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# Cesàro sequence spaces of non-absolute type

**1. Introduction.** In [1] Jagers has determined the Köthe duals of the Cesàro sequence spaces $\mathrm{ces}_p$, $1 < p < \infty$, where $\mathrm{ces}_p$ consist of all real sequences $x = (x_k)_{k \in N}$ such that

$$\|x\|_p = \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right]^{1/p} < \infty.$$

The case when $p = 1$ is trivial (see [4]). The case when $p = \infty$ has been considered by the present authors [3]. In this note, we shall define the Cesàro sequence spaces in a different way, that is, with norms $\|x\|_p$ which do not satisfy the absolute property: $\||x|\|_p = \|x\|_p$, where $|x| = (|x_k|)_{k \in N}$, and try to determine their associate spaces, i.e., the Köthe duals.

**2. Normed Köthe sequence spaces.** Let $X$ be the set of all real sequences $x = (x_k)_{k \in N}$. A functional $\varrho$ from $X$ into the non-negative extended real number system is called a *semi-norm* if

(i) $\varrho(0) = 0$,

(ii) $\varrho(ax) = |a|\varrho(x)$,

(iii) $\varrho(x+y) \leqslant \varrho(x) + \varrho(y)$.

If instead of (i) $\varrho$ satisfies the condition that $\varrho(x) = 0$ if and only if $x = 0$, then $\varrho$ is called a *norm*. We denote by $X_\varrho$ the collection of all sequences $x$ satisfying $\varrho(x) < \infty$. Obviously $X_\varrho$ is a linear space, and we call $X_\varrho$ the *normed Köthe sequence space* of non-absolute type with the semi-norm $\varrho$. If $X_\varrho$ is complete with respect to the norm $\varrho$, then $X_\varrho$ is called a *Banach sequence space* of non-absolute type since we did not assume the absolute property.

From now on, let us assume that $X_\varrho$ is a Banach sequence space of non-absolute type. Given a semi-norm $\varrho$, we define a new semi-norm $\varrho'$ as follows:

$$\varrho'(x) = \sup\left\{\left|\sum_{k=1}^{\infty} x_k y_k\right|;\ \varrho(y) \leqslant 1\right\}$$

and we put $\varrho'(x) = \infty$ if the series $\sum_{k=1}^{\infty} x_k y_k$ does not converge for some $y$ satisfying $\varrho(y) \leqslant 1$. The semi-norm $\varrho'$ is called the *associate semi-norm* of $\varrho$. The space $X_{\varrho'}$ consisting of all sequences $x \in X$ with $\varrho'(x) < \infty$ is called the *associate space* of $X_\varrho$. For any $x = (x_k)_{k\in N} \in X_\varrho$ and $y = (y_k)_{k\in N} \in X_\varrho$ we have

$$\Big|\sum_{k=1}^{\infty} x_k y_k\Big| \leqslant \varrho(x)\varrho'(y).$$

A semi-norm $\varrho$ is said to be *saturated* if for every non-empty subset $E$ of $N$, there exists a non-empty subset $F$ of $E$ such that $\varrho(x_F) < \infty$, where the sequence $x_F = (x_k)_{k\in N}$ is defined as $x_k = 1$ if $k \in F$ and $x_k = 0$ if $k \notin F$. It is easy to see that $\varrho$ is saturated if and only if $X_\varrho$ contains all sequences of finite terms, i.e., sequences having only finitely many non-zero terms. The following theorem is a consequence of the Banach–Steinhaus theorem:

THEOREM 2.1. *Let $\varrho'$ be saturated, and $y \in X$. Then $y \in X_{\varrho'}$ if and only if $\sum_{k=1}^{\infty} x_k y_k < \infty$ for all $x \in X_\varrho$.*

**3. Cesàro sequence spaces.** Let $X_p$ $(1 \leqslant p < \infty)$ and $X_\infty$ be respectively the spaces of all $x \in X$ with

$$\|x\|_p = \Big(\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n} x_k\Big|^p\Big)^{1/p} < \infty \quad \text{for } 1 \leqslant p < \infty,$$

and

$$\|x\|_\infty = \sup\Big\{\Big|\frac{1}{n}\sum_{k=1}^{n} x_k\Big|;\ n \in N\Big\} < \infty.$$

Note that the above norms are saturated except for $p = 1$.

THEOREM 3.1. *The space $X_p$ $(1 \leqslant p \leqslant \infty)$ is a Banach sequence space of non-absolute type.*

Proof. Let $(x^{(i)})_{i\in N}$ be a Cauchy sequence in $X_p$ so that for given $\varepsilon > 0$ we have $\|x^{(i)} - x^{(j)}\|_p < \varepsilon$ for all $i, j \geqslant N(\varepsilon)$. We write $x^{(i)} = (x_k^{(i)})_{k\in N}$. Then for fixed $k$, $(x_k^{(i)})_{i\in N}$ converges. If $\lim_{i\to\infty} x_k^{(i)} = x_\kappa$, then

$$\sum_{n=1}^{m}\Big|\frac{1}{n}\sum_{k=1}^{n}(x_k^{(i)} - x_k)\Big|^p \leqslant \varepsilon^p \quad \text{for } i \geqslant N(\varepsilon) \text{ and all } m.$$

Hence letting $m \to \infty$ we have $\|x^{(i)} - x\|_p \leqslant \varepsilon$ for all $i \geqslant N(\varepsilon)$. This proves that $X_p$ is complete for $1 \leqslant p < \infty$. The completeness of $X_\infty$ follows similarly.

THEOREM 3.2. *$\operatorname{ces}_p \subset X_p$ for $1 \leqslant p \leqslant \infty$ and $\operatorname{ces}_p \neq X_p$.*

Proof. The inclusion is trivial. To see that $\operatorname{ces}_p \neq X_p$, we define a sequence $(x_k)_{k\in N}$ by $x_1 = -1$ and $x_k = 2(-1)^k$, $k = 2, 3, \ldots$ which is an element of $X_p$ but not of $\operatorname{ces}_p$ for $1 < p < \infty$. Further, the sequence $(x_k)_{k\in N}$ defined by $x_k = k(-1)^k$ is a member of $X_\infty$ but not of $\operatorname{ces}_\infty$. Finally the sequence $(x_k)_{k\in N}$ defined by $x_1 = -1$ and $x_k = (-1)^k(2k-1)/k(k-1)$ for $k = 2, 3, \ldots$ is an element of $X_1$ but not of $\operatorname{ces}_1$.

THEOREM 3.3. *Let $1 \leqslant p \leqslant \infty$ and $\sigma$ be defined on $X_p$ by $\sigma(x) = (\sigma_n(x))_{n\in N}$, where $\sigma_n(x) = \frac{1}{n}\sum_{k=1}^{n} x_k$. Then $\sigma$ is an one-to-one bounded linear transformation from $X_p$ onto the sequence space $l_p$ with operator norm 1.*

The proof is easy. The result will be used in an essential way in a proof in the next section.

**4. The associate space of $X_p$.** Let $Y_q$ be the space of all sequences $y \in X$ such that

(1) $|ky_k| \leqslant M$ for all $k \in N$,

(2) $\lambda_q(y) = \left(\sum_{k=1}^{\infty} |k(y_k - y_{k+1})|^q\right)^{1/q} < \infty$ for $1 \leqslant q < \infty$,

and $\lambda_\infty(y) = \sup\{|k(y_k - y_{k+1})|;\ k \in N\} < \infty$.

We shall show that $Y_q$ is the associate space $X_p'$ of $X_p$, where $1/p + 1/q = 1$ and with $\lambda_q = \|\cdot\|_p'$ the associate norm of $\|\cdot\|_p$.

LEMMA 4.1. *If $y = (y_k)_{k\in N}$ is an element in the associate space $X_p'$ of $X_p$, then the sequence $(ky_k)_{k\in N}$ is bounded. In particular, when $p = \infty$, $ky_k \to 0$ as $k \to \infty$.*

Proof. Let $y = (y_k)_{k\in N}$ be an element of $X_p'$. Then $\sum_{k=1}^{\infty} x_k y_k < \infty$ for every $x = (x_k)_{k\in N}$ in $X_p$. This implies that $x_k y_k \to 0$ as $k \to \infty$. Since $x = (k(-1)^k)_{k\in N}$ is an element of $X_\infty$, we have $ky_k \to 0$ as $k \to \infty$. In general, write $s_n = \frac{1}{n}\sum_{k=1}^{n} x_k$, and we have

$$y_k[ks_k - (k-1)s_{k-1}] \to 0 \quad \text{as } k \to \infty \tag{A}$$

for all sequences $(s_k)_{k\in N}$ in $l_p$ by Theorem 3.3. So (A) is also true for all sequences $((-1)^k|s_k|)_{k\in N}$ in $l_p$, i.e.,

$$(-1)^k y_k[k|s_k| + (k-1)|s_{k-1}|] \to 0 \quad \text{as } k \to \infty.$$

It follows that $ky_k s_k \to 0$ as $k \to \infty$. Now we claim that $(ky_k)_{k\in N}$ is bounded. For $p = 1$, it is easy. Suppose that the sequence is not bounded for $1 < p < \infty$. Then there exists a subsequence $(k_j y_{k_j})_{j\in N}$ such that $|k_j y_{k_j}| > j$ for $j \in N$. Take $r$ such that $0 < r < 1$ and $pr > 1$, and define a sequence

$(s'_k)_{k\in N}$ in $l_p$ by

$$s'_k = \begin{cases} (ky_k)^{-r} & \text{when } k = k_j,\ j = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $k_j y_{k_j} s'_{k_j} = (k_j y_{k_j})^{1-r}$ does not tend to zero as $j\to\infty$ which leads to a contradiction. Therefore the sequence must be bounded.

THEOREM 4.2. *The associate space $X'_p$ of $X_p$ $(1 \leqslant p \leqslant \infty)$ is the space $Y_q$ with the norm $\lambda_q$, where $1/p+1/q = 1$.*

Proof. Let $y \in X'_p$. Then $\sum_{k=1}^{\infty} x_k y_k$ is convergent for all $x \in X_p$. Now we apply Abel transformation and obtain

$$\sum_{k=1}^{m} x_k y_k = \sum_{k=1}^{m-1} k(y_k - y_{k+1})\sigma_k + m y_m \sigma_m,$$

where $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} x_k$. Since $x \in X_p$, it follows that $\sigma_n \to 0$ as $n\to\infty$ for $1 \leqslant p < \infty$ and that $(\sigma_n)_{n\in N}$ is bounded for $p = \infty$. In view of Lemma 4.1, the last term in the above equality tends to zero as $m\to\infty$ and hence

$$\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} k(y_k - y_{k+1})\sigma_k.$$

Therefore applying Theorem 3.3 we see that the above series converges for all sequences $\sigma = (\sigma_k)_{k\in N}$ belonging to $l_p$. It is known that $\big(k(y_k - y_{k+1})\big)_{k\in N}$ belongs to $l_q$, where $1/p+1/q = 1$ and that

$$\lambda_q(y) = \sup\Big\{\Big|\sum_{k=1}^{\infty} k(y_k - y_{k+1})\sigma_k\Big|;\ \|\sigma\|_{l_p} \leqslant 1\Big\}$$
$$= \sup\Big\{\Big|\sum_{k=1}^{\infty} x_k y_k\Big|;\ \|x\|_p \leqslant 1\Big\} = \|y\|'_p.$$

Hence, together with Lemma 4.1, we have proved that $X'_p \subset Y_q$.

Conversely, let $y = (y_k)_{k\in N}$ be an element in $Y_q$. Since the norm $\lambda_q$ is saturated, then by Theorem 2.1 and Abel transformation we need only to prove that $my_k\sigma_k \to 0$ as $k\to\infty$ for $y \in Y_q$ and $x \in X_p$. The case when $1 \leqslant p < \infty$ follows easily from the facts that $(ky_k)_{k\in N}$ is bounded and $\sigma_k \to 0$ as $k\to\infty$. For $p = \infty$, $(\sigma_k)_{k\in N}$ is bounded and $\lim_{k\to\infty} y_k = 0$. It follows that

$$|ky_k| \leqslant \sum_{j=k}^{\infty} k\,|y_j - y_{j+1}| \leqslant \sum_{j=k}^{\infty} j\,|y_j - y_{j+1}|.$$

By letting $k \to \infty$, we obtain that $ky_k \to 0$ as $k \to \infty$. This completes the proof.

We remark that in defining $X_p$, the $l_p$ norm involved may be replaced by Orlicz or Luxemburg norm for an Orlicz sequence space. Hence we have defined an Orlicz space of non-absolute type. Its associate space can also be found. The details, together with its general theory, are being worked out and will appear in a separate paper.

**References**

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