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On the convergence of Haar series to infinity

Abstract. It is proved that for any series of Haar functions, its partial sum \( \{S_{2^k-1}\} \) cannot converge to \( +\infty \) on a set of positive measure. Immediately we have: For any series of Walsh functions, its partial sum \( \{S_{2^k-1}\} \) cannot converge to \( +\infty \) on a set of positive measure. Moreover, it is observed that the partial sum \( \{S_n\} \) of a Haar series cannot converge to \( +\infty \) on a set of positive measure with suitable condition on \( n_k \). These results are generalizations for the theorems given by Talaljan and Arutjunjan [5]. Although the basic procedures for proving our theorems are similar to those given by the above authors, they need some substantial modifications, and it seems that the case of Walsh functions stated by them is not the consequence for the case of Haar functions. So it is worthwhile to give these generalizations.

1. Introduction. In answering a question, posed by Lusin [2], in connection with the representation of measurable functions, Men'shov proved the following theorem [2], [3] in 1939.

**Theorem I.** Given a measurable function \( f(x) \), finite almost everywhere (abbreviated a.e.) on \([0, 2\pi]\), there is a trigonometric series

\[
a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

which converges to \( f(x) \) a.e. on \([0, 2\pi]\).

The analogous result for Haar system proved by Bary [4] is as follows.

**Theorem II.** If \( f(x) \) is measurable and finite a.e. on \([0, 1]\), then there is a series in terms of Haar functions which converges to \( f(x) \) a.e. on \([0, 1]\).

Up to now it is still unknown whether there is a trigonometric series that converges to \( +\infty \) on a set of positive measure. In this connection Men'shov proved the following theorem [3] in 1947.

**Theorem III.** For each function \( f(x) \) that is either finite a.e. on \([0, 2\pi]\) or equal to \( +\infty \) or \( -\infty \) on a set of positive measure, there is a trigonometric series which converges in measure to \( f(x) \) on \([0, 2\pi]\).

Later, Talaljan [4] proved that Theorem III holds for any normal basis in \( L_p[a, b] \), \( p > 1 \), and in particular, for any complete orthonormal system. More precisely, he proved...
Theorem IV. If \( \{\varphi_n(x)\} \) is a complete orthonormal system in \( L_2[a, b] \), then for each measurable function \( f(x) \), finite a.e. on \( [a, b] \) or equal to \( +\infty \) or \( -\infty \) on a set positive measure, there is a series
\[
\sum_{n=1}^{\infty} a_n \varphi_n(x) \quad (a_n \text{ is real})
\]
which converges in measure to \( f(x) \) on \( [a, b] \).

It follows that not only a trigonometric series but also a series with respect to any complete orthonormal system can converge in measure to \( +\infty \) on a set of positive measure. However, Theorem IV cannot be strengthened by replacing convergence in measure by convergence a.e. In particular, for the complete Haar system, Talaljan and Arutjunjan [5] proved the following theorem.

Theorem V. A series \( \sum_{n=1}^{\infty} a_n \chi_n(x) \) (\( a_n \text{ is real} \)) of Haar functions cannot converge to \( +\infty \) on a set of positive measure.

They also mentioned that Theorem V holds for Walsh series. It seems that the case of Walsh series is not the consequence for the case of Haar series in that paper. So it is interesting to ask whether a subsequence \( \{S_{n_k}(x)\} \) of the partial sums of a Haar series \( \sum_{n=1}^{\infty} a_n \chi_n(x) \) cannot converge to \( +\infty \) on a set of positive measure. In this paper we obtain a generalization for a suitable condition on \( n_k \). For convenience and for notational simplicity, we give the proof of our theorem explicitly for the special case \( n_k = 2^{k-1} \). It follows that the partial sums \( S_{2n-1}(x) \) of a Walsh series cannot converge to \( +\infty \) on a set of positive measure. Therefore, Theorem V holds for the case of Walsh series.

2. Preliminaries and notation. The Haar system \( \{\chi_n^{(k)}(x)\} \) is defined on \([0, 1)\) as follows [1], [4]:
\[
\chi_0^{(0)}(x) = 1 \quad \text{for } x \in [0, 1),
\]
\[
\chi_0^{(1)}(x) = \begin{cases} 
1 & \text{for } x \in [0, \frac{1}{2}), \\
-1 & \text{for } x \in [\frac{1}{2}, 1),
\end{cases}
\]
\[
\chi_n^{(k)}(x) = \begin{cases} 
\sqrt{2^n} & \text{for } x \in \left[ \frac{2k-2}{2^n+1}, \frac{2k-1}{2^n+1} \right), \\
-\sqrt{2^n} & \text{for } x \in \left[ \frac{2k-1}{2^n+1}, \frac{2k}{2^n+1} \right), \\
0 & \text{elsewhere},
\end{cases}
\]
where \( n = 0, 1, 2, 3, \cdots \) and \( k = 1, 2, 3, \cdots, 2^n \).
We denote the Haar system arranged lexicographically by $\chi_m(x)$, i.e.,

$$\chi_1(x) = \chi_0^{(0)}(x), \quad \chi_m(x) = \chi_k^{(k)}(x) \quad \text{if} \quad m = 2^n + k \geq 2.$$ 

The Walsh system $\{W_n(x)\}$ is defined on $[0, 1)$ as follows [1], [4]:

$$W_0(x) = r_0(x) = \chi_0^{(0)}(x) = 1, \quad \text{on} \quad [0, 1).$$

If $n \geq 1$ and $2^v_1 + 2^v_2 + \cdots + 2^v_p$ ($v_1 < v_2 < \cdots < v_p$) is the dyadic representation of $n$, then the functions $W_n(x)$ are defined by $W_n(x) = r_{v_1+1}(x)r_{v_2+1}(x)\cdots r_{v_p+1}(x)$, where $r_i(x)$ are the Rademacher functions.

It is well known that both the Haar system and the Walsh system are complete orthonormal systems. Also it is known that each $W_m(x)$ with $m = 2^n + k$, $0 \leq k < 2^n$, is a linear combination of $\chi_k^{(k)}(x)$ [1].

Hence we have the following remark.

**Remark.** Every Walsh series of the form

$$a_0 W_0(x) + a_1 W_1(x) + \sum_{n=1}^{\infty} \left( \sum_{k=2^n}^{2^{n+1}-1} a_k W_k(x) \right)$$

coincides with a Haar series of the form

$$a_0 \chi_0^{(0)}(x) + a_1 \chi_0^{(1)}(x) + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{2^n} c_k \chi_k^{(k)}(x) \right).$$

3. **Main theorems.** In this section we shall prove our main result by the parallel argument procedure to that in Talaljan and Arutjunjan [5]. In fact, Theorem V is a consequent result of the following theorem.

**Theorem 1.** The subsequence $\{S_{2n-1}(x)\}$ of the partial sums of a series

$$\sum_{k=1}^{\infty} a_k \chi_k(x) \quad (a_k \text{ is real})$$

of Haar functions cannot converge to $+\infty$ on a set of positive measure.

**Proof.** Let

$$T_n(x) = S_{2n-1}(x) = \sum_{k=1}^{2^{n-1}} a_k \chi_k(x).$$

Assume that the sequence $\{T_n(x)\}$ converges to $+\infty$ on a measurable set $E \subset [0, 1)$, with $m(E) > 0$. By Egoroff theorem, there is a closed set $F \subset E$, $m(F) > 0$, such that $\{T_n(x)\}$ converges to $+\infty$ uniformly on $F$. Consequently there is a constant $C$ such that

$$T_n(x) \geq C, \quad x \in F, \quad n = 1, 2, 3, \ldots$$

We may evidently suppose that $C \geq 0$ and $a_1 > 0$. 
Let \( \{a_{d_i}X_{d_i}(x)\} \) be the terms of the Haar series \( \sum_{k=1}^{\infty} a_k X_k(x) \) that differ from zero on subsets of \( F \) of positive measure.

Set
\[
T_n(x) = \sum_{k=1}^{m_n} a_{d_k}X_{d_k}(x), \quad n = 1, 2, 3, \ldots,
\]
where \( \{d_1, d_2, \ldots, d_{m_p}\} \subseteq \{1, 2, 3, \ldots, 2^{p-1}\}, \quad m_p \leq m_{p+1}, \quad p = 1, 2, \ldots, n. \)
Then the sequence \( \{T_n(x)\} \) converges uniformly to \( +\infty \) on a subset \( G \subset F \), with \( m(G) = m(F) \).

It follows from (2) that
\[
T_n(x) \geq 0, \quad x \in G, \quad n = 1, 2, 3, \ldots
\]

We construct an increasing sequence \( \{i_j\} \) as follows. We take \( i_1 \) to be the smallest integer \( k \) for which the interval \( A_{d_k}^- \), where \( a_{d_k}X_{d_k}(x) \) is negative, intersects \( G \) in a set of measure zero. Supposing that \( i_1 < i_2 < \cdots < i_{j-1} \) have been defined, we take \( i_j \) to be the smallest integer \( k \) such that the interval \( A_{d_k}^- \) lies outside the intervals \( A_{d_{i_1}}^- , A_{d_{i_2}}^- , \ldots , A_{d_{i_{j-1}}}^- \) and intersects \( G \) in a set of measure zero.

From the definition of \( \{i_j\} \) we obtain the following properties:

(a) \( m(A_{d_{i_j}}^- \cap G) = 0, \quad j = 1, 2, 3, \ldots, \);

(b) these intervals \( A_{d_{i_j}}^-, j = 1, 2, 3, \ldots, \) are disjoint, therefore \( \sum_{j=1}^{\infty} m(A_{d_{i_j}}^-) < 1; \)

(c) if \( a_{d_k}X_{d_k}(x) \) is a term of (3) different from \( a_{d_{i_j}}X_{d_{i_j}}(x) \), then \( m(A_{d_k}^- \cap G) > 0. \)

Since if \( m(A_{d_k}^- \cap G) = 0 \) for the function \( a_{d_k}X_{d_k}(x) \) in (c), with \( d_{i_j} < d_k < d_{i_{j+1}} \), it is clear from the definition of the Haar system and the definition of \( \{i_j\} \) that \( a_{d_k}X_{d_k}(x) \) is different from zero only in an interval \( A_{d_s}^-, 1 \leq s \leq j \). Then by (a), we have \( a_{d_k}X_{d_k}(x) = 0 \) a.e. on \( G \), contradicting the definition of (3), so (c) holds.

Using (b), there is a \( j_0 \) such that
\[
\sum_{j=j_0+1}^{\infty} m(A_{d_j}^-) < m(G)/4.
\]
From (3) we delete the functions \( a_{d_{i_j}}X_{d_{i_j}}, j \geq j_0 + 1, \) and the functions that are zero outside one of the intervals
\[
A_{d_{i_j}} = A_{d_{i_j}}^+ \cup A_{d_{i_j}}^-, \quad j \geq j_0 + 1,
\]
where \( A_{d_{i_j}}^+ \) is the interval in which \( a_{d_{i_j}}X_{d_{i_j}}(x) \) is positive.
Since \( m(\Delta_{d_{ij}}^+ - \Delta_{d_{ij}}^-) = m(\Delta_{d_{ij}}^-) \), we have from (5)

\[
\sum_{j=J_0+1}^{\infty} m(\Delta_{d_{ij}}^-) < m(G)/2.
\]

Consequently, the deleted functions are zero on a subset \( H \subset G \), with \( m(H) > m(G)/2 \).

Let the series \( \tilde{T}_n(x) = \sum_{k=1}^{r_n} a_k \chi_{t_k}(x) \) consist of the remaining terms of \( \tilde{T}_n(x) \) in (3), where

\[
\{t_1, t_2, \ldots, t_{r_p}\} \subset \{d_1, d_2, \ldots, d_{m_p}\}, \quad r_p \leq r_{p+1}, \quad p = 1, 2, 3, \ldots, n.
\]

It is clear that

\[
\lim_{n \to \infty} \tilde{T}_n(x) = +\infty \quad \text{uniformly on } H.
\]

We shall show that

\[
\tilde{T}_n(x) \geq M_{j_0} \quad \text{for all } x \in [0, 1) \text{ and } n = 1, 2, 3, \ldots
\]

for some constant \( M_{j_0} \).

Let

\[
b_{d_p} = \sup_{x \in [0, 1)} a_{d_p} \chi_{t_p}(x), \quad p = 1, 2, 3, \ldots, j_0,
\]

and put

\[
M_j = -(b_{d_1} + b_{d_2} + \ldots + b_{d_{t_j}}), \quad j = 1, 2, \ldots, j_0.
\]

Since the first term of \( T_n(x) \) in (2) is the first term of \( \tilde{T}_n(x) \) in (7), we have immediately

\[
\tilde{T}_1(x) = a_{t_1} \chi_{t_1}(x) = a_1 > 0, \quad x \in [0, 1).
\]

We divide the proof of (9) into three steps.

Step I. Suppose that \( \tilde{T}_{n-1}(x) \geq 0, \ x \in [0, 1) \), where \( n \) is such that

\( 1 < t_{r_{n-1}} < d_{t_1} \).

If \( t_{r_n} < d_{t_1} \), we have

\[
\tilde{T}_n(x) = \tilde{T}_{n-1}(x) + \sum_{p=r_{n-1}+1}^{r_n} a_{t_p} \chi_{t_p}(x) \geq 0 \quad \text{for } x \not\in \bigcup_{p=r_{n-1}+1}^{r_n} \Delta_{t_p}^-.
\]

However, if \( x \in \Delta_{t_p}^- \), \( r_{n-1}+1 \leq p \leq r_n \), we have

\[
m(\Delta_{t_p}^- \cap G) > 0 \quad \text{and} \quad \tilde{T}_n(x) = \tilde{T}_n(x).
\]

Since \( \tilde{T}_n(x) \) is constant on \( \Delta_{t_p}^- \), it follows from (4) and (13) that

\[
\tilde{T}_n(x) \geq 0 \quad \text{for } x \in \Delta_{t_p}^-, \ r_{n-1}+1 \leq p \leq r_n.
\]
Consequently, we obtain from (12) and (14)

\[ T_n(x) \geq 0, \quad x \in [0, 1) \quad \text{for which } t_r < d_1. \]  

If \( t_r = d_1 \), we establish

\[ T_n(x) \geq 0, \quad x \notin \Delta_{d_1}. \]

There holds (12) with \( t_r = d_1 \) and note that (14) holds for \( x \in \Delta_p, \)
\[ r_n - 1 + 1 \leq p \leq r_n - 1. \]

On the other hand,

\[ T_n(x) \geq -(b_1 + b_2 + \ldots + t_r) = -(b_d + b_2 + \ldots + b_{d_1}) = M_1 \geq M_0. \]

It follows from (15), (16) and (17) for \( t_r \leq d_1 \) that

\[ T_n(x) > 0 \quad \text{for } x \notin \Delta_{d_1} \]
and

\[ T_n(x) \geq M_0, \quad x \in [0, 1). \]

**Step II.** Suppose that \( t_r \leq d_{ij-1} \), for some \( j, 1 < j \leq j_0 \), we have

\[ T_n(x) \geq 0 \quad \text{for } x \notin \Delta_{d_1} \cup \Delta_{d_2} \cup \ldots \cup \Delta_{d_{ij-1}} \]
and

\[ T_n(x) \geq M_0, \quad x \in [0, 1). \]

Consider the partial sum \( T_{n+1}(x) \); it suffices to treat the case that \( d_{is-1} < t_{r+1} \leq d_i \) for some \( s, j \leq s \leq j_0 \).

First, we consider the case \( d_{is-1} < t_{r+1} < d_i \). It follows from (20) that

\[ T_{n+1}(x) \geq 0 \quad \text{for } x \notin \Delta_{d_1} \cup \Delta_{d_2} \cup \ldots \cup \Delta_{d_{is}} \cup ( \bigcup_{p=r_{n+1}}^{r_{n+1}} \Delta_p). \]

However, if \( x \in \Delta_p, \ r_n + 1 \leq p \leq r_{n+1} \), and \( t_p \) is not of the form \( d_{iq}, \ q = j-1, j, \ldots, s \). From the definition of \( T_{n+1} \) in (7) and (c), we have

\[ T_{n+1}(x) = T_{n+1}(x) \quad \text{and} \quad m(\Delta_p \cap \Delta_{d_1}) > 0. \]

Since \( T_{n+1}(x) \) is constant on \( \Delta_p \), it follows from (4) and (23) that

\[ T_{n+1}(x) \geq 0 \quad \text{for } x \in \Delta_p, \ r_n + 1 \leq p \leq r_{n+1}, \]

and \( t_p \) is not of the form \( d_{iq}, \ q = j-1, j, \ldots, s \). Consequently, we obtain from (22) and (24) for \( t_{r+1} < d_i \)

\[ T_{n+1}(x) \geq 0 \quad \text{for } x \notin \Delta_{d_1} \cup \Delta_{d_2} \cup \ldots \cup \Delta_{d_{is}}. \]

Next, let us consider the case \( t_{r+1} = d_i \). We again have the result that
\(T_{n+1}(x) \geq 0\) for \(x \notin \Delta_{d_{i_1}} \cup \Delta_{d_{i_2}} \cup \ldots \cup \Delta_{d_{i_s}}\).

There holds (22) with \(t_{r_{n+1}} = d_{i_s}\) and note that (24) holds for \(x \in \Delta_{t_p}\), \(r_{n+1} - 1 \leq p \leq r_{n+1}\), \(t_p\) is not of the form \(d_{i_q}\), \(q = j - 1, j, \ldots, s\).

Using the same way as (17), we have

\(T_{n+1}(x) = -(b_{t_1} + b_{t_2} + \ldots + b_{t_{r_{n+1}}}) = -(b_{d_1} + b_{d_2} + \ldots + b_{d_s}) = M_s \geq M_{j_0} \).

Hence we draw the conclusion from (25), (26) and (27) for \(t_{r_n} \leq d_{i_0}\)

\(T_n(x) \geq 0\) for \(x \notin \Delta_{d_{i_1}} \cup \Delta_{d_{i_2}} \cup \ldots \cup \Delta_{d_{i_{j_0}}}\)

and

\(T_n(x) \geq M_{j_0}, \quad x \in [0, 1)\). The same way as (17), we have

\(T_t(x) > -(b_{d_1} + b_{d_2} + \ldots + b_{d_s}) = M_s \geq M_{j_0} \).

Hence we draw the conclusion from (25), (26) and (27) for \(t_{r_n} \leq d_{i_0}\)

\(T_n(x) \geq 0\) for \(x \notin \Delta_{d_{i_1}} \cup \Delta_{d_{i_2}} \cup \ldots \cup \Delta_{d_{i_{j_0}}}\)

and

\(T_n(x) \geq M_{j_0}, \quad x \in [0, 1)\).

For \(t_{r_{n+1}}\), one may have

\[d_{i_{j_0}} < t_{r_{n+1}} < d_{i_{j_0+1}}\quad \text{or} \quad d_{i_s} < t_{r_{n+1}} < d_{i_{s+1}}, \quad \text{where} \quad s \geq j_0 + 1.\]

In the first case, one merely needs to consider \(d_{i_{j_0}} < t_{r_{n+1}} < d_{i_{j_0+1}}\).

By the definition of \(T_{n+1}(x)\) in (7), we have

\[T_{n+1}(x) = T_{n+1}(x)\]

and note that \(m(\Delta_{t_p} \cap G) > 0\), and \(T_{n+1}(x)\) is constant on \(\Delta_{t_p}\), therefore

\(T_{n+1}(x) \geq 0, \quad x \in \Delta_{t_p}, \quad r_{n+1} + 1 \leq p \leq r_{n+1}\).

On the other hand, from (30) we have

\(T_{n+1}(x) \geq 0\) for \(x \notin \Delta_{d_{i_1}} \cup \Delta_{d_{i_2}} \cup \ldots \cup \Delta_{d_{i_{j_0}}} \cup \bigcup_{p=r_{n+1}}^{r_{n+1}} \Delta_{t_p}.\)

It follows from (32) and (33) that

\(T_{n+1}(x) \geq 0\) for \(x \notin \Delta_{d_{i_1}} \cup \Delta_{d_{i_2}} \cup \ldots \cup \Delta_{d_{i_{j_0}}} \).
Using (31), we have
\[
\bar{T}_{n+1}(x) \geq M_{j_0} \quad \text{for} \quad x \notin \bigcup_{p=r_{n+1}+1}^{r_{n+1}} \Delta_{i_p}^{-}.
\]
Therefore, by (32) and (35), we obtain
\[
\bar{T}_{n+1}(x) \geq M_{j_0}, \quad x \in [0, 1).
\]
For the second case, \( d_{i_s} < t_{r_{n+1}} < d_{i_{s+1}} \), \( s \geq j_0 + 1 \). Using (30), we have
\[
\bar{T}_{n+1}(x) \geq 0 \quad \text{for} \quad x \notin \Delta_{d_{i_1}}^{-} \cup \Delta_{d_{i_2}}^{-} \cup \ldots \cup \Delta_{d_{i_{j_0}}}^{-} \cup \bigcup_{p=r_{n+1}+1}^{r_{n+1}} \Delta_{i_p}^{-}.
\]
Since \( t_p \) is not of the form \( d_{i_j} \), \( j \geq j_0 + 1 \), where \( r_{n+1} + 1 \leq p < r_{n+1} + 1 \), by (c) we have
\[
m(\Delta_{i_p}^{-} \cap G) > 0.
\]
We can show that
\[
\bar{T}_{n+1}(x) = \bar{T}_{n+1}(x) \quad \text{for} \quad x \in \Delta_{i_p}^{-}, \ r_{n+1} + 1 \leq p \leq r_{n+1},
\]
\[
\bar{T}_{n+1}(x) \quad \text{is constant on} \quad \Delta_{i_p}^{-}, \ r_{n+1} + 1 \leq p \leq r_{n+1}.
\]
So it is the conclusion from (4), (38), (39) and (40) that
\[
\bar{T}_{n+1}(x) \geq 0 \quad \text{for} \quad x \in \Delta_{i_p}^{-}, \ r_{n+1} + 1 \leq p \leq r_{n+1}.
\]
It follows from (37) and (41) that
\[
\bar{T}_{n+1}(x) \geq 0 \quad \text{for} \quad x \notin \Delta_{d_{i_1}}^{-} \cup \Delta_{d_{i_2}}^{-} \cup \ldots \cup \Delta_{d_{i_{j_0}}}^{-}.
\]
Using (31) and the same procedures as (35) and (36), we obtain
\[
\bar{T}_{n+1}(x) \geq M_{j_0}, \quad x \in [0, 1).
\]
Consequently, inequality (9) follows from the three steps, by comparing (11), (18), (19), (28), (29), (42) and (43).

Using (8) and (9), we conclude that
\[
\lim_{n \to +\infty} \int_{0}^{1} \bar{T}_{n}(x) \, dx = \lim_{n \to +\infty} \int_{0}^{1} \bar{T}_{n}(x) \, dx + M_{j_0} = + \infty.
\]

On the other hand, we have by the definition of Haar system
\[
\lim_{n \to +\infty} \int_{0}^{1} \bar{T}_{n}(x) \, dx = \lim_{n \to +\infty} \int_{0}^{1} \sum_{k=1}^{r_n} a_{i_k} \chi_{t_k}(x) \, dx = a_1.
\]

This contradiction shows that Theorem 1 is true.
In order to complete our proof, it remains to prove (39). In fact, it suffices to show that each deleted term in \( T_{n+1}(x) \) is zero on \( A_{r_n+1} \), \( r_n + 1 \leq p \leq r_{n+1} \), \( d_i < t_{r_n+1} < d_i + 1 \).

We divide the deleted terms into the following two types.

(A) The deleted term is \( a_{d_{ij}} \chi_{d_{ij}}(x) \), \( j_0 + 1 \leq j \leq s \). If \( d_{ij} < t_{r_n} \), by the definition of Haar system, we have for \( r_n + 1 \leq p \leq r_{n+1} \)

\[ \Delta_{t_p} \cap \Delta_{d_{ij}} = \emptyset \quad \text{or} \quad \Delta_{t_p} \subseteq \Delta_{d_{ij}}. \]

Suppose that \( \Delta_{t_p} \subseteq \Delta_{d_{ij}} \), then \( a_{t_p} \chi_{t_p}(x) \) is zero outside \( \Delta_{d_{ij}} \), \( j \geq j_0 + 1 \). Hence the term must be deleted, which contradicts the definition of \( T_{n+1}(x) \) in (7). Consequently, we have

\[ \Delta_{d_{ij}} \cap \Delta_{t_p} = \emptyset, \quad r_n + 1 \leq p \leq r_{n+1}. \]

If \( d_{ij} > t_r \), since the term \( a_{d_{ij}} \chi_{d_{ij}}(x) \) has been deleted in (7), it follows that

\[ \Delta_{d_{ij}} \cap \Delta_{t_p} = \emptyset, \quad r_n + 1 \leq p \leq r_{n+1}. \]

So it is clear from (46) and (47) that the deleted terms of the form

\[ a_{d_{ij}} \chi_{d_{ij}}(x), \quad j \geq j_0 + 1 \]

are zero on \( \Delta_{t_p} \), \( r_n + 1 \leq p \leq r_{n+1} \).

(B) The deleted term is \( a_{d_k} \chi_{d_k}(x) \), \( k \) is not of the form \( i_j \), then \( a_{d_k} \chi_{d_k}(x) \) is zero outside some \( \Delta_{d_{ij}} \), \( s \geq j \geq j_0 + 1 \). It follows that

\[ \Delta_{d_k} \subseteq \Delta_{d_{ij}}, \quad s \geq j \geq j_0 + 1. \]

Consequently, we have

\[ \Delta_{d_k} \cap \Delta_{t_p} = \emptyset, \quad r_n + 1 \leq p \leq r_{n+1}. \]

So the deleted terms of the form \( a_{d_k} \chi_{d_k}(x) \) are zero on the interval

\[ \Delta_{t_p} \], \( r_n + 1 \leq p \leq r_{n+1}. \]

Consequently, it follows from (48) and (49) that

\[ \tilde{T}_{n+1}(x) = T_{n+1}(x) \quad \text{on} \quad \Delta_{t_p} \], \( r_n + 1 \leq p \leq r_{n+1}. \]

This finishes the proof of our theorem.

Using Theorem 1 and the remark in preliminaries, we can give a proof for the case of Walsh series. In fact, we can obtain the following generalization.

**Theorem 2.** The subsequence \( \{ S_{2n-1}(x) \} \) of the partial sums of a series

\[ \sum_{k=0}^{\infty} a_k w_k(x) \quad (a_k \text{ is real}) \quad \text{of Walsh functions cannot converge to} \quad +\infty \quad \text{on a set of positive measure}. \]
Furthermore we obtain an immediate consequence of Theorem 2, due to Talaljan and Arutjunjan [5].

**Theorem 3.** A series \( \sum_{k=0}^{\infty} a_k W_k(x) \) (\( a_k \) is real) of Walsh functions cannot converge to \(+\infty\) on a set of positive measure.

4. **Some generalizations.** In the sequel we note that there is an analogue of Theorem 1 for some classes of subsequences of the partial sums of a Haar series. In fact, from the proof of Theorem 1, we can obtain the following generalizations.

**Theorem 4.** The subsequence \( \{S_{2^n-1+p}(x)\} \) of the partial sums of a Haar series cannot converge to \(+\infty\) on a set of positive measure, where \( p \) is a fixed integer.

**Theorem 5.** A sufficient condition that the subsequence \( \{S_{n_k}(x)\} \) of the partial sums of a Haar series cannot converge to \(+\infty\) on a set of positive measure, is the existence of a positive integer \( N \) such that \( k \geq N, \delta_{n_k+1}, \delta_{n_k+2}, \ldots, \delta_{n_k+1} \) are disjoint, where \( \delta_m \) denotes the interval in which \( \chi_m(x) \) is different from zero.

Under the conditions of Theorem 5, as a typical application we have the following corollary.

**Corollary.** The partial sums \( \{S_{n_k+p}(x)\} \) of a Haar series for a fixed integer \( p \) cannot converge to \(+\infty\) on a set of positive measure.

In particular, we can take \( n_k = pk \) for a fixed positive integer \( p \), so the partial sums of the form \( \{S_{pk+q}(x)\}, q = 0, 1, 2, \ldots, p-1, \) of a Haar series cannot converge to \(+\infty\) on a set of positive measure.

**References**


