On weak sequential compactness of generalized Orlicz spaces

In [3] there are given necessary and sufficient conditions for relative $\sigma(L_\Phi, L_\Psi)$-compactness of a subset of an Orlicz space $L_\Phi$. Applying the method of the proof of this theorem and making use of a criterion of weak compactness of a set in $L_1$ (see [2]), we shall prove a theorem of this type for generalized Orlicz spaces $L_\Phi$ generated by convex functions depending on a parameter.

Let $\mu$ be a finite atomless measure in an abstract set $\Omega$. A function $\Phi: \Omega \times R_+ \to R_+ = [0, \infty]$ is called a convex $\varphi$-function, depending on a parameter $t \in \Omega$ if $\Phi(t, u)$ is measurable with respect to $t$ for every $u \geq 0$, and is a convex $\varphi$-function of $u \geq 0$ for every $t \in \Omega$ and $\Phi(t, u)/u \to 0$ as $u \to O^+$, $\Phi(t, u)/u \to \infty$ as $u \to \infty$ for a.e. $t \in \Omega$.

Let $\Psi$ be for every $t \in \Omega$ the function complementary in the sense of Young to the function $\Phi$, with respect to $u$. Then, considering in the Young inequality $u \cdot v \leq \Phi(t, u) + \Psi(t, v)$ the case of equality and writing $\varphi_1 = \frac{1}{2}\varphi$, we have

$$u \cdot \varphi(t, u) = \Phi(t, u) + \Psi(t, \varphi(t, u)) = 2u\varphi_1(t, u),$$

where $\varphi$ and $\psi$ are such that

$$\Phi(t, u) = \int_0^u \varphi(t, s) \, ds,$$

$$\Psi(t, v) = \int_0^v \psi(t, s) \, ds$$

and

$$\frac{1}{2}\Phi(t, u) = \Phi_1(t, u) = \int_0^u \varphi_1(t, s) \, ds.$$

The function $\Psi_1$, complementary to $\Phi_1$ in the sense of Young satisfies the equality $\Psi_1(t, v) = \frac{1}{2}\Psi(t, 2v)$.

We have

$$2u\varphi_1(t, u) = \Phi(t, u) + \Psi(t, \varphi(t, u)),$$

and so

$$u\varphi(t, u) = \Phi(t, u) + \Psi(t, \varphi(t, u)) = 2u\varphi_1(t, u) \leq 2\Phi(t, u) + 2\Psi(t, \varphi_1(t, u)).$$
Hence

\[ \Psi(t, \varphi(t, u)) - 2\Psi(t, \varphi_1(t, u)) \leq \Phi(t, u), \]

and we get

\[ (*) \quad \Psi(t, \varphi_1(t, u)) \cdot \left\{ \frac{\Psi(t, 2\varphi_1(t, u))}{\Psi(t, \varphi_1(t, u))} - 2 \right\} \leq \Phi(t, u) \]

for \( u \geq 0, \ t \in \Omega \). Now, we show that the condition

\[ \int_{\Omega} \Psi(t, 4v_0(t)) \, d\mu < \infty \]

implies \( \int_{\Omega} \Phi(t, \psi(t, 2v_0(t))) \, d\mu < \infty \).

This follows from inequality (2) from [1], which may be applied to functions depending on a parameter \( t \) for every \( t \), separately:

\[ \Phi(t, \psi(u)) < \Psi(t, 2u). \]

Taking in this inequality \( u = 2v_0(t) \) and integrating over \( \Omega \), we obtain the required condition.

The purpose of this paper is to prove the following

**Theorem.** Let \( \Phi \) and \( \Psi \) be convex \( \varphi \)-functions depending on the parameter \( t \in \Omega \) mutually complementary in the sense of Young. Let \( \mu \) be a finite, atomless measure on \( \Omega \). Moreover, let us assume the following condition: there exists a non-negative, integrable over \( \Omega \) function \( h(t) \) such that

\[(**) \quad \forall \ R > 0 \ \exists \ v_0(t) > 0 \ \forall \ v > v_0(t) \]

where \( v_0(t) \) is a measurable function such that

\[ \int_{\Omega} \Psi(t, 4v_0(t)) \, d\mu < \infty. \]

Then we have the following equivalence:

A set \( A \in L_\phi \) is bounded in the sense of the norm of \( L_\phi \) and

\[ (1) \quad \sup_{f \in A} \int_E (f \cdot g) \, d\mu \to 0 \quad \text{as} \ \mu(E) \to 0 \ \text{for every} \ g \in L_\psi \]

if and only if

\[ (2) \quad \text{there exists a number} \ \lambda_0 > 0 \ \text{such that the family of functions} \ \{ \Phi(t, \lambda_0 |f|) : f \in A \} \ \text{is a weakly sequentially compact subset of} \ L_1. \]

**Proof.** First, we shall prove the sufficiency, i.e. that \( (2) \Rightarrow (1) \).

Let us write \( B = \{ \Phi(t, \lambda_0 |f|) : f \in A \} \). The set \( B \subset L_1 \) is a weakly compact subset of \( L_1 \) if

\[ \lim_{\mu(E) \to 0} \int_E g \, d\mu = 0 \quad \text{uniformly with respect to} \ g \in B \ (\text{see [2],} \]

[1] and [2]
Theorem 8.11 (corollary), p. 294) hence condition (2) says that
\[
\lim_{\mu(E) \to 0} \int_{E} \Phi(t, \lambda_0 |f|) d\mu = 0 \quad \text{uniformly with respect to } f \in A.
\]

(i) We first prove that \( A \) is bounded in \( L_\Phi \). By the assumption, we have
\[
\forall \epsilon > 0 \ \exists \delta(\epsilon) > 0 \ \forall \mu(E) < \delta\epsilon \Rightarrow \int_{E} \Phi(t, \lambda_0 |f|) d\mu < \epsilon.
\]
Let us take \( \epsilon = 1 \) and the respective \( \delta_1 = \delta(1) \). Since the measure \( \mu \) is atomless, there exist measurable and pairwise disjoint sets \( E_1, E_2, \ldots, E_{n_0} \) such that \( \Omega = E_1 \cup E_2 \cup \ldots \cup E_{n_0} \) and \( \mu(E_i) < \delta_1 \) for \( i = 1, 2, \ldots, n_0 \). Then
\[
\int_{E_i} \Phi(t, \lambda_0 |f|) d\mu < 1 \quad \text{for every } f \in A, \ i = 1, 2, \ldots, n_0.
\]
Hence
\[
\int_{\Omega} \Phi(t, \lambda_0 |f|) d\mu < \frac{\lambda_0}{n_0} \int_{\Omega} \Phi(t, \lambda_0 |f|) d\mu < \frac{1}{n_0} \cdot n_0 = 1
\]
for every \( f \in A \). Thus \( \|f\|_\Phi < n_0/\lambda_0 \) for every \( f \in A \), and this means the boundedness of \( A \) in \( L_\Phi \).

(ii) Now, we are going to prove that
\[
\sup_{f \in A} \int_{E} |f \cdot g| d\mu \to 0, \quad \text{as } \mu(E) \to 0 \quad \text{for every } g \in L_\Psi.
\]
Applying the Young inequality, we obtain
\[
\lambda \int_{E} |f \cdot g| d\mu \leq \int_{E} \Phi(t, \lambda_0 |f|) d\mu + \int_{E} \psi\left(t, \frac{\lambda}{\lambda_0} |g|\right) d\mu
\]
for every \( \lambda > 0 \) and \( f \in A \). Let us choose \( \lambda > 0 \) so small that
\[
\int_{\Omega} \psi\left(t, \frac{\lambda}{\lambda_0} |g|\right) d\mu < \infty;
\]
such a \( \lambda \) exists, because \( g \in L_\Psi \). By absolute continuity of the integral as a set function, for every \( \epsilon > 0 \) there exists a \( \delta'(\epsilon) > 0 \) such that if \( \mu(E) < \delta'(\epsilon) \), then
\[
\int_{E} \psi\left(t, \frac{\lambda}{\lambda_0} |g|\right) d\mu < \frac{1}{2} \lambda \epsilon.
\]
Thus
\[
\int_{E} |f \cdot g| d\mu \leq \frac{1}{\lambda} \int_{E} \Phi(t, \lambda_0 |f|) d\mu + \frac{1}{2} \epsilon \quad \text{for every } f \in A.
\]
But applying (3) and taking \( \mu(E) < \delta(\frac{1}{2}\lambda \cdot \varepsilon) \), we have

\[
\int_E \Phi(t, \lambda_0 |f|) d\mu < \frac{1}{2}\lambda \varepsilon
\]

for every \( f \in A \).

Hence, writing \( \delta = \min(\delta(\varepsilon), \delta(\frac{1}{2}\lambda \varepsilon)) \) (\( \delta \) does not depend on \( f \)), we obtain

\[
\int_E |f \cdot g| d\mu < \frac{1}{2} \cdot \frac{\lambda \varepsilon}{\lambda} + \frac{1}{2} \varepsilon
\]

for \( \mu(E) < \delta \) and every \( f \in A \).

Thus

\[
\sup_{f \in A} \int_E |f \cdot g| d\mu < \varepsilon \quad \text{for} \quad \mu(E) < \delta.
\]

Hence

\[
\sup_{f \in A} \int_E |f \cdot g| d\mu \to 0 \quad \text{as} \quad \mu(E) \to 0 \quad \text{for every} \quad g \in L_\Phi.
\]

This concludes the proof of the implication (2) \( \Rightarrow \) (1).

Now, we proceed to the proof of necessity, i.e. (1) \( \Rightarrow \) (2). We may suppose without loss of generality that \( Q_\Phi(f) < 1 \) for \( f \in A \). We indirectly prove that \( B \) is a weakly sequentially compact subset of \( L_1 \), i.e. that

\[
\lim_{\mu(E) \to 0} \int E \Phi(t, \lambda_0 |f|) d\mu = 0
\]

uniformly with respect to \( f \in A \); this means that (3) holds for some \( 0 < \lambda_0 < 1 \).

So, let us suppose that the family \( \{ \Phi(t, \lambda_0 |f|): f \in A \} \) is for all \( 0 < \lambda_0 < 1 \) not uniformly absolutely continuous. Thus, writing \( g(t) = \Phi(t, \lambda_0 |f(t)|) \) for \( f \in A \), the relation \( \sup_{g \in B} \int_E g(t) d\mu \to 0 \) as \( \mu(E) \to 0 \) does not hold. In other words,

\[
\exists \forall \exists \exists \{ \mu(E) < \delta(\varepsilon) \text{ and } \int_E g d\mu \geq \varepsilon \}.
\]

We shall show that

\[
\lim_{n \to \infty} \sup_{g \in B} \int_E (g(t) = u_n(t))^+ d\mu > \varepsilon_0 > 0
\]

for every sequence of integrable functions \( 0 \leq u_n(t) \uparrow \infty \).

Let us suppose that if (5) is not true, then there is a sequence of measurable functions \( 0 \leq u_n \uparrow t \) such that \( \lim_{n \to \infty} \sup_{g \in B} \int_E (g(t) - u_n(t))^+ d\mu = 0 \), and so there exists a sequence \( 0 \leq u_n(t) \uparrow \infty \) for which \( \sup_{g \in B} \int_E (g(t) - u_n(t))^+ d\mu \to 0 \), i.e.

\[
\forall \exists \int_E (g(t) - u_n(t))^+ d\mu < \eta \quad \text{for} \quad n > N_\eta \quad \text{where} \quad N_\eta \quad \text{does not depend on} \quad g \in B.
\]

Let us write \( h_n(t) = (g(t) - u_n(t))^+ \); then \( \int h_n(t) d\mu < \eta \), \( n > N_\eta \) uniformly with respect to \( g \in B \). But

\[
h_n(t) = \begin{cases} 
g(t) - u_n(t) & \text{if} \quad g(t) > u_n(t), \\
0 & \text{if} \quad g(t) \leq u_n(t),
\end{cases}
\]

whence

\[
\eta > \int h_n(t) d\mu \geq \int h_n(t) d\mu = \int_{E \cap \{ t : g(t) > u_n(t) \}} g(t) d\mu - \int_{E \cap \{ t : g(t) > u_n(t) \}} u_n(t) d\mu.
\]
However, we have for \( t \in E \cap \{ t : g(t) > u_n(t) \} = G_n, g(t) > u_n(t), \) and the right-hand side of the last inequality is non-negative. Hence

\[
\left| \int_{G_n} g(t) \, d\mu - \int_{G_n} u_n(t) \, d\mu \right| < \eta \quad \text{for } n > N_\eta
\]

and for an arbitrary measurable set \( E \subset \Omega \). Let us denote by \( F_n(E) \) the expression under the sign of absolute value on the left-hand side of the last inequality. Then

\[
\int_E g \, d\mu = \int_{E - G_n} g \, d\mu + \int_{G_n} g \, d\mu \leq \int_{E - G_n} u_n(t) \, d\mu + F_n(E) + \int_{G_n} u_n(t) \, d\mu
\]

Let us take any \( \eta > 0 \) and a respective \( N_\eta \) such that \( |F_n(E)| < \eta \) for \( n > N_\eta \). Let us denote by \( n_\eta \) the least index \( n \) such that \( n > N_\eta \). Then \( |F_n(E)| < \eta \) for every \( E \). Fix \( n_\eta \); then

\[
\int_E g \, d\mu < \int_E u_n(t) \, d\mu + \eta.
\]

From the integrability of \( u_n(t) \) it follows that \( \int_E u_n(t) \, d\mu \) is an absolutely continuous set function. Hence there exists a \( \delta(\eta) > 0 \) such that if \( \mu(E) < \delta(\eta) \), then \( \int_{G_n} u_n(t) \, d\mu < \eta \). Thus, if \( \mu(E) < \delta(\eta) \), then \( \int_E g \, d\mu < 2\eta \) for every \( g \) (uniformly with respect to \( g \)). Consequently, we obtain a contradiction to (4). So we have proved condition (5).

Let \( a^+ = a \) for \( a \geq 0 \), \( a^+ = 0 \) for \( a < 0 \). Since \( u_n(t) \uparrow \) we get \( (g(t) - u_n(t))^+ > (g(t) - u_{n+1}(t))^+ \), and hence

\[
\sup_{g \in B} \int_{\Omega} (g(t) - u_n(t))^+ \, d\mu > \sup_{g \in B} \int_{\Omega} (g(t) - u_n(t))^+ \, d\mu.
\]

Thus

\[
\lim_{n \to \infty} \sup_{g \in B} \int_{\Omega} (g(t) - u_n(t))^+ \, d\mu > \epsilon_0 > 0
\]

implies

\[
\sup_{g \in B} \int_{\Omega} (g(t) - u_n(t))^+ \, d\mu > \epsilon_0 > 0 \quad \text{for every } n.
\]

Hence for each \( n \) there exists an \( f_n \in A \) such that

\[
\int_{\Omega} (\Phi(t, \lambda_0 |f_n(t)|) - u_n(t))^+ \, d\mu > \epsilon_0 > 0 \quad \text{for } n = 1, 2, \ldots
\]

Taking \( R = 2^{n+1} \) in assumption (**), we may find measurable functions \( v_n(t) > 0 \) such that \( \Psi(t, 2v) + h(t) > 2^{n+1} \Psi(t, v) \) for \( v \geq v_n(t) \). Let us write

\[
E_n = \{ t : \Phi(t, \lambda_0 |f_n(t)|) > u_n(t) \},
\]

\[
g_n(t) = \varphi_1(t, \lambda_0 |f_n(t)| \chi_{E_n}(t)) = \varphi_1(t, \lambda_0 |f_n(t)|) \chi_{E_n}(t).
\]
Moreover, let us choose, by definition, integrable functions \( u_n(t) \geq 0 \) such that 
\( u_n(t) \uparrow \infty, u_n(t) \geq 2^n, u_n(t) \geq \Phi(t, \psi(2v_n(t))) \). We have 
\[ \varphi_\Phi(\lambda_0 |f_n|) < \varphi_\Phi(\|f_n\|) \leq 1. \]

Hence 
\[ 1 \geq \int_\Omega \Phi(t, \lambda_0 |f_n|) \, d\mu \geq \int u_n(t) \, d\mu > 2^n(E_n), \]
and so \( \mu(E_n) < \frac{1}{n} \); thus \( \mu(E_n) \to 0 \) as \( n \to \infty \).

Taking \( u = \lambda_0 |f_n(t)| \) and \( t \in E_n \), we have, by inequality (*), that 
\[ \frac{\Psi(t, g_n(t))}{\Psi(t, g_n(t))} = \Phi(t, \lambda_0 |f_n(t)|) \leq \Phi(t, |f_n(t)|). \]

On the other hand, we apply the inequalities appearing in the definition of \( u_n(t) \). Then, taking \( t \in E_n \), we obtain 
\[ [t, \lambda_0 |f_n(t)|] > u_n(t) \geq \Phi[t, \psi(t, 2v_n(t))]; \]

hence 
\[ \lambda_0 |f_n(t)| \geq \psi(t, 2v_n(t)). \]

Since \( \varphi \) is a non-decreasing function, we get 
\[ \varphi[t, \lambda_0 |f_n(t)|] \geq \varphi[\psi(t, 2v_n(t))] = 2v_n(t), \]
\[ g_n(t) = \varphi_1 \{t, \lambda_0 |f_n(t)|\} = \frac{1}{2} \varphi(t, \lambda_0 |f_n(t)|) \geq v_n(t). \]

Hence 
\[ \frac{\Psi(t, 2g_n(t)) + h(t)}{\Psi(t, g_n(t))} \geq 2^{n+1} \geq 2^n + 2, \]
\[ 2^n \Psi(t, g_n(t)) \leq \Psi(t, g_n(t)) \left\{ \frac{\Psi(t, 2g_n(t))}{\Psi(t, g_n(t))} - 2 \right\} + h(t) \]
for \( t \in E_n \). Consequently,
\[ 2^n \Psi(t, g_n(t)) \leq \Phi(t, |f_n(t)|) + h(t) \quad \text{for } t \in E_n. \]

We integrate both sides of this inequality over \( \Omega \):
\[ 2^n \varphi_\Psi(g_n) = 2^n \int_{E_n} \Psi(t, g_n(t)) \, d\mu \leq \int_{E_n} \Phi(t, |f_n(t)|) \, d\mu + \int_\Omega h(t) \, d\mu \]
\[ \leq \varphi_\Phi(\|f_n\|) + \int_\Omega h(t) \, d\mu \leq 1 + \int_\Omega h(t) \, d\mu = C < \infty, \]

since the function \( g_n(t) \) is equal to zero outside the set \( E_n \). Hence \( \varphi_\Psi(g_n) \leq 2^{-n} C \) for \( n = 1, 2, \ldots \).
Let us write \( g(t) = \sup_n g_n(t) \); then
\[
g(t) = \lim_{n \to \infty} \sup_{i \leq n} g_i(t).
\]
Hence
\[
\Psi(t, g(t)) \leq \lim_{n \to \infty} \sup_{i \leq n} \Psi(t, g_i(t)), \quad t \in \Omega,
\]
and so, by the Fatou lemma, we obtain
\[
\varphi_{\Psi}(g) = \int \Psi(t, g(t)) d\mu \leq \lim_{n \to \infty} \int \sup_{i \leq n} \Psi(t, g_i(t)) d\mu
\]
\[
= \lim_{n \to \infty} \varphi_{\Psi}(\sup(g_1, g_2, \ldots, g_n))
\]
\[
\leq \varphi_{\Psi}(g_1) + \varphi_{\Psi}(g_2) + \ldots \leq \sum_{n=1}^\infty 2^{-n} \cdot C = C < \infty.
\]
Thus \( g \in L_{\Psi} \). Applying the case of equality in the Young inequality, we get
\[
2\lambda_0 \int_{E_n} |f_n(t) \cdot g_n(t)| d\mu = \Phi(t, \lambda_0 |f_n(t)| \chi_{E_n}(t)) + \Psi(t, 2g_n(t)).
\]
Hence
\[
2\lambda_0 \int_{E_n} |f_n(t) \cdot g_n(t)| d\mu
\]
\[
= \int_{E_n} \Phi(t, \lambda_0 |f_n(t)|) d\mu + \int_{E_n} \Psi(t, 2g_n(t)) d\mu \geq \int_{E_n} \Phi(t, \lambda_0 |f_n(t)|) d\mu
\]
\[
= \int_{E_n} \left[ \Phi(t, \lambda_0 |f_n(t)|) - u_n(t) \right] d\mu + \int_{E_n} u_n(t) d\mu
\]
\[
> \int_{E_n} \left[ \Phi(t, \lambda_0 |f_n(t)|) - u_n(t) \right]^+ d\mu + \epsilon_0 > 0 \quad \text{(see (7)).}
\]
But \( g(t) = \sup_n g_n(t) \), and so
\[
2\lambda_0 \int_{E_n} |f_n \cdot g| d\mu > 2\lambda_0 \int_{E_n} |f_n \cdot g_n| d\mu > \epsilon_0 > 0,
\]
which yields
\[
\int_{E_n} |f_n \cdot g| d\mu > \frac{\epsilon_0}{2\lambda_0} > 0 \quad \text{for } n = 1, 2, \ldots
\]
However, we have \( g \in L_{\Psi} \). Moreover, \( \mu(E_n) \to 0 \). Since we assume (1), i.e.
\[
sup_{f \in \mathcal{A}} \int_{E_n} |f \cdot g| d\mu \to 0, \quad \text{as } \mu(E) \to 0 \text{ for every } g \in L_{\Psi},
\]
we obtain
\[
\sup_{f \in \mathcal{A}} \int_{E_n} |f \cdot g| d\mu \to 0,
\]
and since $f_n \in A$, we get
\[ \int_{E_n} |f_n \cdot g| \, d\mu \leq \sup_{f \in A} \int_{E_n} |f \cdot g| \, d\mu \to 0, \quad \text{as } n \to \infty, \]
a contradiction to (8). This proves the theorem completely.

Let us remark that assumption (**) in the theorem is needed only in the proof of necessity, i.e. $(1) \Rightarrow (2)$.

References