number of balls with radius} \leq \varepsilon \}.
\]

For the properties of \( \alpha \) and \( \alpha^* \) we refer to [3] and [9] for instance. By a Kamke function we mean a function \( w : I \times \mathbb{R}^+ \to \mathbb{R}^+ \) such that:

(i) \( w \) satisfies the Carathéodory conditions,

(ii) for any \( t \in I \), \( w(t, 0) = 0 \),

(iii) for any \( c \in [0, b] \), \( u \equiv 0 \) is the only absolutely continuous function on \([0, c]\) which satisfies \( w(t, u(t)) \) a.e. on \([0, c]\) and such that \( u(0) = 0 \). Let \( \mathcal{F} : I \to 2^E - \{ \emptyset \} \) be measurable and integrable bounded with weakly compact values. The set of all integrable selections of \( \mathcal{F} \), \( S_{\mathcal{F}} \), is weakly compact in the Banach space, \( L^1(I, E) \), of Lebesgue Bochner integrable functions \( f : I \to E \) endowed with the usual norm [4]. Let \( \mathcal{L}(E) \) be the algebra of all continuous, linear operators from \( E \) to \( E \). If \( \mathcal{S} : I \times I \to \mathcal{L}(E) \) is such that \( \mathcal{S}(t, 0)x_0 \) is a solution of the problem

\[
(i) \begin{cases}
\dot{x}(t) = A(t)x \\
x(0) = x_0
\end{cases}
\]

where \( \{A(t) : t \in I \} \) is a family of densely defined, closed, linear operators on \( E \).

A continuous function \( x : [-d, b] \to E \) is called an integral solution of the problem \( (P) \) if

\[
x = \mathcal{K} \text{ on } [-d, 0] \text{ and } x(t) = \mathcal{S}(t, 0)\mathcal{K}(0) + \int_0^t \mathcal{S}(t, s)f(s)ds \text{ for all } t \in I,
\]

since \( f(s) \in F(s, \theta, x) \) and \( f \in L^1(I, E) \). \( \mathcal{S}(\cdot, \cdot) \) is called a fundamental solution of \( (i) \) or the family \( \{A(t) : t \in I \} \) is a generator of \( \mathcal{S}(\cdot, \cdot) \) see [34] and [33].
3. Main Results.

**Theorem 3.1** Let $\beta$ be either a Kuratowski measure of noncompactness or a Hausdorff measure of noncompactness; $0 \leq M < \infty$ : $w$ a Kamke function and let $F$ be a multifunction from $[0, b] \times C_{E}([-d, 0])$ into the set, $P_{f_{\epsilon}}(E)$, of all nonempty closed convex subsets of $E$ such that

$(F_1)$ for each $\varepsilon > 0$, there exists a closed subset $I_{\varepsilon}$ of $I$ with $\lambda(I - I_{\varepsilon}) < \varepsilon$ such that for any nonempty bounded subset $A$ of $C_{E}([-d, 0])$ and for each closed subset $J \subseteq I_{\varepsilon}$, one has

$$\beta(F(J \times A)) \leq \sup_{t \in J} w(t, \beta(A(0)))$$

$(F_2)$ $\|F(t, K)\| \leq c(t)(1 + \|K(0)\|)$ for each $K \in C_{E}([-d, 0])$ and for some $c \in L^{1}(I, IR)$ a.e. on $I$,

$(F_3)$ $F(\cdot, K)$ has a measurable selection, for each $K \in C_{E}([-d, 0])$

$(F_4)$ for each $t \in I$, $F(t, \cdot)$ is weakly sequentially upper hemi-continuous. Further, let $\{A(t): t \in I\}$ is a generator of a fundamental solution $S : I \times I \to \mathcal{L}(E)$ such that

a) $S(t, t) = id$, $t \in I$, id is the identity function on $E$;

b) $S(t, s)S(s, r) = S(t, r)$, $t, s, r \in I$;

c) $S$ is continuous;

d) $\|S(t, s)\| \leq M$, $t, s \in I$;

e) for each $s \in I$, $S(\cdot, s)$ is uniformly continuous.

Then, for each $K \in C_{E}([-d, 0])$, problem $(P)$ has an integral solution; and the solution set of all integral solutions of $(P)$, $S(K)$, is compact.

**Proof** First we drive a priori bound for the integral solutions of problem $(P)$ on $I$. If $x$ is such a solution, then we have $x$ equal to $K$ on $[-d, 0]$ and $x(t) = S(t, 0)K(0) + \int_{0}^{t} S(t, s)f(s)ds$ for all $t \in I$ with $f(s) \in F(s, \theta, x)$ and $f \in L^{1}(I, E)$. So, for each $t \in I$,

$$\|x(t)\| \leq \|S(t, 0)\|\|K(0)\| + \int_{0}^{t} \|S(t, s)\|\|f(s)\| ds$$

$$\leq M\|K(0)\| + \int_{0}^{t} M c(t)(1 + \|x(s)\|) ds$$

$$\leq M\|K(0)\| + M\|c\| + \int_{0}^{t} M c(t)\|x(s)\| ds.$$

if $M_1 = (M\|K(0)\| + \|c\|)e^{M_1\|c\|}$, then $\|x(t)\| \leq M_1$. Put $\varphi(t) = c(t)(1 + M_1)$. So we may assume without any loss of generality $\|F_{1}(t, x(t))\| \leq \varphi(t)$ a.e. on $I$ since, otherwise, with $B_{M_1} = \{x \in E : \|x(t)\| \leq M_1\}$, we can replace $F$ by $F'$ which is defined by

$$F'(t, x(t)) = \begin{cases} F_{1}(t, x(t)) & \text{if } x \in B_{M_1} \\ F_{1}(t, \frac{M_{1}x(t)}{\|x\|}) & \text{if } x \notin B_{M_1}. \end{cases}$$
For arbitrary $n \in \mathbb{N}$, define $\Phi_1 : [-d, \frac{b}{n}] \times E \to E$ by
\[
\Phi_1(t, x) = \begin{cases} 
K(t) & \text{if } t \in [-d, 0] \\
K(0) + nt(x - K(0)) & \text{if } t \in [0, \frac{b}{n}] 
\end{cases}
\]
and define $F_1 : [0, \frac{b}{n}] \times E \to P_f(E)$ by $F_1(t, x) = F(t, \theta_{\frac{b}{n}}(\Phi_1(.,x)))$. Thus, from Lemma 2.3, for each $v \in C([-d, \frac{b}{n}], E)$ we can find at least one integrable selection $\sigma$ of $F_1(.,v(.,))$. Consequently we can define a multivalued function $G : B_{M_1} \subseteq C([-d, \frac{b}{n}], E) \to 2^{C([-d, \frac{b}{n}], E)}$ by
\[
(Gx)(t) = S(t, 0)K(0) + \int_0^t S(t, s)F(t, \theta_{\frac{b}{n}}(\Phi_1(.,x(s))))ds,
\]
for each $x \in B_{M_1}$, $Gx \neq \emptyset$. Since $S$ is continuous we can define a function $\xi : L^1([-d, \frac{b}{n}], E) \to C([-d, \frac{b}{n}], E)$ by $\xi(f)(t) = S(t, 0)K(0) + \int_0^t S(t, s)f(s)ds$. If we set $V = \{f \in L^1([-d, \frac{b}{n}], E) : ||f|| \leq \varphi(t) \text{ a.e. on } [0, \frac{b}{n}]\}$, then $V$ is uniformly integrable in $L^1([-d, \frac{b}{n}], E)$ and, since $S(., s)$ is uniformly continuous, $\xi(V) = \{x \in C([-d, \frac{b}{n}], E) : x(t) = S(0, t)K(0) + \int_0^t S(t, s)f(s)ds, f \in V\}$ is nonempty equicontinuous subset of $C([-d, \frac{b}{n}], E)$ and so, $\overline{conv}(\xi(V))$ is nonempty convex closed equibounded and equicontinuous subset of $C([-d, \frac{b}{n}], E)$.

Let $(x_m, y_m) \in \text{Graph } G$ such that $x_m \to x$, $y_m \to y$ in $C([-d, \frac{b}{n}], E)$ $y_m : I \to C([-d, \frac{b}{n}], E)$ is given by $y_m(t) = S(t, 0)K(0) + \int_0^t S(t, s)f_m(s)ds$, $f_m \in L^1([-d, \frac{b}{n}], E)$, $f_m(s) \in F_1(s, x_m(s))$ and
\[
f_m(t) = \begin{cases} 
S(t, 0)K(0) & \text{if } 0 \leq t \leq \frac{b}{m} \\
S(t, 0)K(0) + \int_0^t S(t, s)f_m(s)ds & \text{if } \frac{b}{m} \leq t \leq \frac{b}{n}.
\end{cases}
\]
Thus
\[
\lim_{m \to \infty} ||\xi(f_m) - f_m|| = \lim_{m \to \infty} \sup_{t \in [0, \frac{b}{n}]} ||\xi(f_m)(t) - f_m(t)|| 
\leq \lim_{m \to \infty} \left( \sup_{t \in [0, \frac{b}{n}]} ||\xi(f_m)(t) - f_m(t)|| + \sup_{t \in [\frac{b}{m}, \frac{b}{n}]} ||\xi(x_n)(t) - f_m(t)|| \right) 
\leq \lim_{m \to \infty} \left( \sup_{t \in [0, \frac{b}{n}]} \int_0^t ||S(t, s)f_m||ds + \sup_{t \in [\frac{b}{m}, \frac{b}{n}]} \int_0^t ||S(t, s)f_m||ds 
- \int_0^t \frac{b}{n} S(t, s)f_m ds \right) 
\leq \lim_{m \to \infty} \left( \sup_{t \in [0, \frac{b}{n}]} \int_0^t M \varphi(s) ds + \sup_{t \in [\frac{b}{m}, \frac{b}{n}]} \int_0^t ||S(t, s)f_m||ds \right) 
\leq \lim_{m \to \infty} \left( \sup_{t \in [0, \frac{b}{n}]} \int_0^t M \varphi(s) ds + \sup_{t \in [\frac{b}{m}, \frac{b}{n}]} \int_0^t \frac{b}{n} M \varphi(s) \right) = 0.
\]
Obviously the sets $H := \{f_m : m \in \mathbb{N}\}$ and $G := \{\xi(f_m) : m \in \mathbb{N}\}$ are equicontinuous. Let $\rho(t) := \beta(H(t))$, $t \in [0, \frac{b}{n}]$. Then $\rho(0) = 0$. We claim that $\rho$ is
differentiable a.e. on $[0, \frac{b}{n}]$. Since $\|f_m - \xi(f_m)\| \to 0$ as $m \to \infty$ so, from Lemma 2.1, $\beta((Id - \xi)H) = 0$ which given that

$$\beta(\{f_m : m \in \mathbb{N}\}) = \beta(\{\xi(f_m) : m \in \mathbb{N}\}).$$

Since for all $t, \tau \in [0, \frac{b}{n}],$

$$\beta\{\xi(f_m)(\tau) : m \in \mathbb{N}\} \leq \beta\{\xi(f_m)(t) : m \in \mathbb{N}\} + \beta\{\xi(f_m)(\tau) - \xi(f_m)(t) : m \in \mathbb{N}\}$$

and

$$\beta\{\xi(f_m)(t) : m \in \mathbb{N}\} \leq \beta\{\xi(f_m)(\tau) : m \in \mathbb{N}\} + \beta\{\xi(f_m)(t) - \xi(f_m)(\tau) : m \in \mathbb{N}\},$$

then $|\rho(\tau) - \rho(t)| \leq 2\beta(B(0, 1)) \int_a^b M \varphi(s) \, ds$. Therefore $\rho$ is absolutely continuous function and thus it is differentiable a.e. on $[0, \frac{b}{n}]$. Let $(t, \tau) \in [0, \frac{b}{n}] \times [0, \frac{b}{n}]$ such that $t \leq \tau$. Since $\rho$ is continuous and $w$ is Caratheodory we can find a closed subset $I_{C}$ of $[0, \frac{b}{n}]$, $\delta > 0$, $\eta > 0$ ($\eta \leq \delta$) and for $s_1, s_2 \in I_{C}$, $r_1, r_2 \in [0, \frac{b}{n}]$ such that if $|s_1 - s_2| < \delta$, $|r_1 - r_2| < \delta$, then $|w(s_1, r_1) - w(s_2, r_2)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\rho(s_1) - \rho(s_2)| < \frac{\varepsilon}{2}$. Consider the following partition, to $[t, \tau]$, $t = t_0 < t_1 < \cdots < t_\beta = \tau$ such that $t_i - t_{i-1} < \eta$ for $i = 1, \cdots, \beta$. Let $A_1 = \{x(s) : x \in H, s \in [t_{i-1}, t_i] \cap I_{C}\}$. Let $Z$ be a bounded subset of $E$ and $A = \{\theta(Z)(\Phi_{\beta}(., x)) : x \in Z\}$. Thus, for each $t \in [0, \frac{b}{n}]$, $\beta(F(t, \{t\} \times Z)) = \beta(F(t, \{t\} \times A))$. From Condition (1) we can find a closed subset $J_{\beta}$ of $[0, \frac{b}{n}]$ such that $\lambda(J \cap J_{\beta}) < \varepsilon$ and that for any compact subset $C$ of $J_{\beta}$ $\beta(F(C \times Z)) = \beta(F(C \times A)) \leq \sup_{s \in C} w(s, \beta(Z))$. Let $T_i = J_{\beta} \cap [t_{i-1}, t_i] \cap I_{C}$, $P = \sum_{i=1}^{\beta} T_i = [t, \tau] \cap I_{C} \cap I_{C}$ and $Q = [t, \tau] - P$. Thus

$$\int P F_1(s, H(s)) \, ds \subset \sum_{i=1}^{\beta} \int_{T_i} F_1(s, H(s)) \, ds \subset \sum_{i=1}^{\beta} \lambda(T_i) \text{conv} F_1(T_i \times A_1)$$

and by the mean value theorem we obtain
Now, we have
\[
\beta(\int F_1(s, H(s)) \, ds) \leq \sum_{i=1}^{m} \lambda(T_i) \beta(F_1(T_i \times A_i)) \\
\leq \sum_{i=1}^{m} \lambda(T_i) \sup_{s_i \in T_i} w(s_i, \beta(A_i)) \\
= \sum_{i=1}^{m} \lambda(T_i) w(q_i, \rho(p_i)); \quad (q_i, p_i \in T_i) \\
\leq \sum_{i=1}^{m} \int_{T_i} w(s, \rho(s)) \, ds + \varepsilon \lambda(T_i) \\
= \int_{P} w(s, \rho(s)) \, ds + \varepsilon \lambda(P) \\
\leq \int_{\tau} \tau w(s, \rho(s)) \, ds + \varepsilon (\tau - t).
\]
Moreover, we get \( \beta(\int Q F_1(s, p_H(s)) \, ds) \leq 2 \int_Q \varphi(s) \, ds \). As \( \lambda(Q) < 2 \varepsilon \) and since \( \varepsilon \) is arbitrary, then

\[
\beta(\int_{t}^{\tau} F_1(s, H(s)) \, ds) \leq \int_{t}^{\tau} w(s, \rho(s)) \, ds.
\]

On the other hand, we have

\[
\beta(\xi(H)(\tau)) \leq \beta(\xi(H)(t)) + \beta(\int_{t}^{\tau} F_1(s, H(s)) \, ds).
\]

By relations (1) and (2) we get
\[
\rho(\tau) - \rho(t) \leq \beta(\int_{t}^{\tau} F_1(s, H(s)) \, ds) \leq \int_{t}^{\tau} w(s, \rho(s)) \, ds.
\]

Therefore \( \dot{\rho}(t) \leq w(t, \rho(t)) \) a.e. on \([0, \frac{b}{n}]\). Since \( \rho(0) = 0 \) and \( w \) is a Kamke function, then \( \rho \equiv 0 \). Thus the weak closure of \((f_m)_{m \in \mathcal{K}} \) is weakly compact and so we can suppose that the sequence \((f_m)_{m \in \mathcal{K}} \) converges to a continuous function \( x_1 \) such that \( x_1 = K \) on \([-d, 0]\) and for each \( t \in [0, \frac{b}{n}] \)

\[
x_1(t) = S(t, 0)K(0) + \int_{0}^{t} S(t, s)l_1(s) \, ds
\]

where \( l_1(s) \in F(s, \theta_{\frac{b}{n}}(\Phi(., x_1(t)))) \) a.e. on \([0, \frac{b}{n}]\).

Now, by the mathematical induction for some \( k \in \{2, 3, \ldots, n\} \), we can assume that there exists the function \( x_{k-1} \) such that \( x_{k-1} = K \) on \([-d, 0]\) and for each \( t \in [0, \frac{(k-1)b}{n}] \)

\[
x_{k-1}(t) = S(t, 0)K(0) + \int_{0}^{t} S(t, s)l_{k-1}(s) \, ds.
\]
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\[ l_{k-1}(s) \in F(t, \theta_{\frac{(k-1)b}{n}} \Phi_{k-1}(., x_{k-1}(s))) \text{ a.e. and } l_{k-1} \in L^1([0, \frac{(k-1)b}{n}], E) \]

also let \( \Phi_k : [-d, \frac{kb}{n}] \times E \to E \) be such that

\[
\Phi_k(t, x) = \begin{cases} 
    x_{k-1}(t) & \text{if } t \notin [-d, \frac{(k-1)b}{n}], \\
    x_{k-1}(\frac{(k-1)b}{n}) + n(t - \frac{(k-1)b}{n})(x - x_{k-1}(\frac{(k-1)b}{n})) & \text{if } t \in [\frac{(k-1)b}{n}, \frac{kb}{n}]. 
\end{cases}
\]

Arguing as in above, for the multifunction \( F_k : [\frac{(k-1)b}{n}, \frac{kb}{n}] \times E \to P_{fc}(E) \) which is defined by \( F_k(t, x) = F(t, \theta_{\frac{k}{n}}(\Phi_k(., x))) \), we have a continuous function \( x_k \) defined on \( [\frac{(k-1)b}{n}, \frac{kb}{n}] \) by

\[
x_k(t) = S(t, \frac{(k-1)b}{n})x_{k-1}(\frac{(k-1)b}{n}) + \int_{\frac{(k-1)b}{n}}^{t} S(t, s)l_k(s)ds 
\]

where \( l_k : [s, \frac{kb}{n}] \times E \to E \) is defined by

\[
l_k(s) = S(s,0)K(0) + \int_{0}^{s} S(s, t)h_n(t)dt,
\]

with \( h_n(t) \in F(t, \theta_{\frac{k}{n}} \Phi_{k}(., v_{n}(t))) \text{ a.e. on } I \).
where \( t \in \left[\frac{(k-1)b}{n}, \frac{kb}{n}\right] \subset I \), for \( k \in \{1, 2, 3, \ldots n\} \). Now we claim that the set \( L = \{v_n : n \in \mathbb{N}\} \) is an equicontinuous set. So let \( t_1, t_2 \in I \) with \( t_1 < t_2 \). Then

\[
\begin{align*}
&\int_0^{t_1} \left\| v_n(t_1) - v_n(t_2) \right\| \leq \left\| S(t_1, 0) - S(t_2, 0) \right\| \left\| K(0) \right\| \\
&+ \int_{t_1}^{t_2} \left( \int_0^{t_1} \left\| S(t_1, s) - S(t_2, s) \right\| h_n(s) \right) \, ds + \int_{t_1}^{t_2} \left\| S(t_2, s) \right\| h_n(s) \, ds \\
&\leq \left\| S(t_1, 0) - S(t_2, 0) \right\| \left\| K(0) \right\|
\end{align*}
\]

and, since \( v_n = K \) on \([-d, 0]\), this shows that \( L \) is equicontinuous in \( C_E[-d, b] \).

Moreover the set \( \beta(L(t)) = \beta(\{v_n(t) : n \in \mathbb{N}\}) \) is such that \( \beta(L(0)) = 0 \) and, by the same as above, we get \( \beta(L(t)) = 0 \) for all \( t \in I \). Thus by Ascoli’s theorem we may the sequence \( \{v_n : n \in \mathbb{N}\} \) converges uniformly to a function \( v \in C_E([-d, b]) \) such that \( y = K \) on \([-d, 0]\).

Thus by Eberlein-Šmulian Theorem there exists a subsequence \( (h_{n_k}) \) of \( (h_n) \) such that \( h_{n_k} \to l \) weakly, \( l \in \delta_F \).

Thus \( v_n \) tends weakly to \( S(t, 0)K(0) + \int_0^t S(t, s)l(s)ds \).

Moreover since, for each \( n \in \mathbb{N} \), \( v_n \in C_E([-d, b]) \), \( v_n \) converges uniformly to \( v \) on each compact subset of \([-d, b]\) and \( v \) is uniformly continuous on \([-d, 0]\); also for each \( t \in I \), there exists \( n > \frac{b}{d} \) with \( t \in \left[\frac{(k-1)b}{n}, \frac{kb}{n}\right] \) for \( k \in \{1, 2, \ldots n-1\} \) so, as \( k' = k - 1 \),

\[
\begin{align*}
&\left\| v_n(t) - v(t) \right\|
\end{align*}
\]

Thus, from Lemma 2.2, we conclude that the solution set, \( S(K) \), of integral solutions of \((P)\) is nonempty. Next if \( \{v_n : n \in \mathbb{N}\} \) is a sequence of \( S(K) \), then argu-
ing as in the proof above we can show that, for each \( t \in I \), \( \beta(\{v_n(t) : n \in \mathbb{N}\}) = 0 \). Thus this sequence has a convergent subsequence, so \( S(K) \) is compact.

If we replace in Theorem 3.1 \( \beta \) by a measure of weak noncompactness, then for each \( K \in C_E([-d,0]) \) the solution set of all integral solutions of Problem \((P)\), \( S(K) \), is nonempty weakly compact subset of \( C([-d,b],E) \). Moreover we can define the multifunction \( S : C_E([-d,0]) \to 2^{C([-d,b],E)} \) such that, for each \( K \in C_E([-d,0]), S(K) \) is the solution set of problem \((P)\).

In the following theorem we assume:

\[ C(H) \{H_n : n \in \mathbb{N}\} \text{ be a sequence of multifunctions from } I \times C_E([-d,0]) \text{ into the set, } P_I(E), \text{ of nonempty closed convex subsets of } E \text{ such that} \]

1. \( H(t,K) = \bigcap_{n=1}^{\infty} H_n(t,K) \),
2. \( H_{n+1}(t,K) \subseteq H_n(t,K) \), for all \( n \in \mathbb{N} \),
3. if \( h \) is the Hausdorff distance, then \( \lim_{n \to \infty} h(H_n(t,K),H(t,K)) = 0 \),
4. for some \( C > 0 \) \( \|H_n(t,K)\| \leq C \),
5. for \( n \in \mathbb{N} \), \( H_n \) satisfies conditions \( F_1,F_2 \) of Theorem 3.1.

**Theorem 3.2** If \( E \) is a separable Banach space, \( H(t,.) \) is weakly sequentially upper semi-continuous, \( H(.,K) \) has a measurable selection and hypotheses \( C(H) \) hold, then for each \( K \in C_E([-d,0]) \) \( S_E(K) = \bigcap_{n=1}^{\infty} S_{H_n}(K) \).

**Proof** Thanks to our assumptions, we obtain the solution set \( S_E(K) \) is nonempty and for any \( n \in \mathbb{N} \), \( S_E(K) \subseteq S_{H_n}(K) \), so \( S_E(K) \subseteq \bigcap_{n=1}^{\infty} S_{H_n}(K) \). Conversely, let \( v \in \bigcap_{n=1}^{\infty} S_{H_n}(K) \). Thus there exists \( h_n \) such that, for each \( t \in I \), \( h_n(t) \in H_n(t,\theta_tv) \) and \( v(t) = S(t,0)K(0) + \int_0^t S(t,s)h_n(s)ds \). From condition \( C(H)(3) \) we have \( h_n(t) \in H(t,\theta_tv) + \varepsilon_n(t)B_1 \) a.e. on \( I \), where \( \varepsilon_n(t) = h(H_n(t,\theta_tv),H(t,\theta_tv)) \to 0 \) as \( n \to \infty \) and \( B_1 \) is the closed unit ball in \( E \). By condition \( C(H)(4) \), the sequence \( \{h_n : n \in \mathbb{N}\} \) is uniformly bounded. We consider a subsequence \( \{h_{n_k}(t) : n_k \in \mathbb{N}\} \) and we can passing to convex combination of \( h_{n_k}(t) \) denoted by \( h_{n_k}(t) \). Thus \( h_{n_k}(t) \to l(t) \in E \) and moreover \( h_{n_k}(t) \in \sum_{m>n} \gamma_m H(t,\theta_tv) + \varepsilon_m(t)B_1 \) a.e. on \( I \), where \( \gamma_m(t) \geq 0 \) and also \( \sum_{m>n} \gamma_m = 1 \). At this point, we let \( n_k \to \infty \) and since \( H \) has convex values, so \( l(t) \in H(t,\theta_tv) \) and hence the result.

**Theorem 3.3** The multifunction \( S \) is upper semicontinuous and both the multifunctions \( S_I : C_E([-d,0]) \to 2^E \), defined by \( S_I(K) = \{v(t) : v \in S(K)\} \) and that \( S_K : I \to 2^E \) which is defined by \( S_K(t) = \{v(t) : v \in S(K)\} \) is upper semicontinuous and has compact values. Further, the set \( \bigcup_{t \in I} S_K(t) \) is compact in \( E \).

**Proof** For each closed subset \( Z \) of \( C_E([-d,b]) \), to show that \( S \) is upper semicontinuous, we claim that \( A = \{K \in C_E([-d,0]) : S_K \cap Z \neq \emptyset \} \) is sequentially closed in \( C_E([-d,0]) \). Let \( \{K_n : n \in \mathbb{N}\} \subseteq A \) such that \( K_n \to K \). Then \( S_{K_n} \cap Z \neq \emptyset \) and hence there exists \( v_n \in S_{K_n} \cap Z \), where \( v_n(t) = S(t,0)K_n(0) + \int_0^t S(t,s)g_n(s)ds \) with \( g_n(s) \in F(s,\theta_tv_n) \) a.e. on \( I \) and \( g_n(.) \in L^1(I,E) \). Now, for each \( t \in I \), we have

\[
\beta(\{v_n(t) : n \in \mathbb{N}\}) \leq M\beta(\{K_n(0) : n \in \mathbb{N}\}) + M\beta(\int_0^t g_n(s)ds : n \in \mathbb{N}\)).
\]
But $\beta(\{K_n(0) : n \in \mathbb{N}\}) = 0$, where $K_n \to K$. Thus

$$
\beta(\{v_n(t) : n \in \mathbb{N}\}) \leq M \beta(\int_0^t g_n(s)ds : n \in \mathbb{N}).
$$

Arguing as in the proof of Theorem 3.1 we have $\beta(\{v_n(t) : n \in \mathbb{N}\}) = 0$. Now since the sequence $\{v_n(t) : n \in \mathbb{N}\}$ is equicontinuous, so from Arzela-Ascoli theorem we can find a subsequence $(v_{n_k})$ converges to $v_0$ in $C_E([-d, b])$. Let $v_{n_k}(t) = S(t, 0)K_{n_k}(0) + \int_0^t S(t, s)g_{n_k}(s)ds$, where $g_{n_k}(s) \in F(s, \theta, v_{n_k})$ a.e. on $I$ and $g_{n_k}(\cdot) \in L^1(I, E)$. Then we can write $g_{n_k} = K$ on $[-d, 0]$ and

$$
g_{n_k}(t) = \begin{cases} S(t, 0)K(0) & \text{if } 0 \leq t \leq \frac{b}{n_k} \\ S(t, 0)K(0) + \int_0^{t - \frac{b}{n_k}} S(t, s)g_{n_k}(s)ds & \text{if } \frac{b}{n_k} \leq t \leq b. \end{cases}
$$

As in the proof of Theorem 3.1 we obtain $\beta(\{g_{n_k}(t) : n_k \in \mathbb{N}\}) = 0$ for $t \in I$, so $g_{n_k} \to g_0 \in L^1(I, E)$ and from Lemma 2.2 $g_0(t) \in F(t, \theta, v_0)$. Thus

$$
v_0(t) = S(t, 0)K(0) + \int_0^t S(t, s)g_0(s)ds
$$

and consequently $A = \{K \in C_E([-d, 0]) : S_K \cap Z \neq \emptyset\}$ is sequentially closed in $C_E([-d, 0])$ thus $S$ is upper semicontinuous. Further, by the same arguments we can show that $P = \{K \in C_E([-d, 0]) : S_I(K) \cap Z \neq \emptyset\}$ is closed so, $S_I(K)$ is upper semicontinuous. Since $S(K)$ is compact, then both $S_K$ and $S_I$ has compact values. Lastly the set $Q = \{t \in I : S_K(t) \cap Z\}$ is closed, then from Berge’s Theorem [4] $\bigcup_{t \in I} S_K(t)$ is compact in $E$.

Now we consider the following control problem

$$
(Q) \begin{cases} \dot{x}(t) \in A(t)x(t) + F(t, \theta, x) \\ x = K \in Z \\ \text{minimise } \omega(x(b)) \end{cases}
$$

where $Z$ is a compact subset of $C_E([-d, 0])$ and $\omega : E \to \mathbb{R}$ is lower semicontinuous. We say that Problem $(P)$ has an optimal solution if there exist $K_0 \in Z$ and $v \in S(K_0)$ such that $\omega(v(b)) = \inf\{\omega(x(b)) : x \in S(K_0)\}$.

**Theorem 3.4** Under the assumptions of Theorem 3.1, Problem $(Q)$ has an optimal solution.

**Proof** If $K_0 \in Z \subseteq C_E([-d, 0])$, then there exists a continuous function $v \in S(K_0)$ and so, $v(b) \in S_0(K_0)$. But $S_I$ is upper semicontinuous and has compact values thus, from Berge’s Theorem [4], $S_I(Z)$ is compact and so, $\omega$ has its minimum $b_0$ on $S_0(Z)$. Thus there exists $K_1 \in Z$ such that $v_0 \in S_I(K_1)$, where $\omega(v_0) = b_0$ and $v_0 \in S_0(Z)$, thus $v_0 \in S_K(b)$ which means that $v_0 = v(b)$ for some $v \in S(K_1)$. Therefore $\omega(v(b)) = \inf\{\omega(x(b)) : x \in S(K_1)\}$.
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