On minimal projections generated by isometries of Banach spaces

Let $B$ be a real Banach space. Let $A$ be a linear isometry in $B$ which has a fixed point. The present paper is inspired on a fact that is known to some degree but not precisely discussed in any source, i.e., if the space $B$ is represented as a direct (topological) sum of $\text{Im}(I - A)$ and $\text{Ker}(I - A)$ [where $I$ is the identity map of $B$], then the projection $P$ onto $\text{Im}(I - A)$ along $\text{Ker}(I - A)$ is the minimal projection (cf. below, Theorem 1.1).

In connection with this fact two problems arise. The first problem concerns a possibility of the representation of a space $B$ as a direct topological sum of $\text{Im}(I - A)$ and $\text{Ker}(I - A)$.

In Section 1 we give necessary and sufficient conditions of this representation for any linear operation in $B$ (cf. [8], [9]).

The second problem concerns the estimation of a norm of a minimal projection $\hat{P}$. The answer is given in Section 2 with the help of the concave function $g$ (which has no name for the present) and a Chebyshev radius, which were before used by the estimation of a norm of a minimal projection onto hyperspaces [6].

As the examples showing the obtained results we take the vector-valued Orlicz space $B = L_{M_1}([0, 1]; L_{M_2}(0, 1))$, the Banach space with symmetric norm (in particular, $l_1, c_0$) and the space $C(S^1)$ of all continuous functions on the circle $S^1$.

Note that in the last two examples, our results can also be obtained in a more complicated way with the help of the theory of operators acting on compact topological groups (cf. for example [11], [12]).

1. **A condition of the representation of $B$ as $B = \text{Im}(I - A) \oplus \text{Ker}(I - A)$**. As usually the Banach space $B$ is called (topological) direct sum of the
subspaces $D$ and $K$ if each $x \in B$ can be uniquely expressed as $x = y + z$, where $y \in D$, $z \in K$, and the linear operator $P : B \to D$, $P(x) = y$ is bounded. We shall write $B = D \oplus K$. The relative projection constant of a complemented subspace $D$ in a Banach space $B$ is the number $\lambda(B, D) = \inf \{ ||P|| : P \text{ projects } B \text{ onto } D \}$.

Let $A$ be a linear bounded map: $B \to B$. Let $B^A = \text{Ker}(I - A)$, $B_A = \text{Im}(I - A)$, let $\theta$ be the zero element in $B$, let $N$ be the set of natural numbers. If a set $D \subset B$ and $f$ is a map on $B$, then by $f|_D$ we denote the restriction of $f$ to the set $D$.

**Theorem 1.1.** Let $A$ be a linear isometric operator of the Banach space $B$ onto itself, and $B = \text{Im}(I - A) \oplus \text{Ker}(I - A)$. Let $\tilde{P}$ be the projection from $B$ onto $\text{Im}(I - A)$ annihilated on $\text{Ker}(I - A)$. Then $\tilde{P}$ is a minimal projection and it can be defined by

$$\tilde{P} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A^{-k} PA^k,$$

where $P$ is a projection from $B$ onto $\text{Im}(I - A)$; $A^0 = I$.

**Proof.** Let $P$ be a projection on subspace $B_A$. Let $\tilde{P}$ be the map defined by (1).

We shall show that $\tilde{P}$ is defined correctly for each $x \in B$ and it is a projection from $B$ onto $B_A$ along $B^A$. Indeed, if $z \in B_A$, then there exists a $x \in B$ such that $z = x - Ax$. Then $A^k(z) = (I - A)(A^k z) \in B_A$ for all $k \in N$. Hence $P(A^k(z)) = A^k(z)$ for all $k \in N$ and $\tilde{P}(z) = z$.

If $z \in B^A$, then $A^k(z) = z$ for all $k \in N$. Since $P(z) \in B_A$ and $P(z) = x' - Ax'$ for a $x' \in B$, we have

$$\tilde{P}(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A^{-k}(x' - Ax') = \lim_{n \to \infty} \frac{1}{n}(A^{-n-1}(x') - A(x')) = 0,$$

because $||A^{-1}|| = 1$.

By triangle inequality, we obtain $||\tilde{P}|| \leq ||P||$. Therefore, by linearity and boundedness of the map $\tilde{P}$ defined by (1), $\tilde{P}$ is the minimal projection from $B$ onto $B_A$ along $B^A$.

**Remark 1.1.** The proof of this theorem is essentially a proof of some ergodic statistic theorem (cf. [5]). Note that some ergodic statistic theorems were used in fact before, in connection with the investigation of a minimal projection (cf. [1], [13], [16]).

If $\text{Im}(I - A)$ is a reflexive subspace, then the existence of minimal projection follows from the Isbell-Semadeni results ([10]), i.e., if a comple-
mented subspace $D$ of a Banach space $B$ is isometrically isomorphic to a space $Z^*$, then there exists a minimal projection from $B$ onto $D$.

For using of Theorem 1.1 we ought to have a condition for a decomposition of $B$ as a direct sum $B = B_A \oplus B^A$. The next proposition gives the conditions of this decomposition (if $A$ is a linear continuous map) in terms of a convergence of averaging operators $A(n)$:

$$A(n) = \frac{1}{n} \sum_{k=0}^{n-1} A^k \quad (n = 1, 2, \ldots), \quad A^0 = I.$$  

**Example 1.1 (Fürstenberg [7]).** Let $C(S^1)$ be the space of all continuous functions on the circle $S^1$. Then for every linear isometric surjective operator $A: C(S^1) \to C(S^1)$ and for every $x \in S^1$ there exists $\lim_{n \to \infty} (A(n))x$, where $A(n)$ is defined by (2).

**Proposition 1.1.** Suppose that a linear continuous operator $A$ in a Banach space $B$ is such that $B_A$ is closed and

$$\lim_{n \to \infty} \frac{||A^n||}{n} = 0.$$  

In order that $B$ be a direct sum $B_A \oplus B^A$ it is necessary and sufficient that

$$\lim_{n \to \infty} (A(n))(x) \quad \text{for any} \ x \in B.$$  

**Proof.** Necessity. Notice that if $x \in B^A$, then $(A(n))(x) = x$, and therefore $\lim_{n \to \infty} (A(n))(x) = x$. If $x \in B_A$, then there exists $y \in B$ such that $x = y - Ay$, i.e.,

$$\lim_{n \to \infty} (A(n))(x) = (1/n)(y - A^n(y)).$$  

Hence, by condition (3), $\lim_{n \to \infty} (A(n))x = \theta$. Thus, for each $x \in B$, (4) is true.

The proof of sufficiency follows directly from [5] (Chapter VIII, § 5.2).

**Remark 1.2.** (a) If $A$ is an isometry, i.e., a linear isometric operator in $B$, then from the proof of necessity we have $B_A \cap B^A = \{\theta\}$. In general case the last equation is not true. For example, if $B = \text{span} \{e^x, xe^x\} \subset C([0, 1])$ and $A$ is the differentiation operator in $B$, then $B_A = B_A^A = \text{span} \{e^x\}$.

(b) If $A$ is an isometry of $B$ onto itself and $B$ is a reflexive space, then $B = \overline{B_A} \oplus B^A$ (cf. [5]), where $\overline{B_A}$ is the closure of $B_A = (I - A)B$.

(c) For the convergence of the sequence $(A(n))_1^\infty$ it is not sufficient that $A$ can be an isometry of $B$ onto itself. Indeed, let $\mathcal{A} = (\alpha_{ij})$, $1 \leq j, i \leq 2$, be a matrix of order two, $\alpha_{ij} \in \mathbb{N}$, $1 \leq i, j \leq 2$, det $\mathcal{A} = 1$, $|\text{tr } \mathcal{A}| > 2$. (For example,

$$\mathcal{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$
Let $T$ be a standard automorphism of a torus $V = S^1 \times S^1$ corresponding to $\mathfrak{H}$, i.e., for a point $v = (z, w) \in V$, $z = e^{2\pi i x}$, $w = e^{2\pi i y}$, $x, y \in \mathbb{R}$, we have

$$Tv = (e^{2\pi i (a_1 x + a_2 y)}, e^{2\pi i (a_3 x + a_4 y)}).$$

Let $A$ be an isometry of the space $C(V)$ of all real continuous functions on $V$, generated by $T$ (i.e., $A\varphi(v) = \varphi(Tv)$, where $\varphi \in C(V)$, $v \in V$). Then by a result of H. Fürstenberg from [7] (Theorem 3.3) in $C(V)$ there exists a function $f$ for which the sequence $\{(A(n)f)_1\}^\infty_{n=1}$ is not convergent, though $A$ is a surjection. By Proposition 1.1,

$$C(V) \neq (C(V))_A \oplus (C(V))^A.$$ 

**Proposition 1.2.** Let $A : B \to B$ be a linear continuous operator in $B$. Then the following statements are equivalent:

(i) $B = B_A \oplus B^A$;

(ii) $B_A$ is a subspace of $B$ and $(I - A)|_{B_A}$ is a one-to-one operator of $B_A$ onto $B_A$;

(iii) for each $x \in B$ there exists $x' \in B$ such that $(I - A)x = (I - A)^2 x'$ and for each sequence $(x_k)_1^\infty$ such that $\lim_{k \to \infty} (I - A)^2 x_k = \theta$, there holds $\lim_{k \to \infty} (I - A)x_k = \theta$.

**Proof.** (ii) $\Rightarrow$ (i). We shall prove that for every element $x \in B$ there exists $y \in B^A$ and $z \in B_A$ such that $x = y + z$.

Let $x \in B$. Since $(I - A)|_{B_A}$ is a surjective map onto $B_A$, there exists $z \in B_A$ such that $x - Ax = z - Az$. Let $y = x - z$; then $y - Ay = (x - Ax) - (z - Az) = \theta$. Hence $y \in B^A$. Next, assume that $y = y' + z'$, where $y, y' \in B^A$, $z, z' \in B_A$. Then $z - z' = y - y' \in B^A$. Hence, $(I - A)(z - z') = \theta$. Since the operator $(I - A)|_{B_A}$ is one-to-one, we have $z = z'$ and $y = y'$.

Next note that by the Banach Open Mapping Principle (cf. [2], p. 33) the operator $((I - A)|_{B_A})^{-1} : B_A \to B_A$ is a continuous linear operator. Therefore for each sequence $(z_k)_1^\infty \subset B_A$ such that $(I - A)z_k \to \theta$, we get $\lim_{k \to \infty} z_k = \theta$.

We define here the operator $P : B \to B_A$ by the formula $Px = z$, where $z \in B_A$ is such that $x = y + z$ ($y \in B^A$).

Next, we prove that $P$ is a continuous operator. Indeed, let $(x_k)_1^\infty$ be a sequence such that $x_k \to \theta$, and $(y_k)_1^\infty, (z_k)_1^\infty$ are such that $x_k = y_k + z_k$ ($y_k \in B^A$, $z_k \equiv B_A$, $k = 1, 2, \ldots$). Then

$$||(I - A)x_k|| \leq \left((1 + ||A||) ||x_k||\right) \to 0.$$ 

Hence,

$$(I - A)x_k \to \theta.$$
By

\[(I - A) x_k = y_k + z_k - y_k -Az = (I - A)z_k,\]

we obtain as above \(z_k = P(x_k) \rightarrow \theta.\)

Since \(P\) is evidently a linear operator, \(P\) is a projection from \(B\) onto \(B_A.\)

(i) \Rightarrow (iii). Let \(x \in B, x = y + z,\) where \(y \in B^A, z \in B_A.\) Then \(z = (I - A)x',\)
where \(x' \in B.\) Hence \((I - A)x = (I - A)(y + z) = (I - A)z = (I - A)^2 x'.\) Next it is easy to verify that \((I - A)_{B_A}\) is a one-to-one surjective operator. Finally, we note that by the Banach Open Mapping Principle the operator \((I - A)_{B_A}^{-1}\) is a continuous linear operator on \(B_A.\)

The implication (iii) \Rightarrow (ii) is obvious.

Remark 1.3. Let \(A: B \rightarrow B\) be a linear continuous operator. It is easy to see that

(a) If \(A(B) = B\) and \(B = B_A \oplus B^A,\) then \(A(B_A) = B_A\) and \(B^{-1} = B^A,\)
\(B_A^{-1} = B_A,\) where \(A^{-1}\) is the inverse operator to \(A.\)

(b) If \(\dim B_A < \infty,\) then \(B = B_A \oplus B^A \iff B_A \cap B^A = \{\theta\}.\)

Example 1.2. Let \(M_i (i = 1, 2)\) be two Orlicz functions: \([0, +\infty) \rightarrow [0, +\infty),\) i.e., continuous convex non-decreasing functions with \(M_i (0) = 0\) and \(M_i \neq 0.\) Let \(L_{M_2} (0, 1)\) be an Orlicz space of equivalence classes of such measurable functions \(h: [0, 1] \rightarrow (-\infty, +\infty)\) for which

\[
\|h\| = \inf \{t > 0: \int_0^1 M_2(|h(x)|/t) dx \leq 1\} < +\infty.
\]

Let \(B = L_{M_1} ([0, 1]; L_{M_2} (0, 1))\) be the Orlicz space of the equivalence classes of strongly measurable functions \(f: [0, 1] \rightarrow L_{M_2} (0, 1)\) for which

\[
\|f\|_1 = \inf \{t > 0: \int_0^1 M_1(|f(x)|/t) dx \leq 1\} < +\infty.
\]

Now, we shall define for every \(n \in \mathbb{N}\) a map \(\tau_n: [0, 1] \rightarrow [0, 1]\) as follows: if \(n = 1,\) then \(\tau_1(x) = x\) for every \(x \in [0, 1];\) if \(n \geq 2,\) then

\[
\tau_n(x) = \begin{cases} 
  x + \frac{1}{n} & \text{if } x \in \left(0, \frac{n - 1}{n}\right); \\
  x - \frac{n - 1}{n} & \text{if } x \in \left[\frac{n - 1}{n}, 1\right]; \\
  1 & \text{if } x = 0.
\end{cases}
\]

An operator \(Q(x): L_{M_2} (0, 1) \rightarrow L_{M_2} (0, 1)\) defined for every \(x \in (0, 1]\) by
\(Q(x)h = h(\tau_{[1/x]}(y))\) (where \(h \in L_{M_2} (0, 1),\) \([1/x]\) is the greatest integer of the number \(1/x,\) \(y \in [0, 1]\), and \(Q(0)h = h,\) for \(x = 0,\) is a linear isometry.
Then the operator \( A : B \to B \), such that \( Af(x) = Q(x)(f(\tau_2(x))) \) for each \( f \in B \), \( x \in [0, 1] \) is a linear isometry of \( B \) onto itself.

\( B^A \) will be a subspace of all functions \( f \in B \), satisfying for every \( n \in N \) the property:

\[
(f(x))(\tau_n(y)) = (f(\tau_2(x)))(y) \quad \text{for } \mu\text{-almost every } x \in \left( \frac{1}{n+1}, \frac{1}{n} \right)
\]

and \( y \in [0, 1] \) relative to Lebesgue measure \( \mu \).

\( B_A \) will be a subspace of all functions \( f \in B \) satisfying for every \( n \in N \setminus \{1\} \) the property:

\[
\sum_{k=0}^{n-1} (f(x) + f(x + \frac{1}{n}))(y + k/n) = 0
\]

for \( \mu\text{-almost every } x \in (1/(n+1), 1/n] \) and \( y \in [0, 1/n] \) relative to Lebesgue measure \( \mu \).

It is easy to verify that the subspaces \( B^A \) and \( B_A \) are infinite-dimensional and that the operator \( A \) satisfies condition (iii) from Proposition 1.2.

Now, let \( P \) be a projection from \( B \) onto \( B_A \) which can be defined in the following manner: \( Pf(x) = f(x) \) if \( x \in [0, 1] \); if \( n \geq 2 \) and \( x - \frac{1}{n} \in (1/(n+1), 1/n] \), then

\[
(Pf(x))y = \begin{cases} 
(f(x))y & \text{if } y \in [0, \frac{n-1}{n}], \\
- \sum_{k=0}^{n-1} (f(x - \frac{1}{n}) + f(x))(y - \frac{k}{n}) & \text{if } y \in \left[ \frac{n-1}{n}, 1 \right].
\end{cases}
\]

By Theorem 1.1 and Proposition 1.2 the projection \( \tilde{P} : B \to B_A \) defined by (1) is a minimal one. By virtue of results of Section 2, \( \|P\| \leq 2 \).

2. Evaluation of norm of minimal projection, generated by isometry. In this section, \( S_X \) (resp. \( U_X \)) will denote the unit sphere (resp. the unit ball) of a real Banach space \( X \).

Let \( D, K \) be subspaces of a Banach space \( B \) such that \( B = D \oplus K \). Let \( x \in S_K \) and \( D_x = D \oplus \text{span} \{x\} \). Let \( f \in S_{D^*_x} \) be such that \( f^{-1}(0) = D \).

For every \( a \in [0, 1] \), let

\[
W^x_a = f^{-1}(a), \quad C^x_a = U_{D_x} \cap W^x_a, \quad \varrho^x_D(a) = \inf_{z \in W^x_a, y \in C^x_a} ||z - y||,
\]

\( C^x_a \) and \( \varrho^x_D(a) \) will be called, respectively, the hypercircle in \( D \) and the Chebyshev radius of \( C^x_a \) (cf. [6]).

Next write \( C^x_x \) for \( \sup_{a \in (0, 1)} \varrho^x_D(a) \) and \( C^x_K \) for \( \sup_{x \in S_K} C^x_x \). Consider now the
function \( g : [1, 2] \to [1, 2] \) (cf. [6]) defined as

\[
g(t) = \begin{cases} 
1 + \frac{1}{2}((t-1) + \sqrt{(t-1)^2 + 8(t-1)}) & \text{if } 1 \leq t \leq \sqrt{17} - 3, \\
1 + \frac{8(t-1)}{t^2 + 4(t-1)} & \text{if } \sqrt{17} - 3 < t \leq 2.
\end{cases}
\]

Note that \( g \) is strictly increasing and concave. Moreover, \( g(1) = 1, g(2) = 2 \), \( g(t) \geq t \) for each \( t \in [1, 2] \). In terms of the function \( g \) and the number \( C_B^\delta \) we can evaluate the norm of the minimal projection \( \tilde{P} \) defined by (1).

**Theorem 2.1.** Let \( A \) be a linear isometry of Banach space \( B \) onto itself such that \( B = \text{Im}(I-A) \oplus \text{Ker}(I-A) \). Let \( D = \text{Im}(I-A), K = \text{Ker}(I-A) \). Let \( \tilde{P} \) be a projection from \( B \) onto \( D \) along \( K \). Then

\[
1 \leq C_B^\delta \leq \lambda(D_x, D) = ||\tilde{P}|| \leq g(C_B^k) \leq 2.
\]

**Proof.** Let \( x \in S_K \) and \( D_x = D \oplus \text{span}\{x\} \). By Remark 1.3 the operator \( A_x = A|_{D_x} \) is an isometry of \( D_x \) onto itself with \( \text{Im}(I-A) = D, \text{Ker}(I-A) = \text{span}\{x\} \). By Theorem 1.1, the projection \( P^\delta_x : D_x \to D \) along \( \text{span}\{x\} \) is a minimal projection, i.e., \( ||P^\delta_x|| = \lambda(D_x, D) \). In view of the fact that \( \text{codim}_{D_x} D = 1 \) and by a result of C. Franchetty ([6], Theorem 3) we have

\[ 1 \leq C_B^\delta \leq \lambda(D_x, D) \leq g(C_B^k) \leq 2. \]

Observe also that in view of the inequality \( \lambda(D_x, D) \leq \lambda(B, D) \), we obtain \( 1 \leq C_B^\delta \leq \lambda(B, D) \). Next we use the fact that the projection \( \tilde{P} \) is minimal and \( ||\tilde{P}|| = \lambda(B, D) \) (cf. Theorem 1.1).

If \( ||\tilde{P}|| = 1 \), then the theorem follows from the identity \( g(1) = 1 \).

Next assume that \( ||\tilde{P}|| > 1 \). Then for any \( \varepsilon > 0 \) with \( \varepsilon < ||\tilde{P}|| - 1 \), there exists \( x_0 \in S_K \) such that \( ||\tilde{P}(x_0)|| + \varepsilon > ||\tilde{P}|| \). Let \( y_D \) and \( y_K \) be such that \( x_0 = y_D + y_K \) with \( y_D \in D \) and \( y_K \in K \). Evidently, \( y_K \neq \theta \).

Let \( z = y_K/||y_K|| \). Then \( ||\tilde{P}(x_0)|| = ||P^\delta_k(x_0)|| \leq ||P^\delta_k|| = \lambda(D_z, D) \), where \( P^\delta_k \) is the projection from \( D_z = D \oplus \text{span}\{z\} \) onto \( D \) along \( \text{span}\{z\} \). Clearly, \( \lambda(B, D) < \lambda(D_z, D) + \varepsilon \). Since the function \( g \) is strictly increasing, we get

\[
\lambda(B, D) \leq \sup_{x \in S_K} \lambda(D_x, D) \leq \sup_{x \in S_K} g(C_B^\delta) \leq g\left(\sup_{x \in S_K} C_B^\delta\right) = g\left(\sup_{x \in S_K} C_B^k\right) \leq 2.
\]

By Theorem 2.1 and a result of Franchetty in [6] (Theorem 4), taking into account the form of function \( g \), we get directly:

**Corollary 2.1.** Let \( A \) be a linear isometry of a Banach space \( B \) onto itself and \( B = \text{Im}(I-A) \oplus \text{Ker}(I-A) \). Let \( D = \text{Im}(I-A), K = \text{Ker}(I-A) \). Then

(1) \( \lambda(B, D) = 1 \iff C_B^\delta = 1 \iff \forall x \in S_K: C_B^\delta = 1 \iff \forall a \in (0, 1) \)

and \( \forall x \in S_K: \varphi_B^\delta(a) \leq 1 \),

(II) \( \lambda(B, D) < 2 \iff C_B^\delta < 2 \iff \forall x \in S_K \exists a_x \in (0, 1): \varphi_B^\delta(a_x) < 1 + a_x \),

(III) \( \lambda(B, D) = 2 \iff C_B^\delta = 2 \).
In the next example we give a realization of cases (II) and (III) of Corollary 2.1. For case (I), see [11] (Proposition 3.a.4).

**Example 2.1** Let $B$ be a Banach space with the symmetric norm (2) (relative to a normal basis $(e_i)_i$ ([17], [18])). Let $(j(i))_{i=1}^\infty$ be a strictly increasing sequence of natural numbers so that $j(1) = 1$ and $k(i) = j(i+1) - j(i) \geq 3$. Now, let $A: B \to B$ be a linear isometry such that $A(e_s) = e_{s+1}$ for every $s \in \mathbb{N}$ and $j(i) \leq s < j(i+1) - 1$ and $A_{j(i)+1} = e_{j(i)} (i = 1, 2, \ldots)$.

Let $D_{k(i)} = \{x = (\alpha_1, \alpha_2, \ldots) \in B: \sum_{v=j(i)}^{j(i)+1-1} \alpha_v = 0, \alpha_v = 0$ if $v < j(i)$ or $v \geq j(i+1)\}$, $B_{k(i)} = \text{span} \{e_{j(i)}, \ldots, e_{j(i)+1-1}\}$, $i = 1, 2, \ldots$

It is easy to see that $\dim(B_{k(i)}/D_{k(i)}) = 1$. From Theorem 1.1 it follows immediately that the isometry $A$ generates in the subspace $B_{k(i)}$ the minimal projection $P_i$ from $B_{k(i)}$ onto $D_{k(i)}$ and $\|P_i\| = \lambda(B_{k(i)}, D_{k(i)})$.

Now, in view of Proposition 1.2 it is easy to check that its condition (ii) holds. Hence, by Theorem 1.1 the projection $\tilde{P}$ from $B$ onto $D = \text{Im}(I - A) = \bigoplus_{i=1}^\infty D_{k(i)}$ along $\text{Ker}(I - A)$ is a minimal projection.

It is obvious that

$$\|\tilde{P}\| = \sup_i \|P_i\|$$

(because for each $x = \sum_{i=1}^\infty x_i e_i$ we have $\|x\| \geq \|\sum_{v=j(i)}^{j(i)+1-1} \alpha_v e_v\|$, $i = 1, 2, \ldots$, [18]).

Now, let $B = l_1$ or $B = c_0$. We prove that $\|P_i\| = 2 - 2/k(i)$ ($i = 1, 2, \ldots$). Indeed, if $B = l_1$, then there exists a linear isometry $F_1: B_{k(i)} \overset{\text{onto}}{\to} l_1^{|k(i)|}$, so $F_1(D_{k(i)}) = f_{1,i}^{-1}(0)$, where $f_{1,i} = (1, \ldots, 1) \in (l_1^{|k(i)|})^\ast$. If $B = c_0$, then there exists a linear isometry $F_2: B_{k(i)} \overset{\text{onto}}{\to} l_\infty^{|k(i)|}$, so $F_2(D_{k(i)}) = f_{2,i}^{-1}(0)$, where $f_{2,i} = (1/k(i), \ldots, 1/k(i)) \in (l_\infty^{|k(i)|})^\ast$.

By the result of [3] we get in both cases: $\lambda(B_{k(i)}, D_{k(i)}) = 2 - 2/k(i)$. By (8), $\|\tilde{P}\| = 2 - \inf_i (2/k(i))$.

Therefore, $\|\tilde{P}\| = 2$ and $C_{K}^i = 2$, where $K = \text{Ker}(I - A)$, if $\sup k(i) = +\infty$. If $\sup k(i) < +\infty$, then for each $x \in S_K$ there exists $a_x \in (0, 1)$ such that $\gamma_{ii}(a_x) < 1 + a_x$.

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(2) Let $E$ be the set all $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ with $\varepsilon_i \in \{-1, 1\}$ ($i = 1, 2, \ldots$). Let $\Pi$ be the set of all permutations $\sigma: \mathbb{N} \to \mathbb{N}$. A Banach space $B$ with a normal basis $(e_i)_i$ is said to be symmetric iff $\|\varepsilon \sigma x\| = \|x\|$ for every $x \in B$, $\sigma \in \Pi$, $\varepsilon \in E$ (cf. [18]).
Remark 2.1. Note, if $B = c_0$, then the projection $\tilde{P}$ from Example 2.1 is the unique minimal projection onto $D$.

If $B = l_1$, then the uniqueness of the minimal projection $\tilde{P}$ onto $D$ (from Example 2.1) fails, although the projections $P_i$ are unique minimal projections from $B_{k(i)}$ onto $D_{k(i)}$, $i = 1, 2, \ldots$ (Cf. [14], [15].)

From Theorem 2.1 we get immediately:

**Corollary 2.2.** Suppose $D$ be a complemented subspace of Banach space $B$ such that $\lambda(B, D) > 2$. Then there exists no linear isometry $A$ of $B$ onto itself such that $D = \text{Im}(I - A)$ and $B = D \oplus \text{Ker}(I - A)$.

**Example 2.2.** Let $B = \tilde{C}([0, 2\pi])$ be the space of all continuous $2\pi$-periodic real-valued functions defined on $[0, 2\pi]$, and $D_n$ ($n \geq 1$), be the subspace in $B$ consisting of all trigonometric polynomials of degree $\leq n$.

We shall prove that for $n \geq 8$ there exists no linear isometry $A$ of $B$ onto itself such that $\text{Im}(I - A) = D_n$. Suppose that for some $n \geq 8$ there exists such a linear isometry $A$.

By Proposition 1.1 and Example 1.1 it follows that $B = D_n \oplus \text{Ker}(I - A)$.

Hence, by Theorem 2.1, $\lambda(B, D_n) \leq 2$. On the other hand, $\lambda(B, D_n)$ is equal to the Lebesgue Constant $\varrho_n$ such that $\varrho_n = (4/\pi^2) \log n + 1.27033 + \varepsilon_n$, where $0.166 > \varepsilon_n \downarrow 0$ (cf., for example, [4]). Taking into account that $n \geq 8$, we have $\varrho_n > 2$, a contradiction.

**References**


FINANCE-ECONOMICS INSTITUTE, LENINGRAD
and
INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY OF BYDGOSZCZ