On \( m \)-paracompact spaces II

In a recent paper [2], we have obtained some results on \( m \)-paracompact spaces. In the present note, we obtain some characterizations of \( m \)-paracompact spaces and show that the corresponding known results for paracompact spaces obtained by Tamano [4], Vaughan [5], Michael [1] and authors [3] follow as corollaries to our results.

**Definitions.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two families of subsets of a space \( X \). \( \mathcal{A} \) is said to be **cushioned** in \( \mathcal{B} \) if there exists a cushion map \( f: \mathcal{A} \to \mathcal{B} \) such that for every subfamily \( \mathcal{A}' \) of \( \mathcal{A} \) we have

\[
\bigcup \{ A': A' \in \mathcal{A}' \} \subseteq \bigcup \{ f(A'): A' \in \mathcal{A}' \}
\]

\( \mathcal{A} \) is said to be **linearly-cushioned** in \( \mathcal{B} \) with cushion map \( f: \mathcal{A} \to \mathcal{B} \) if there is a linear ordering \( ' \prec ' \) on \( \mathcal{A} \) such that for every subfamily \( \mathcal{A}' \) of \( (\mathcal{A}, ' \prec ') \) for which there exists an \( A \in \mathcal{A} \) such that \( A' \prec A \) for all \( A' \in \mathcal{A}' \) we have

\[
\bigcup \{ A': A' \in \mathcal{A}' \} \subseteq \bigcup \{ f(A'): A' \in \mathcal{A}' \}.
\]

\( \mathcal{A} \) is said to be **order cushioned** in \( \mathcal{B} \) with cushion map \( f: \mathcal{A} \to \mathcal{B} \) if there exists a well ordering \( ' \prec ' \) such that for every subfamily \( \mathcal{A}' \) of \( \mathcal{A} \) and \( A \in \mathcal{A} \) such that \( A' \prec A \) for all \( A' \in \mathcal{A}' \) we have

\[
\text{Cl}_A \left( \bigcup \{ A' \cap A: A' \in \mathcal{A}' \} \right) \subseteq \bigcup \{ f(A'): A' \in \mathcal{A}' \}.
\]

The concept of cushioned families was introduced by Michael [1]. The notion of linearly cushioned families is due to Tamano [4], who assumed the ordering to be a well ordering. However, it was proved by Vaughan [5] that if \( \mathcal{A} \) is linearly cushioned in \( \mathcal{B} \) with respect to a linear order, then there also exists a well ordering with respect to which \( \mathcal{A} \) is linearly-cushioned in \( \mathcal{B} \).

The concept of order cushioned families has been introduced by the authors [3].

**Theorem.** For a space \( X \), the following are equivalent:

(a) \( X \) is normal and \( m \)-paracompact.
(b) Every open covering of $X$ of cardinality $\leq m$ has an open, cushioned refinement.

c) Every open covering of $X$ of cardinality $\leq m$ has a $\sigma$-cushioned open refinement.

d) Every open covering of $X$ of cardinality $\leq m$ has a linearly-cushioned, open refinement.

e) Every open covering of $X$ of cardinality $\leq m$ has an order cushioned, open refinement.

(f) Every open covering of $X$ of cardinality $\leq m$ has a cushioned refinement.

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be any open covering of $X$ of cardinality $\leq m$. Since $X$ is $m$-paracompact, there exists a locally-finite, open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$. Since $X$ is normal there exists a locally-finite, open refinement $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of $\mathcal{V}$ such that $W_\lambda \subseteq V_\lambda$ for each $\lambda \in \Lambda$. Then $\{W_\lambda : \lambda \in \Lambda\}$ is an open cushioned refinement of $\mathcal{U}$.

(b) $\Rightarrow$ (c). Obvious.

c) $\Rightarrow$ (d). Let $\mathcal{U}$ be any open covering of $X$ of cardinality $\leq m$. Let $\mathcal{V} = \bigcup_{n=1}^\infty \mathcal{V}_n$ be a $\sigma$-cushioned, open refinement of $\mathcal{U}$ so that each $\mathcal{V}_n$ is cushioned in $\mathcal{U}$. Let each $\mathcal{V}_n$ be well ordered and define an order $\prec$ in $\mathcal{V}$ as follows: if $V_1, V_2 \in \mathcal{V}$, then $V_1 < V_2$ if (i) $V_1 < V_2$ in the ordering of $\mathcal{V}_n$ in case both $V_1, V_2 \in \mathcal{V}_n$ (it being assumed without any loss of generality that different $\mathcal{V}_n$'s do not have any members in common) and (ii) $V_1 < V_2$ if $V_1 \prec V_2$ in $\mathcal{V}_n$ and $n < m$. With this ordering, it is easy to verify that $\mathcal{V}$ is a linearly-cushioned, open refinement of $\mathcal{U}$.

d) $\Rightarrow$ (e). Let $\mathcal{U}$ be any open covering of $X$ of cardinality $\leq m$. Then, there exists a linearly-cushioned, open refinement $\mathcal{V}$ of $\mathcal{U}$ with cushion map $f: \mathcal{V} \rightarrow \mathcal{U}$. It can be assumed that the ordering of $\mathcal{V}$ is a well-ordering (cf. Vaughan [5], Theorem 1). We shall show that $\mathcal{V}$ is order cushioned in $\mathcal{U}$. Let $\mathcal{V}'$ be any subfamily of $\mathcal{V}$ and let $V \in \mathcal{V}'$ be such that $V' < V$ for all $V' \in \mathcal{V}'$. Then $\text{Cl}_V[\bigcup\{V' \cap V : V' \in \mathcal{V}'\}] = \text{Cl}_V[\bigcup\{V' : V' \in \mathcal{V}'\}] \cap V = \bigcup\{V' : V' \in \mathcal{V}'\} \cap V = \bigcup\{V' : V' \in \mathcal{V}'\} \subseteq \bigcup\{f(V') : V' \in \mathcal{V}'\}$.

Hence $\mathcal{V}$ is an order cushioned refinement of $\mathcal{U}$.

e) $\Rightarrow$ (f). Let $\mathcal{U}$ be any open covering of $X$ of cardinality $\leq m$. Let $\mathcal{V}$ be an open, order cushioned refinement of $\mathcal{U}$ with cushion map $f: \mathcal{V} \rightarrow \mathcal{U}$. We shall show that there exists a cushioned refinement of $\mathcal{U}$. Let $W_v = V \sim \bigcup\{V' : V' < V\}$ for each $V \in \mathcal{V}$ and $\mathcal{W} = \{W_v : V \in \mathcal{V}\}$. Let $g: \mathcal{V} \rightarrow \mathcal{U}$ be defined as $g(W_v) = f(V)$ for each $W_v \in \mathcal{W}$. It will be shown that $\mathcal{W}$ is cushioned in $\mathcal{U}$ with cushion map $g$. First, let $x \in X$. 

Then there exists a smallest $V \in \mathcal{V}$ such that $x \in V$. Obviously then $x \in W_V$. So $\mathcal{V}$ is a covering of $X$ each member of which is clearly contained in some member of $\mathcal{U}$. Let $\mathcal{V}'$ be any subfamily of $\mathcal{V}$. We shall show that

$$\bigcup \{W_V: W_V \in \mathcal{V}'\} \subset \bigcup \{g(W_V): W_V \in \mathcal{V}'\}.$$  

Let $y \in \bigcup \{W_V: W_V \in \mathcal{V}'\}$. There exists $V \in \mathcal{V}$ such that $y \in V$. Let $M$ be any other open set containing $y$. Then $M \cap V$ is an open set containing $y$. Also, since $V \cap W_{V'} = \emptyset$ for all $V' > V$, therefore $(M \cap V) \cap \bigcup \{W_V: V' < V, W_V \in \mathcal{V}'\} \neq \emptyset$ and thus $y \in \bigcup \{W_V: V' < V, W_V \in \mathcal{V}'\}$. Let $\mathcal{V}'' = \{W_{V'}: V' < V, W_{V'} \in \mathcal{V}'\}$ and let $\mathcal{V}''' = \{V' \in \mathcal{V}: W_V \in \mathcal{V}''\}$. Since $V' < V$ for all $V' \in \mathcal{V}'''$, therefore $\mathcal{V}'''$ is majorized and hence we have

$$\text{Cl}_V[\bigcup \{V': V' < V, V' \in \mathcal{V}''\}] \cap V = \text{Cl}_V[\bigcup \{V' \cap V: V' \in \mathcal{V}''\}] \leq \bigcup \{f(V'): V' \in \mathcal{V}''\} = \bigcup \{g(W_V): W_V \in \mathcal{V}'\} \leq \bigcup \{g(W_V): W_V \in \mathcal{V}'\}.$$

Now, since $V$ is an open set, therefore we have

$$\text{Cl}_V[\bigcup \{V': V' \in \mathcal{V}''\}] \cap V = \bigcup \{V': V' \in \mathcal{V}''\} \cap V.$$

But $y \in \bigcup \{W_{V'}: W_{V'} \in \mathcal{V}''\} \subset \bigcup \{V': V' \in \mathcal{V}''\}$ and also $y \in V$. Therefore,

$$y \in \text{Cl}_V[\bigcup \{V': V' \in \mathcal{V}''\}] \cap V \subseteq \bigcup \{g(W_V): W_V \in \mathcal{V}'\}.$$

Thus, $\mathcal{V}$ is a cushioned refinement of $\mathcal{U}$.

(f) $\Rightarrow$ (a). This follows in view of a theorem obtained by authors (cf. [2], Theorem 1.10).

From the above theorem follows the following important corollary:

**Corollary 1.** For a space $X$, the following are equivalent:

(a) $X$ is normal and paracompact.

(b) Every open covering of $X$ has an open, cushioned refinement.

(c) Every open covering of $X$ has an open, $\sigma$-cushioned refinement.

(d) Every open covering of $X$ has a linearly-cushioned, open refinement.

(e) Every open covering of $X$ has an order cushioned, open refinement.

(f) Every open covering of $X$ has a cushioned refinement.

In view of the fact that every paracompact, regular space is normal, we have the following known results which follow from the above corollary:

**Corollary 2** (Michael [1]). A regular space is paracompact iff every open covering has a cushioned (or open, $\sigma$-cushioned) refinement.

**Corollary 3** (Tamano [4]). A completely-regular space is paracompact iff every open covering has a linearly-cushioned open refinement.
COROLLARY 4 (Vaughan [5]). A regular space is paracompact iff every open covering has a linearly-cushioned, open refinement.

COROLLARY 5 (Authors [3]). A regular space is paracompact iff every open covering has an order cushioned, open refinement.

References


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