Z. DOMAŃSKI and Z. ROJEK (Warsaw)

Application of Fisher–Riesz–Kupradze method
to solving the second Fourier problem

Abstract. In this work we construct the solution of the second Fourier problem using Fisher–Riesz–Kupradze method.

1. Introduction. Let Ω be a bounded domain in Euclidean space $E^3$ the boundary of which $^\dagger$ is a closed surface $S$ of class $C^{2,1}$. We shall denote the points of $\Omega$ as $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, ..., the point of the surface $S$ as $\xi = (\xi_1, \xi_2, \xi_3)$, $\eta = (\eta_1, \eta_2, \eta_3)$, ... The symbol $|x - y|$ denotes the Euclidean distance between points $x$ and $y$: $\Omega_T = \Omega \times (0, T)$; $S_T = S \times (0, T)$; $n_\eta = [\alpha_1(\eta), \alpha_2(\eta), \alpha_3(\eta)]$ is the inner unit normal to surface $S$ at $\eta$.

Let $\chi$ be a function defined in the domain $\Omega$ and let this function have partial derivatives $\partial\chi/\partial x_j$, $j = 1, 2, 3$, uniformly continuous in $\Omega$. $(\chi)$ will be the function defined on the surface $S$ by

$$(\chi)(\eta) = \lim_{x \to \eta} \chi(x).$$

DEFINITION 1. Let a function $\mu$ defined on surface $S$ be given and let $(\chi) = \mu$. We define the derivatives $D_{x_j} \mu$, $j = 1, 2, 3$, of the function $\mu$ (see [3], p. 15–20) by the formula

$$(1) \quad D_{x_j} \mu = \left(\frac{\partial \chi}{\partial x_j}\right) - \alpha_j \frac{d\chi}{dn}, \quad j = 1, 2, 3,$$

where

$$\frac{d\chi}{dn} = \sum_{j=1}^{3} \left(\frac{\partial \chi}{\partial x_j}\right) \alpha_j.$$

The function $\mu$ for which $D_{x_j} \mu$, $j = 1, 2, 3$, exist will be said to be differentiable in Hugoniot–Hadamard sense, simply in H–H sense.

$\dagger$ For the definition of the surface of class $C^{2,1}$ see [3], p. 96–98, where $L_2(B, \lambda) = C^{2,1}$. 

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We know that (see [3], p. 19)

\[ \sum_{j=1}^{3} \alpha_j D_{x_j} \mu = 0. \]  

Instead of using \( D_{x_j} \) we shall use \( D_j \). If the function \( \mu \) defined on \( S_T \) has in \( H-H \) sense derivatives of order \( l \geq 0 \) with respect to space variables satisfying H"older condition with exponent \( h \) with respect to these variables, \( 0 \leq h \leq 1 \), and if the function \( \mu \) satisfies the H"older condition with exponent \( \alpha \), \( 0 \leq \alpha \leq 1 \), with respect to time variable, we shall say that \( \mu \in C^{1,h,\alpha}(S_T) \).

II. Properties of the Poisson–Weierstrass integral and heat potentials. We know (see [5], p. 529) that for the equation

\[ \Delta u - \frac{\partial u}{\partial t} = g \]

we have the fundamental formula

\[
\int_{\Omega} u(y, 0) v(x-y, t) dy - \int_{\Omega} g(y, \tau) v(x-y, t-\tau) dy d\tau -
\int_{\partial \Omega} \frac{du}{dn} (x-\eta, t-\tau) dS_\eta d\tau +
\int_{\partial \Omega} u(\eta, \tau) \frac{dv}{dn} (x-\eta, t-\tau) dS_\eta d\tau
\]

\[
= \begin{cases} 
0 & \text{for } x \notin \bar{\Omega}, \ t > 0, \\
n(\xi, t) & \text{for } (\xi, t) \in S_T, \\
u(x, t) & \text{for } (x, t) \in \Omega_T,
\end{cases}
\]

where

\[ v(x, t) = \frac{1}{(4\pi t)^{3/2}} \exp \left( -\frac{x^2}{4t} \right) \text{ for } t > 0. \]

The integrals in (4) play the main role when initial value problems are concerned with (3). Some properties of these integrals will be given.

Definition 2. The Poisson–Weierstrass integral with density \( \phi \) is (see [11], p. 21–23; [9], p. 282–283) the integral of the form

\[ I(x, t) = \int_{\Omega} \phi(y) v(x-y, t) dy. \]

Theorem 1. If the function \( \phi \) belongs to \( C^2_0(\Omega) \), then the Poisson–Weierstrass integral \( I(\xi, t) \) for set \( S_T \) belongs to the class \( C^{2,0;1}(S_T) \).

We can prove this theorem using the classical method.
DEFINITION 3. The heat volume potential with density $g$ is (see [11], p. 23–34) the integral of the form

$$J(x, t) = \int_0^t \int g(y, \tau) v(x - y, t - \tau) dy d\tau.$$  

THEOREM 2. If $g(\cdot, t) \in C^2_0(\Omega)$ for every $t \in (0, T)$ and $g(x, \cdot) \in C((0, T))$ for every $x \in \Omega$, then $J(\cdot, \cdot) \in C^{2,0:1}(S_T)$. The proof of this theorem is analogous to the proof of Theorem 1.

DEFINITION 4. The heat potential of the simple layer with density $\psi$ is (see [10], p. 81–112) the integral of the form

$$U(x, t) = \int_0^t \int_{S_T} \psi(\eta, \tau) v(x - \eta, t - \tau) dS_n d\tau.$$  

THEOREM 3. If the function $\psi$ is defined on $S_T$ and if $\psi(\cdot, t)$ is differentiable in $H^2$ sense at every value $\tau \in (0, T)$ and if the derivatives $D_j \psi, j = 1, 2, 3$, are bounded and integrable on $S_T$, then the derivatives $D_j U, j = 1, 2, 3$, exist and satisfy the H"older condition with respect to space variables with an exponent arbitrarily smaller than 1.

Proof. If $x \notin S$, then

$$\frac{\partial U}{\partial x_j}(x, t) = \int_0^t \int_{S_T} \psi(\eta, \tau) \frac{\partial v(x - \eta, t - \tau)}{\partial x_j} dS_n d\tau$$

$$= - \int_0^t \int_{S_T} \psi(\eta, \tau) \frac{\partial v(x - \eta, t - \tau)}{\partial n_j} dS_n d\tau.$$

Using formula (1), we obtain

$$\frac{\partial U}{\partial x_j}(x, t) = - \int_0^t \int_{S_T} \psi(\eta, \tau) D_j v dS_n d\tau - \int_0^t \int_{S_T} \psi(\eta, \tau) \frac{dv}{dn_j} dS_n d\tau.$$  

Taking into account the equality $\sum_{k=1}^{3} \alpha_k^2(\eta) = 1$ and that $\sum_{k=1}^{3} \alpha_k(\eta) D_j \alpha_k(\eta) = 0$, we have

$$D_j(\psi v) = \sum_{k=1}^{3} \alpha_k D_j(\alpha_k \psi v).$$  

Using the equality $\sum_{k=1}^{3} \alpha_k D_k(\psi v) = 0$ (see (2)), we get

$$\sum_{k=1}^{3} \alpha_j [D_k(\alpha_k \psi v) - \psi v D_k \alpha_k] = 0.$$
\( \psi D_j v = D_j(\psi t) - v D_j \psi \): therefore from (10) and (11) we get

\[
(12) \quad - \int_0^t \int_0^v (D_j \psi - \alpha_j \psi \sum_{k=1}^3 D_k \alpha_k) v dS_n d\tau + \int_0^t \int_0^v \left[ \sum_{k=1}^3 D_k (\alpha_k \psi v) - \sum_{k=1}^3 \alpha_k D_j (\alpha_k \psi v) \right] dS_n d\tau.
\]

From the Stokes theorem and from the fact that \( S \) is a closed surface we see that the last integral in (12) vanishes. Therefore we can write (9) in the form

\[
(13) \quad \frac{\partial U}{\partial x_j} (x, t) = \int_0^t \int_0^v (D_j \psi - \alpha_j \psi \sum_{k=1}^3 D_k \alpha_k) v dS_n d\tau - \int_0^t \int_0^v \alpha_j (\xi) \psi (\xi, t) \frac{dv}{dn} dS_n d\tau.
\]

The first of these integrals is the heat potential of the simple layer with density \( D_j \psi - \alpha_j \psi \sum_{k=1}^3 D_k \alpha_k \), and the second of these integrals is the heat potential of the double layer with density \( \alpha_j \psi \). Using the known boundary property of the heat potential of the double layer (see [11], p. 13–21), we get

\[
\lim_{x \to \xi} - \int_0^t \int_0^v \frac{\partial U}{\partial x_j} (\xi, t) = \int_0^t \int_0^v (D_j \psi - \alpha_j \psi \sum_{k=1}^3 D_k \alpha_k) v dS_n d\tau - \int_0^t \int_0^v \alpha_j (\xi) \psi (\xi, t) \frac{dv}{dn} dS_n d\tau + \frac{1}{2} \alpha_j (\xi) \psi (\xi, t) = I_j - W_j + \frac{1}{2} \alpha_j \psi.
\]

Therefore

\[
(14) \quad D_j U = I_j \alpha_j \sum_{k=1}^3 \alpha_k I_k - (W_j - \alpha_j \sum_{k=1}^3 \alpha_k W_k).
\]

The heat potential of the simple layer \( I_j \) and the heat potential of the double layer \( W_j \) satisfy the Hölder condition (see [11], p. 6–10, 18–21) with respect to the space variable with an arbitrary exponent smaller than 1 with the previously given assumptions about function \( \psi \), therefore (14) we see that \( D_j U \) satisfies the Hölder condition with the same exponent.

**Theorem 4.** If the density \( \psi \) is a bounded integrable function in \( S_T \) and satisfies the Hölder condition with respect to the time variable \( t \) with exponent \( x \), \( 0 < x \leq 1 \), and the condition

\[
\lim_{t \to 0^+} \psi (\eta, t) = \psi (\eta, 0) = 0,
\]

then the heat potential of the simple layer \( U (\xi, t) \) satisfies the Hölder condition with respect to the variable \( t \) with exponent \( x \).

The proof of this theorem is easy.
III. Problem. Using Fisher–Riesz–Kupradze method, we shall solve the following Fourier problem:

We seek the function \( u(x, t) \) satisfying in \( \Omega_T \) the equation

\[
\Delta u - \frac{\partial u}{\partial t} = g
\]

and the following conditions:

\[
\lim_{t \to 0^+} u(x, t) = \varphi(x),
\]

\[
\lim_{x \to \xi} \frac{du}{dn_x}(x, t) = \psi(\xi, t).
\]

We assume that \( S \) is of class \( C^{2,\lambda} \),

\[
g(\cdot, t) \in C^2_0(\Omega) \quad \text{for any } t \in (0, T),
\]

\[
g(x, \cdot) \in C((0, T)) \quad \text{for any } x \in \Omega, \varphi \in C^2_0(\Omega),
\]

\[
\psi \in C^{1,0,\lambda}(\Sigma_T) \quad \text{and} \quad \psi(\eta, 0) = 0.
\]

Solution of the problem. Using the basic formulas (4) we can write the solution (if it exists) in the form

\[
u(x, t) = \int_{\Omega} \varphi(y) v(x - y, t) dy - \int_{\Omega} \int g(y, \tau) v(x - y, t - \tau) dy d\tau -
\]

\[
- \int_{\Sigma} \int \psi(\eta, \tau) v(x - \eta, t - \tau) dS_\eta d\tau + \int_{\Sigma} \int f(\eta, \tau) \frac{dv}{dn_\eta} (x - \eta, t - \tau) dS_\eta d\tau,
\]

where the known function \( f(\eta, \tau) \) is the solution of the functional equation

\[
\int_{\Sigma} f(\eta, \tau) \frac{dv}{dn_\eta} (x - \eta, t - \tau) dS_\eta d\tau = F(x, t) \quad (x \notin \Omega, \ t \in (0, T)).
\]

In (19) we denoted

\[
F(x, t) = - \int_{\Omega} \varphi(y) v(x - y, t) dy + \int_{\Omega} \int g(y, \tau) v(x - y, t - \tau) dy d\tau +
\]

\[
+ \int_{\Sigma} \int \psi(\eta, \tau) v(x - \eta, t - \tau) dS_\eta d\tau.
\]

Theorem 5. The functional equation (19) has exactly one solution belonging to class \( C^{1,h,\lambda}(\Sigma_T) \).

Proof. Let us assume that (19) has got a solution. Using the properties of the heat potential of the double layer (see [11], p. 15), we get the integral
equation

\[
- \frac{1}{2} f(\xi, t) + \int_0^t \int_S f(\eta, \tau) \frac{dv}{dn_{\eta}} (\xi - \eta, t - \tau) dS_{\eta} d\tau = F(\xi, t).
\]

The integral equation (21) is a Volterra equation with a continuous right side; it has exactly one continuous solution (see [12], p. 129–135). Taking into account the assumptions, we infer on the basis of Theorems 1–4 that the function \( F(\xi, t) \in C^{1, \lambda; \varphi}(S_T) \) and \( F(\xi, 0) = 0 \). We shall show that the solution of the integral equation (19) is of class \( C^{1, \lambda; \varphi}(S_T) \).

We shall use the following lemmas.

**Lemma 1.** If a function \( F \) is continuous and bounded on \( S_T \), it satisfies the Hölder condition with exponent \( \lambda \) with respect to variable \( t \) and \( F(\xi, 0) = 0 \) for \( \xi \in S \), then the solution of the integral equation (21) satisfies the Hölder condition with the same exponent \( \lambda \) with respect to variable \( t \).

We get the proof of Lemma 1 by showing that any of the terms of the Neumann series (which is the solution of (21)) satisfies the Hölder condition with respect to variable \( t \) with exponent \( \lambda \), and that the sum of the Neumann series satisfies the Hölder condition with respect to variable \( t \) with exponent \( \lambda \).

**Lemma 2.** If the function \( F(\cdot, t) \in C^{1, \lambda}(S) \) for any \( t \in (0, T) \), then the solution \( f(\cdot, t) \) of the integral equation (21) belongs to class \( C^{1, \lambda}(S) \) for any \( t \in (0, T) \).

Lemma 2 is proved by application of a localization principle based on the appropriate partition of unity (see [1], p. 63–88).

The function \( F(\xi, t) \) satisfies assumptions of Lemmas 1 and 2, therefore on the basis of these lemmas we conclude that the solution of the integral equation (21) is of class \( C^{1, \lambda; \varphi}(S_T) \). We shall show that the solution \( f \) of the integral equation (21) is a solution of the functional equation (19).

Let us suppose that the function \( f \) is not a solution of the functional equation (19). Let us denote

\[
\Phi(x, t) = \int_0^t \int_S f(\eta, \tau) \frac{dv}{dn_{\eta}} (x - \eta, t - \tau) dS_{\eta} d\tau - F(x, t),
\]

where the function \( f(\eta, \tau) \) is a solution of the integral equation (21). We have \( \Phi(x, t) \neq 0 \) for \( x \notin \overline{\Omega} \). It is easy to check that

\[
\Delta \Phi - \frac{\partial \Phi}{\partial t} = 0,
\]

\[
\lim_{t \to 0} \Phi(x, t) = 0,
\]

\[
(22) \quad \Delta \Phi - \frac{\partial \Phi}{\partial t} = 0,
\]

\[
(23) \quad \lim_{t \to 0} \Phi(x, t) = 0,
\]
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(24) \[ \lim_{x \to x_0} \Phi(x, t) = 0, \]

(25) \[ \lim_{|x| \to +\infty} \Phi(x, t) = 0. \]

Looking for the solution of problems (22), (23), (24), (25) in the form of the heat potential of the double layer with continuous and bounded density \( \mu \)

(26) \[ \Phi(x, t) = \int_0^t \int_S \mu(\eta, \tau) \frac{dv}{dn} (x - \eta, t - \tau) dS_\eta d\tau \]

and using condition (24) and the property of the heat potential of the double layer, we get an integral equation of the form

(27) \[ -\frac{1}{2} \mu(\xi, t) + \int_0^t \int_S \mu(\eta, \tau) \frac{dv}{dn} (\xi - \eta, t - \tau) dS_\eta d\tau = 0. \]

The equation (27) has exactly one solution \( \mu(\eta, \tau) \equiv 0 \). Therefore, \( \Phi(x, t) = 0 \). The contradiction we arrived at ends the proof of Theorem 5.

Because the solution of (19) belongs to class \( C^{1, h, \lambda}(S_T) \) the Lapunov-Taubers theorem is valid for the function \( u \) given by formula (18) (see [2], corollary, p. 142). Taking the above into account, it is easy to check that the function \( u \) is a solution of problems (15), (16), (17).

Construction of the solution. Let \( S^1 \) be an arbitrary closed Lapunov’s surface which is the boundary of a domain \( \Omega^1 \) which includes \( \bar{\Omega} \) in its interior.

Let us write

\[ \Omega_T^1 = \Omega^1 \times (0, T); \quad S_T^1 = S^1 \times (0, T). \]

Let us take a countable dense set of points \( \{(\xi^k, t_i)\}, k, i = 1, 2, \ldots \), on the surface \( S_T^1 \) and let us consider the set of the functions

(28) \[ \Gamma(\xi^k - \eta, t_i - \tau) = \begin{cases} \frac{dv}{dn}(\xi^k - \eta, t_i - \tau) & \text{for } \tau < t_i, \\ 0 & \text{for } t_i \leq \tau < T, \end{cases} \]

\( k, i = 1, 2, \ldots \)

Ordering the set (28) in a certain way, we write

(29) \[ \Gamma(\xi^{kj} - \eta, t_{ij} - \tau) = \Gamma_j(\eta, \tau), \quad j = 1, 2, \ldots, \]

and consider the space \( L_2(S_T) \) with the norm

\[ \|\varphi\| = (\int_0^t \int_S |\varphi(\eta, \tau)|^2 dS_\eta d\tau)^{1/2}. \]

Lemma 3. The set of functions \( \{\Gamma_j(\cdot, \cdot)\}, j = 1, 2, \ldots, \) is linearly independent.
Lemma 4. The set of functions \( \{\Gamma_j(\cdot, \cdot)\}, j = 1, 2, \ldots \), is complete in the domain \( L_2(S_T) \).

The proofs of Lemmas 3 and 4 are similar to those in [7], [8]. Subjecting the set \( \{\Gamma_j(\cdot, \cdot)\} \) to the process of orthonormalization, we get the set \( \{\omega_j(\cdot, \cdot)\} \) of orthonormal functions. The elements of the set \( \{\omega_j(\cdot, \cdot)\}, j = 1, 2, \ldots \), are linear combinations of the elements of the set \( \{\Gamma_j(\cdot, \cdot)\} \) and vice versa (see [4], p. 72-73), i.e.,

\[
\omega_j(\eta, \tau) = \sum_{k=1}^{j} A_{kj} \Gamma_k(\eta, \tau); \quad \Gamma_j(\eta, \tau) = \sum_{k=1}^{j} B_{kj} \omega_k(\eta, \tau).
\]

We denote Fourier coefficients of the function \( f(\cdot, \cdot) \) with respect to the set \( \{\omega_j(\cdot, \cdot)\}, j = 1, 2, \ldots \), by

\[
\Phi_j = \int \int f(\eta, \tau) \omega_j(\eta, \tau) dS_{\eta} d\tau, \quad j = 1, 2, \ldots
\]

Putting \( x = \xi^k_j, \ t = t_j \) into equation (19), multiplying this equation by \( A_{rj} \) and taking the sum from \( r \) equal one to \( r \) equal \( j \), we get

\[
\sum_{r=1}^{j} \int \int f(\eta, \tau) A_{rj} \Gamma_r(\eta, \tau) dS_{\eta} d\tau = \sum_{r=1}^{j} A_{rj} F(\xi^k_r, t_j)
\]

and therefore

\[
\Phi_j = \sum_{r=1}^{j} A_{rj} F(\xi^k_r, t_j).
\]

Because \( F(x, t) \) is a given function the coefficients \( A_{rj} \) are the normalization constants, therefore all Fourier coefficients \( \Phi_j \) are determined. Since \( f \in L_2(S_T) \), we have

\[
\lim_{N \to +\infty} \left\| f - \sum_{k=1}^{N} \Phi_k \omega_k \right\| = 0.
\]

Let us write

\[
f^N(\eta, \tau) = \sum_{k=1}^{N} \Phi_k \omega_k(\eta, \tau),
\]

\[
u^N(x, t) = \int \int f^N(\eta, \tau) \frac{dv}{dn_{\eta}} (x - \eta, t - \tau) dS_{\eta} d\tau - F(x, t).
\]

Theorem 6. For any \( x \in \Omega \) and for any \( t \) from the interval \( (0, T) \) and for any \( \varepsilon > 0 \) there exists a \( N_0 \) such that for \( N > N_0 \) we have

\[
|u(x, t) - u^N(x, t)| < \varepsilon.
\]

Proof. We have

\[
|u(x, t) - u^N(x, t)| \leq \int \int \left| f(\eta, \tau) - f^N(\eta, \tau) \right| \left| \frac{dv}{dn_{\eta}} (x - \eta, t - \tau) \right| dS_{\eta} d\tau.
\]
Using the estimate
\[
\left[ \frac{dv}{dn}(x - \eta, t - \tau) \right]^2 \leq M \left[ (t - \tau)^\sigma \sigma^{6 - 2v} \right]^{-1},
\]
where \(0 < v < 1\), \(\sigma = \inf_{\eta \in \overline{S}}|x - \eta|\) and \(M\) is a positive constant, we get
\[
\int_0^t \int_{\overline{S}} \left[ \frac{dv}{dn}(x - \eta, t - \tau) \right]^2 dS \ dt \leq \frac{(\text{mes } S) MT^{1-v}}{\sigma^{6-2v}(1-v)} = M_1^2.
\]

On the basis of (30) we can choose an \(N\) such that
\[
\int_0^t \int_{\overline{S}} |f(\eta, \tau) - f^N(\eta, \tau)|^2 dS \ dt \leq \|f - f^N\|^2 \leq \frac{\varepsilon^2}{M_1^2}.
\]

Using the Schwarz inequality in (34) and taking into account (35) and (36), we get (33).

On the basis of Theorem 6 we get the solution of problems (15), (16), (17)
\[
u(x, t) = \lim_{N \to \infty} \int_0^t \int_{\overline{S}} \left( \sum_{k=1}^N \Phi_k \omega_k(\eta, \tau) \right) \frac{dv}{dn}(x - \eta, t - \tau) dS \ dt - F(x, t).
\]

References


