Baire classification and multivalued maps

A set $Y$ with two topologies $\tau_1$ and $\tau_2$ is called a bitopological space $[8], [14]$. For a set $M \subseteq Y$ by $M^{(\tau_1)}$ we denote the $\tau_1$-closure of $M$.

In $(Y, \tau_1, \tau_2)$ the topology $\tau_2$ is perfectly normal with respect to $\tau_1$ if each $\tau_2$-closed set $M \subseteq Y$ is of the form $M = \bigcap_{n=1}^{\infty} W_n$, where $W_n$ are $\tau_1$-open sets such that $\overline{W_n}^{(\tau_2)} \subseteq W_n$ for $n \geq 1$ $[6]$. Equivalently, $\tau_2$ is perfectly normal with respect to $\tau_1$ if each $\tau_2$-open set $W$ is of the form $W = \bigcup_{n=1}^{\infty} W_n$, where $W_n \in \tau_2$ and $\overline{W_n}^{(\tau_1)} \subseteq W_n$.

This property is not symmetrical. For instance, if $Y$ is the set of real numbers, $\tau_1 = \{(a, \infty): a \in Y\} \cup \{\emptyset, Y\}$ and $\tau_2$ is the natural topology on $Y$, then $\tau_1$ is perfectly normal with respect to $\tau_2$ but converse does not hold.

In the case $\tau_1 = \tau_2$ we have a perfectly normal topological space $[5]$.

In the sequel by $\mathcal{C}(Y, \tau_i)$ and $\mathcal{K}(Y, \tau_i)$ we shall denote the class of all non-empty $\tau_i$-closed or $\tau_i$-compact subsets of $Y$, respectively.

Let $X$ be a topological space. If $F: X \to Y$ is a multivalued map, then for any sets $A \subseteq X$ and $B \subseteq Y$ we denote $[3]$:

$$F(A) = \bigcup \{F(x): x \in A\},$$
$$F^+(B) = \{x \in X: F(x) \subseteq B\},$$
$$F^-(B) = \{x \in X: F(x) \cap B \neq \emptyset\}.$$

For any countable ordinal number $\alpha$, a multivalued map $F: X \to Y$ is said to be of $\tau_i$-lower or $\tau_i$-upper Baire class $\alpha$ if for each $\tau_i$-open set $V \subseteq Y$ the set $F^-(V)$ or $F^+(V)$, respectively, is of the additive class $\alpha$ in $X$. We shall use $LB_{\alpha}(\tau_i)$ and $UB_{\alpha}(\tau_i)$ to denote the $\tau_i$-lower and $\tau_i$-upper Baire classes $\alpha$ of multivalued maps. Thus $LB_0(\tau_i)$ and $UB_0(\tau_i)$ are classes of $\tau_i$-lower and $\tau_i$-upper semicontinuous maps, respectively.

Now let $F_n, F: X \to Y$ be multivalued maps such that $F_n(x), F(x) \in \mathcal{C}(Y, \tau_i)$ for $n \geq 1$, $x \in X$. We write $F \in \tau_i$-$\text{lim} F_n$ if for each $x \in X$ the sequence $\{F_n(x): n \geq 1\}$ converges to $F(x)$ in the Vietoris topology on $\mathcal{C}(Y, \tau_i)$. 

Theorem 1.1. ([6], Theorem 2.1). Let $X$ be a topological space and let $(Y, \tau_1, \tau_2)$ be a bitopological space such that $\tau_1 \subset \tau_2$ and $\tau_2$ is perfectly normal with respect to $\tau_1$. Suppose that $F_n, F : X \to Y$ are multivalued maps such that $F_n(x), F(x) \in \mathcal{C}(Y, \tau_1)$ for each $n \geq 1, x \in X$, and $F = \tau_2^- \lim F_n$. Then

(a) For every $\tau_2$-closed set $M \subset Y$,

$$F^+(M) = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} F_{n+k}^+(W_n),$$

where $W_n$ are $\tau_1$-open sets such that $M = \bigcap_{n=1}^{\infty} W_n$ and $\mathcal{V}_{n+1}^{(2)} \subset W_n$ for $n \geq 1$.

(b) If $F_n(x), F(x) \in \mathcal{K}(Y, \tau_2)$, then

$$F^-(M) = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} F_{n+k}^-(W_n).$$

This theorem implies the following results:

Corollary 1.2. Let $X$ be a topological space and let $(Y, \tau_1, \tau_2)$ be a bitopological space such that $\tau_1 \subset \tau_2$ and $\tau_2$ is perfectly normal with respect to $\tau_1$. Suppose that $F_n, F : X \to Y$ are multivalued maps such that $F_n(x), F(x) \in \mathcal{C}(Y, \tau_1)$ for $n \geq 1, x \in X$ and $F = \tau_2^- \lim F_n$.

(a) If $F_n \in UB_2(\tau_1)$ for $n \geq 1$, then $F \in LB_{2+1}(\tau_2)$.

(b) If $F_n(x), F(x) \in \mathcal{K}(Y, \tau_2)$ for $n \geq 1, x \in X$ and $F_n \in LB_2(\tau_1)$, then $F \in UB_{2+1}(\tau_2)$.

When $\tau_1 = \tau_2$, Corollary 1.2 coincides with the result of Garg [7], Theorem 3.1.

Corollary 1.3. Let $X$ be a topological space and let $(Y, \tau_1, \tau_2)$ be a bitopological space such that $\tau_1 \subset \tau_2$ and $\tau_2$ is perfectly normal with respect to $\tau_1$. If $F : X \to Y$ is a multivalued map such that $F(x) \in \mathcal{C}(Y, \tau_1)$ for $x \in X$, then the following is satisfied:

(a) If $F \in UB_2(\tau_1)$, then $F \in LB_{2+1}(\tau_2)$.

(b) If $F \in LB_2(\tau_1)$ and $F(x) \in \mathcal{K}(Y, \tau_2)$ for $x \in X$, then $F \in UB_{2+1}(\tau_2)$.

If $\tau_1 = \tau_2$ and $Y$ is a compact metric space, then Corollary 1.3 gives the theorem of Kuratowski [12].

By $(E, \tau_w, \tau_s)$ we denote a separable Banach space with the weak topology $\tau_w$ and the topology $\tau_s$ determined by the norm on $E$.

Theorem 1.4 ([6], Theorem 5.1). In the bitopological space $(E, \tau_w, \tau_s)$, $\tau_s$ is perfectly normal with respect to $\tau_w$.

Corollary 1.5. Let $X$ be a topological space and let $F_n, F : X \to E$ be multivalued maps such that $F_n(x), F(x) \in \mathcal{C}(E, \tau_w)$ for $n \geq 1, x \in X$ and $F = \tau_s^- \lim F_n$.
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(a) If $F_n \in \text{UB}_a(\tau_w)$, then $F \in \text{LB}_a(\tau_s)$.
(b) If $F_n \in \text{LB}_a(\tau_w)$ and $F_n(x), F(x) \in \mathcal{H}(E, \tau_s)$, then $F \in \text{UB}_a(\tau_s)$.

**Corollary 1.6.** Suppose that $X$ is a topological space and $F: X \to E$ is a multivalued map such that $F(x) \in \mathcal{H}(E, \tau_w)$ for $x \in X$. Then
(a) If $F \in \text{UB}_a(\tau_w)$, then $F \in \text{LB}_a(\tau_s)$.
(b) If $F \in \text{LB}_a(\tau_w)$ and $F(x) \in \mathcal{H}(E, \tau_s)$ for $x \in X$, then $F \in \text{UB}_a(\tau_s)$.

Any singlevalued map $f: X \to Y$ can be considered as a multivalued map $F$ defined by $F(x) = \{ f(x) \}$. In this case for each set $D \subseteq Y$ we have $F^-(D) = F^+(D) = f^{-1}(D)$. Moreover, $\text{UB}_a = \text{LB}_a = \text{BA}_a$, i.e., it is the Baire class $\alpha$ of singlevalued maps [2], [11].

Therefore from Corollaries 1.5 and 1.6 we obtain the following

**Corollary 1.7.** ([1], Theorems 3 and 2). Let $X$ be a topological space.
(a) If $f_n: X \to E$ are maps of the weak class $\alpha$ and $f = \tau_s\lim_{n \to \infty} f_n$, then $f$ is in $\text{B}_a(\tau_s)$.
(b) If a map $f: X \to E$ is in $\text{B}_a(\tau_w)$, then $f \in \text{B}_a(\tau_s)$.

**II**

In this section we consider multivalued maps of two variables. For a map $F: X \times Y \to Z$ by $F_x$ and $F^y$ we denote the maps defined by $F_x(y) = F(x, y) = F^y(x)$ for $x \in X$, $y \in Y$.

The paper of Engelking [4] contains the following

**Theorem.** In a metric space the union of a locally finite family of sets of an additive (multiplicative) class $\alpha$ is the set of the same class.

Let us note that the proof of that theorem gives more.

**Theorem 2.1.** In a perfect space having a $\sigma$-locally finite base the union of a locally finite family of sets of an additive (multiplicative) class $\alpha$ is the set of the same class.

**Corollary 2.2.** In a perfect space possessing a $\sigma$-locally finite base the union of a $\sigma$-locally finite family of sets of an additive class $\alpha$ is the set of the same class.

**Theorem 2.3.** Suppose that $X$ is a metric space, $Y$ is a perfect space possessing a $\sigma$-locally finite base and $(Z, \tau_1, \tau_2)$ is a bitopological space in which $\tau_2$ is perfectly normal with respect to $\tau_1$. If $F: X \times Y \to Z$ is a multivalued map such that $F_x \in \text{UB}_a(\tau_1)$ for $x \in X$ and $F^y \in \text{UB}_0(\tau_1) \cap \text{LB}_0(\tau_2)$ for $y \in Y$, then $F \in \text{LB}_a(\tau_2)$.

**Proof.** Any $\tau_2$-closed set $M \subseteq Z$ is of the form $M = \bigcap_{n=1}^{\infty} W_n$, where $W_n \subseteq \tau_1$ and $\overline{W_n}^{(2)} \subseteq W_n$ for $n \geq 1$. Let $\{ B_s: s \in S \}$ be a $\sigma$-locally finite base for $X$, let $S_n = \{ s \in S: \text{diam } B_s < 1/n \}$ and $\{ z_s: z_s \in B_s, s \in S \}$. We shall show that
(1) \[ F^+ (M) = \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F_{z_s}^+ (W_n)). \]

Let us take a point \((x, y) \in F^+ (M).\) Then \(F(x, y) \subset W_n\) for each \(n \geq 1.\) Because \(F \in UB_0 (\tau_1),\) there exists a neighbourhood \(B_{s(m)} \) of \(x\) such that \(\operatorname{diam} B_{s(m)} < 1/n\) and \(F \in B_{s(m)} \subset W_n.\) Hence \(F(z_{s(n)}^+, y) \subset W_n\) and in consequence

\[ F^+ (M) \subset \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F_{z_s}^+ (W_n)). \]

Now let \((x, y) \in \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F_{z_s}^+ (W_n))\) and \((x, y) \notin F^+ (M).\) There exists a sequence \(\{z_{s_n}: s_n \in S_n, n \geq 1\}\) such that

(2) \[ x = \lim_{n \to \infty} z_{s_n}, \]

(3) \[ F(z_{s_n}, y) \subset W_n \quad \text{for} \quad n \geq 1. \]

On the other hand, for some \(m\) we have \(F(x, y) \cap (Z \setminus \bar{W}_m) \neq \emptyset.\) The condition \(F \in LB_0 (\tau_2)\) and (2) imply that there exists \(n_0\) such that \(F(z_{s_n}, y) \cap (Z \setminus \bar{W}_m) \neq \emptyset\) for \(n \geq n_0.\) So for \(n \geq \max \{n_0, m\}\) we obtain \(F(z_{s_n}, y) \cap (Z \setminus \bar{W}_n) \neq \emptyset,\) what is a contradiction to (3). Thus (1) is proved.

For any fixed \(n \geq 1,\)

\[ \{(B_s \times Y) \cap (X \times F_{z_s}^+ (W_n)): s \in S\} \]

is a \(\sigma\)-locally finite family of sets of the additive class \(\alpha\) in the space \(X \times Y.\) According to Corollary 2.2 the set \(\bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F_{z_s}^+ (W_n))\) is of the same class. Thus (1) implies that \(F^+ (M)\) is of the multiplicative class \(\alpha + 1\) and \(F \in LB_{\alpha + 1} (\tau_2).\)

**Theorem 2.4.** Suppose that \(X\) is a metric space, \(Y\) is a perfect space with a \(\sigma\)-locally finite base and \((Z, \tau_1, \tau_2)\) is a bitopological space such that \(\tau_2\) is perfectly normal with respect to \(\tau_1.\) If \(F: X \times Y \rightarrow Z\) is a multivalued map such that \(F(x, y) \in \mathcal{U}(Z, \tau_2)\) for \((x, y) \in X \times Y,\) \(F \in LB_0 (\tau_1)\) for \(x \in X,\) and \(F \in LB_0 (\tau_1) \cap UB_0 (\tau_2)\) for \(y \in Y,\) then \(F \in UB_{\alpha+1} (\tau_2).\)

**Proof.** Let \(\{B_s: s \in S\}\) be a \(\sigma\)-locally finite base for \(X,\) \(D = \{z_s: z_s \in B_s, s \in S\}\) and \(S_n = \{s \in S: \operatorname{diam} B_s < 1/n\}.\) A \(\tau_2\)-closed set \(M \subset Z\) is of the form \(M = \bigcap_{n=1}^{\infty} W_n,\) where \(W_n \in \tau_1\) and \(\bar{W}_{n+1} = W_n\) for \(n \geq 1.\)

We shall prove the expression

(4) \[ F^-(M) = \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F_{z_s}^- (W_n)). \]

If \((x, y) \in F^- (M),\) then \(F(x, y) \cap W_n \neq \emptyset\) for each \(n.\) It follows from the
condition $F^y \in \text{LB}_0(\tau_1)$ that there exists a neighbourhood $B_x$ of $x$ such that $\text{diam} B_x < 1/n$ and $F(x', y) \cap W_n \neq \emptyset$ for $x' \in B_x$. Thus $F(z_s, y) \cap W_n \neq \emptyset$ for some $s \in S_n$ and

\begin{equation}
(x, y) \in \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} (B_s \times Y) \cap (X \times F^{-}_s(W_n)).
\end{equation}

Now let $(x, y)$ be a point satisfying (5) and let us suppose that $F(x, y) \cap M = \emptyset$. According to (5) there exists a sequence $\{z_{s_n}: s_n \in S_n\}$ such that

\begin{equation}
x = \lim_{n \to \infty} z_{s_n}, \quad F(z_{s_n}, y) \cap W_n \neq \emptyset \quad \text{for } n \geq 1.
\end{equation}

On the other hand, $F(x, y) \subset Z \setminus M = \bigcup_{n=1}^{\infty} (Z \setminus W_n^{(2)})$. The $\tau_2$-compactness of $F(x, y)$ implies the inclusion $F(x, y) \subset Z \setminus W_m^{(2)}$ for some $m \geq 1$. Because $F^y$ is $\tau_2$-upper semicontinuous we can take $n_0$ such that $F(z_{s_n}, y) \subset Z \setminus W_m^{(2)}$ for $n \geq n_0$. Hence $F(z_{s_n}, y) \subset Z \setminus W_m^{(2)} \subset Z \setminus W_n$ for $n \geq \max \{n_0, m\}$, what is a contradiction to (6). So (4) is proved. The rest of the proof is analogous as in Theorem 2.3.

If $\tau_1 = \tau_2$, $X$, $Y$ and $Z$ are metric spaces and $f: X \times Y \to Z$ is a singlevalued map, then each of Theorems 2.3 and 2.4 gives the theorems of Montgomery [13] and Kuratowski [10] (for separable space [9]). Moreover, applying Theorem 1.4, we obtain the following

**Corollary 2.5.** Assume that $X$ is a metric space, $Y$ is a perfect space with a $\sigma$-locally finite base. If $f: X \times Y \to E$ is a singlevalued map such that $f^y$ is continuous for $y \in Y$ and $f_x$ is of the weak class $\alpha$ for $x \in X$, then $f$ is of the class $\alpha + 1$.

Let $u_a$ and $l_a$ denote the set of all real functions $f$ such that for each real number $r$ the set $\{x: f(x) < r\}$ or $\{x: f(x) > r\}$ is of the additive class $\alpha$ [15].

**Theorem 2.6.** Let $X$ be a metric space and let $Y$ be a perfect space with a $\sigma$-locally finite base. If $f: X \times Y \to R$ is a real function such that $f_x \in u_a$ ($f_x \in l_a$) for $x \in X$ and $f^y$ is continuous for $y \in Y$, then $f \in l_{a+1} \cup (f \in u_{a+1})$.

**Proof.** Let us put $\tau_1 = \{(a, \infty): a \in R\} \cup \{\emptyset, R\}, \quad \tau_2 = \{\emptyset, R\} \cup \{(-\infty, a): a \in R\}$. Then in the bitopological space $(R, \tau_1, \tau_2)$ the topology $\tau_i$ is perfectly normal with respect to $\tau_j$, $i \neq j$, $i, j = 1, 2$. So the conclusion follows from Theorem 2.3 or 2.4.

**III**

For a multivalued map $F: X \to Y$, the graph of $F$ is denoted by $\Gamma(F)$, i.e., $\Gamma(F) = \{(x, y) \in X \times Y: y \in F(x)\}$.

We present here characterizations of the graphs of multivalued maps belonging to UB or LB.
At first we formulate some properties of bitopological spaces.

A bitopological space is called pairwise Hausdorff [8] if for each distinct points \( x, y \in Y \) there exist disjoint subsets \( U \in \tau_i, V \in \tau_j \) such that \( x \in U, y \in V \) for \( i, j = 1, 2; i \neq j \).

In \((Y, \tau_1, \tau_2)\) the topology \( \tau_i \) is perfect with respect to \( \tau_j \) if each \( \tau_i \)-open set is \( F_\sigma \) in \((Y, \tau_j)\) for \( i \neq j \) [14].

The bitopological space \((E, \tau_w, \tau_s)\) is pairwise Hausdorff and \( \tau_s \) is perfect with respect to \( \tau_w \) [6].

**Theorem 3.1.** Let \((X, \tau)\) be a perfect space with a \( \sigma \)-locally finite base and let \((Y, \tau_1, \tau_2)\) be a pairwise Hausdorff bitopological space in which \( \tau_2 \) is perfect and has a \( \sigma \)-locally finite base. If \( F: X \to Y \) is a multivalued map such that \( F(x) \in K(Y, \tau_1) \) for \( x \in X \) and \( F \in UB_2(\tau_1) \), then \( \Gamma(F) \) is of the multiplicative class \( \alpha \) in \((X \times Y, \tau \times \tau_2)\).

**Proof.** Let us denote by \( \mathcal{B} = \{V_s: s \in S\} \) a \( \sigma \)-locally finite base of the topology \( \tau_2 \). If \((x, y) \notin \Gamma(F)\), then \( y \notin F(x) \). Since \( F(x) \) is \( \tau_1 \)-compact and \((Y, \tau_1, \tau_2)\) is pairwise Hausdorff, there exist sets \( U(x, y) \in \tau_1 \) and \( V_s \in \mathcal{B} \) such that

\[
F(x) \subseteq U(x, y), \quad y \in V_s \quad \text{and} \quad U(x, y) \cap V_s = \emptyset.
\]

Let \( U_s \) be the union of all sets \( U(x, y) \) satisfying (7). Thus we obtain \( X \times \{y \in V_s \} \) \( U_s \times V_s \). The converse inclusion is evident, so we have

\[
X \times Y \setminus \Gamma(F) = \bigcup_{s \in S} F^+(U_s) \times V_s,
\]

where \( U_s \) and \( V_s \) satisfy (7).

\( \{F^+(U_s) \times V_s: s \in S\} \) is a \( \sigma \)-locally, finite family of sets of the additive class \( \alpha \) in \((X \times Y, \tau \times \tau_2)\). Applying Corollary 2.2 to (8) we have that \( \Gamma(F) \) is of the multiplicative class \( \alpha \) in \((X \times Y, \tau \times \tau_2)\).

Let us note that for a map \( F \in UB_0(\tau_1) \) with \( \tau_1 \)-compact values it is sufficient to assume \((X, \tau)\) any topological space and \((Y, \tau_1, \tau_2)\) pairwise Hausdorff [6], Theorem 4.1.

If \( \tau_1 = \tau_2 \), all spaces are metrizable and \( F \) is a singlevalued map, then Theorem 3.1 coincides with the result of Kuratowski [10], p. 541, and Montgomery [13], Theorem 4.

**Corollary 3.2.** Let \((X, \tau)\) be a perfect space with a \( \sigma \)-locally finite base. If \( F: X \to E \) is a multivalued map such that \( F(x) \in K(E, \tau_w) \) for \( x \in X \) and \( F \in UB_2(\tau_w) \), then \( \Gamma(F) \) is of the multiplicative class \( \alpha \) in \((X \times E, \tau \times \tau_2)\).

**Theorem 3.3.** Let \((X, \tau)\) be a perfect space with a \( \sigma \)-locally finite base and let \((Y, \tau_1, \tau_2)\) be a bitopological space such that \( \tau_1 \subset \tau_2 \), \( \tau_2 \) is perfect with respect to \( \tau_1 \) and \( \tau_2 \) has a \( \sigma \)-locally finite base. If \( F: X \to Y \) is a multivalued map, \( F(x) \in K(Y, \tau_1) \) for \( x \in X \) and \( F \in LB_2(\tau_1) \), then \( \Gamma(F) \) is of the multiplicative class \( \alpha + 1 \) in \((X \times Y, \tau \times \tau_2)\).

**Proof.** Let us denote by \( \{V_{sn}: s \in S_n, n \geq 1\} \) a \( \sigma \)-locally finite base of the
topology $\tau_2$ (for each $n \geq 1$, the family $\{V_{sn}: s \in S_n\}$ is locally finite). If $(x, y) \notin \Gamma(F)$, then $y \in Y \setminus F(x) \in \tau_1$. Thus there exist $V_{sn}$ and $\tau_1$-open set $U_{sn}$ such that $y \in V_{sn} \subset U_{sn} \subset Y \setminus F(x)$. So we have

$$(x, y) \in \bigcup_{n=1}^{\infty} \bigcup_{s \in S_n} (Y \setminus U_{sn}) \times V_{sn}.$$ 

It is easy to verify that

(9) \[ X \times Y \setminus \Gamma(F) = \bigcup_{n=1}^{\infty} \bigcup_{s \in S_n} F^+ (Y \setminus U_{sn}) \times V_{sn}. \]

For each fixed $n$, $\{F^+(Y \setminus U_{sn}) \times V_{sn}: s \in S_n\}$ is a locally finite family of sets of the multiplicative class $\alpha$ in $(X \times Y, \tau \times \tau_2)$. According to Theorem 2.1 the union

$$\bigcup_{s \in S_n} F^+(Y \setminus U_{sn}) \times V_{sn}$$

is of the same class. Therefore, (9) implies that $\Gamma(F)$ is of the additive class $\alpha + 1$ in $(X \times Y, \tau \times \tau_2)$. 

**Corollary 3.4.** Let $(X, \tau)$ be a perfect space. If $F: X \to E$ is a multivalued map such that $F(x) \in \mathcal{C}(E, \tau_w)$ for $x \in X$ and $F \in \mathcal{LB}_2(\tau_w)$, then $\Gamma(F)$ is of the multiplicative class $\alpha + 1$ in $(X \times E, \tau \times \tau_s)$.

**References**


