On the spectrum of the Laplace operator

1. Introduction. Let us consider in the Euclidean space $\mathbb{R}^k$, $k > 1$, a bounded domain $U$ and let its boundary be denoted by $\partial U$. In various physical considerations one is led to the eigenvalue problem

$$\frac{1}{2} \Delta u = \mu u, \quad "u = 0 \text{ on } \partial U".$$

For $U$ with smooth boundary the condition "$u = 0$ on $\partial U$" simply means that $u(x) \to 0$ as $x \to z \in \partial U$, $x \in U$. It is well known that under certain smoothness hypothesis on $\partial U$ the domain $D(\Delta)$ of the Laplace operator corresponding to this boundary condition can be chosen in such a way that $\frac{1}{2}\Delta$ as an operator in $L^2(U)$ is symmetric, has discrete spectrum and the orthonormal set \{\varphi_j\} of eigenfunctions corresponding to the eigenvalues \{\mu_j\} is complete.

H. Weyl established in 1915 (cf. [13], pp. 41-45) the asymptotic formula

$$(1.1) \quad \lim_{n \to \infty} \frac{-\mu_n}{n^{2/k}} = 2\pi \left( \frac{\Gamma( \frac{1}{2} k + 1) }{|U|} \right)^{2/k},$$

where $|U|$ is the Lebesgue measure of $U$.

About twenty years later T. Carleman succeeded in proving (cf. [3])

$$(1.2) \quad \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} q_j^2(x) = \frac{1}{|U|} \quad \text{for } x \in U.$$
2. Potential theory. Most of the results and definitions mentioned here can be found in [2].

The fundamental solution for $\frac{1}{2} \Delta$ in $\mathbb{R}^k$ is known to be

\[
(2.1) \quad h(x) = \begin{cases} 
\frac{1}{\pi} \log |x| & \text{for } k = 2, \\
\frac{1}{2 \pi^{k/2}} \Gamma(\frac{1}{2} k - 1) |x|^{2-k} & \text{for } k > 2;
\end{cases}
\]

i.e. in the distribution sense

\[
(2.2) \quad \frac{1}{2} \Delta h_y = -\delta_y, \quad y \in \mathbb{R}^k,
\]

where $h_y(x) = h(x - y)$ and $\delta_y$ is the $\delta$-Dirac distribution concentrated at $y$.

The letter $U$ is reserved for fixed but arbitrary bounded domain in $\mathbb{R}^k$.

The harmonic measure for $U$ corresponding to $x \in U$ is denoted by $\mu_x$; $C(\partial U)$ denotes the set of all continuous functions on $\partial U$. The Wiener generalized solution of the Dirichlet problem with the boundary function $f \in C(\partial U)$ is then equal to

\[
Hf(x) = \int_{\partial U} f(z) \mu_x(dz), \quad x \in U.
\]

A point $z \in \partial U$ is said to be regular if and only if for all $f \in C(\partial U)$

\[
(2.3) \quad Hf(x) \to f(z) \quad \text{as } x \to z, \quad x \in U.
\]

The set of all regular points is denoted by $\partial_r U$. Now we are ready to write the formula for the Green function of $U$

\[
(2.4) \quad G(x, y) = h(x - y) - H_{h_y}(x), \quad x, y \in U.
\]

It can be shown that

\[
H_{h_y}(x) = H_{h_x}(y) \quad \text{for } x, y \in U,
\]

whence the symmetry of $G(x, y)$ follows.

The set of all real valued bounded and continuous functions on $U$ is denoted by $C(U)$ and the set of all bounded Borel functions on $U$ by $B(U)$. For $f \in B(U)$ we define $\|f\| = \sup \{|f(x)| : x \in U\}$. Clearly $[C(U), \|\|]$ and $[B(U), \|\|]$ are Banach spaces.

Let us write

\[
Hf(x) = \int_U h(x - y)f(y) dy, \quad x \in \mathbb{R}^k.
\]
It can be seen from (2.1) that

\begin{equation}
H: B(U) \rightarrow C(R^k),
\end{equation}

where $C(R^k)$ is the set of all continuous functions on $R^k$.

The Green operator $G$ is defined by formula

\begin{equation}
Gf(x) = \int_{U} G(x, y)f(y)\,dy, \quad x \in U.
\end{equation}

This is a good place to introduce the set

\[ C_0(U) = \{ f \in C(U) : \text{for each } z \in \partial U, f(x) \rightarrow 0 \text{ as } x \rightarrow z, x \in U \}, \]

it is immediate that $C_0(U)$ is a closed subspace of the Banach space $C(U)$.

Definitions (2.1) and (2.4) can be used to show that $G$ is a continuous operator on $B(U)$ and, moreover, that $G: B(U) \rightarrow C(U)$.

Now, let $f \in B(U)$. According to (2.6), (2.4) and (2.5) we have

\[ Gf(x) = g(x) - H_0(x) \quad \text{with } g = Hf \]

whence by (2.3) $Gf \in C_0(U)$. Thus,

\begin{equation}
G: B(U) \rightarrow C_0(U).
\end{equation}

This property will still hold after we replace $B(U)$ by the set of all bounded Lebesgue measurable functions on $U$.

In the sequel we are going to employ the following maximum principle: If $h$ is harmonic on $U$ and $h \in C_0(U)$, then $h = 0$.

The domain of the Laplace operator is defined as follows

\begin{equation}
D(\Delta) = \{ f \in C_0(U) \cap C^2(U) : \Delta f \in C(U) \},
\end{equation}

where $C^2(U)$ is the set of all functions $f$ on $U$ with continuous partial derivatives of order two.

It is very convenient to state in this place the following identity

\begin{equation}
f = G(-\frac{1}{2} \Delta f) \quad \text{for } f \in D(\Delta).
\end{equation}

For the proof let $f \in D(\Delta)$ and $g = G(-\frac{1}{2} \Delta f)$. It follows from (2.7) and (2.8) that $f - g \in C_0(U)$. According to (2.2) and (2.4) we have $\Delta g = \Delta f$ in the distribution sense and therefore $g - f$ is harmonic. Thus, the above maximum principle gives the required equality $g = f$.

3. Heat equation. In the time-space $(0, \infty) \times R^k$ the fundamental solution for the heat equation

\begin{equation}
\frac{1}{2} \Delta u = \frac{\partial u}{\partial t}
\end{equation}
is known to be
\begin{equation}
(3.2) \quad p(t, x, y) = (2\pi t)^{-\frac{1}{2}} \exp \left( -\frac{1}{2t} |x-y|^2 \right).
\end{equation}

For fixed \( y \in \mathbb{R}^k \) the function is the unique positive solution corresponding to the initial distribution \( \delta_y \) and to the boundary condition \( p(t, x, y) \to 0 \) as \( x \to \infty \) for each positive \( t \).

Now let \( U \) be bounded domain in \( \mathbb{R}^k \). We are interested in the fundamental solution for equation (3.1) on \((0, \infty) \times U \) corresponding to the initial distribution \( \delta_y \) and to the boundary values zero, i.e. we would like to have for each \( y \in U \) on \((0, \infty) \times U \) a positive function \( q(\cdot, \cdot, y) \) satisfying (3.1) and such that \( q(t, \cdot, y) \to \delta_y \) with \( t \to 0_+ \) and for each \( t > 0 \), \( q(t, x, y) \to 0 \) with \( x \to z \in \partial_x U, \ x \in U \). The existence and uniqueness of such \( q(t, x, y) \) can be established either probabilistically or with the aid of the axiomatic potential theory of H. Bauer (cf. [8], [9], [5] and [1]). Either approach can be used to derive the following properties of \( q(t, x, y) \):

1° \( 0 < q(t, x, y) < p(t, x, y) \) on \((0, \infty) \times U \times U \).

2° \( q(t, x, y) \) is symmetric in \( x \) and \( y \).

3° For \( t > 0, s > 0 \) and \( x, y \in U \) we have
\[ q(t+s, x, y) = \int_U q(t, x, z)q(s, z, y)\,dz. \]

4° For fixed \( t > 0 \) and \( x \in U \), \( q(t, x, \cdot) \) is in \( C_0(U) \).

5° Let
\[ Q_t f(x) = \int_U f(y)q(t, x, y)\,dy, \]
and let \( f \in C(U) \). Then for each \( x \in U \) we have
\begin{equation}
(3.3) \quad Q_t f(x) \to f(x) \quad \text{with} \quad t \to 0_+. \tag{3.3}
\end{equation}

6° \( \{Q_t, t > 0\} \) is a semigroup of operators in \( B(U) \) with \( \|Q_t\| \leq 1 \) and, moreover,
\begin{equation}
(3.4) \quad Q_t : B(U) \to C(U). \tag{3.4}
\end{equation}

7° For \( x, y \in U \) we have
\begin{equation}
(3.5) \quad G(x, y) = \int_0^\infty q(t, x, y)\,dt. \tag{3.5}
\end{equation}

8° For each \( x \in U \) there exists on \((0, \infty) \times U \) a Borel probability measure \( \nu_x \) such that
\begin{equation}
(3.6) \quad q(t, x, y) = p(t, x, y) - \int_{E_t} p(t-s, z, y)\nu_x(ds, dz), \tag{3.6}
\end{equation}
where \( E_t = (0, t) \times U \).
In the Banach space $B(U)$ we can introduce in a natural way the notion of weak convergence (cf. [6], pp. 77-79) which is characterized as follows: $f = \text{wlim} f_n$ for $f_n, f \in B(U)$, i.e. $f$ is a weak limit of the sequence $\{f_n\}$ if and only if the sequence $\{\|f_n\|\}$ is bounded and $f_n(x) \to f(x)$ for $x \in U$.

For the semigroup $Q_t: B(U) \to B(U)$ the set

$$B_0(U) = \{f \in B(U): f = \text{wlim}_{t \to 0^+} Q_t f\}$$

is called the invariant subspace.

Notice that from (3.3) and from the inequality $\|Q_t\| \leq 1$ it follows

(3.7)

$$C(U) \subseteq B_0(U).$$

The weak infinitesimal operator of the semigroup $\{Q_t\}$ is defined by the formula

(3.8)

$$Af = \text{wlim}_{t \to 0^+} \frac{Q_t f - f}{t}.$$}

The domain of $A$ is denoted by $D(A)$ and it is defined as the set of all $f \in B_0(U)$ for which the right-hand side of (3.8) exists and belongs to $B_0(U)$.

The resolvent operator $R_\lambda$ for the semigroup $\{Q_t\}$ is defined for $f \in B(U)$ as follows

(3.9)

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x) dt, \quad \lambda > 0, \ x \in U.$$

The case of $\lambda = 0$ needs to be discussed separately. Notice that for $f \geq 0$ according to (3.9) and (3.5) we have $R_\lambda f \neq Gf$ as $\lambda \searrow 0$ and therefore for arbitrary $f \in B(U)$, $R_\lambda f(x) \to Gf(x)$ as $\lambda \to 0^+$ and $x \in U$. Moreover,

$$\|R_\lambda f\| \leq \|R_\lambda 1\| \leq \|G\| \|f\| \leq \|G1\| \|f\|.$$

Thus,

$$Gf = \text{wlim}_{\lambda \to 0^+} R_\lambda f, \quad f \in B(U),$$

but this means that $G$ is the potential operator of the semigroup $\{Q_t\}$. Since $G$ is bounded we can apply a known result on weak infinitesimal operators for semigroups of contractions (cf. [6], p. 65) to obtain

(3.10)

$$G: B_0(U) \to D(A),$$

$$-A: D(A) \to B_0(U),$$

and that $G$ is the inverse to $-A$, i.e.

(3.11)

$$(-A)Gf = f \quad \text{for } f \in B_0(U),$$

$$G(-A)f = f \quad \text{for } f \in D(A).$$
Combining (3.7) and (3.10) we find that $G[C(U)] \subset D(A)$ whence by (2.9) we get
\[(3.12) \quad D(A) \subset D(A).\]

Now we would like to check
\[(3.13) \quad Af = \frac{1}{2} Af \quad \text{for} \quad f \in D(A).\]

This can be seen as follows. Let $f \in D(A)$. According to (2.9) $f = G(-\frac{1}{2} Af)$, where $-\frac{1}{2} Af \in C(U) \subset B_0(U)$ and therefore (3.11) gives $Af = AG(-\frac{1}{2} Af) = \frac{1}{2} Af$. Thus we have shown that $A$ is an extension of $A$.

4. The eigenvalue problem. Consider the Hilbert space $L^2(U)$ with the scalar product
\[
(f, g) = \int_U f(x)g(x)dx, \quad \|f\|_2 = \sqrt{(f, f)}.
\]

The Green operator $G$ is a self-adjoint weakly singular integral operator in $L^2(U)$ with the kernel $G(x, y)$. The weak singularity follows from (2.1) and (2.4), i.e. we have
\[
0 \leq G(x, y) \leq \frac{C}{|x-y|^3}, \quad x, y \in U
\]
with some constants $C$ and $0 < \delta < k$. Defining
\[
G^{(0)}(x, y) = G(x, y), \quad G^{(m+1)}(x, y) = \int_U G(x, z)G^{(m)}(z, y)dz
\]
we find that (cf. [12], p. 75)
\[(4.1) \quad G^{(m)}(x, y) \leq C_m, \quad C_m = \text{const}, \quad x, y \in U.
\]

This inequality allows to establish the compactness of $G$ in $L^2(U)$ and in particular that its spectrum is discrete with 0 as the only limit point.

Using Property 3° of Section 3 and (3.5) one shows (cf. [4]) that $G$ is positive, i.e. $(Gf, f) > 0$ for $f \neq 0, f \in L^2(U)$. This implies that 0 is not an eigenvalue and that the spectrum of $G$ is non-negative. Thus, there exists in $L^2(U)$ an orthonormal complete set of eigenvectors $\{\varphi_j\}$ such that the corresponding eigenvalues $\lambda_j$ satisfy the inequalities $\lambda_{j+1} \leq \lambda_j$.

It is clear that $G^m \varphi_j = \lambda_j^m \varphi_j$ and therefore (4.1) gives
\[(4.2) \quad \|\varphi_j\| \leq \frac{C_m}{\lambda_j^m} |U|^{1/2}.
\]
Consequently the eigenfunctions $\varphi_j$ are bounded and Lebesgue measurable and therefore by the modified property (2.7) we get

$$G\varphi_j = \lambda_j \varphi_j \quad \text{and} \quad \varphi_j \in C_0(U) \quad \text{for all } j. \tag{4.3}$$

Applying to both sides of (4.3) the Laplace operator in the distribution sens we obtain

$$\frac{1}{2} \Delta \varphi_j + \frac{1}{\lambda_j} \varphi_j = 0. \tag{4.4}$$

Since the operator $\frac{1}{2} \Delta - \lambda_j^{-1}$ is elliptic we can apply the H. Weyl Lemma ([7], p. 140, Corollary 4.1.2) to find that the functions $\varphi_j$ are in $C^\infty(U)$. Moreover, (4.2) gives $\frac{1}{2} \Delta \varphi_j = \lambda_j^{-1} \| \varphi_j \| < \infty$ whence $\frac{1}{2} \Delta \varphi_j \in C(U)$ and therefore $\varphi_j \in D(\Delta)$. Consequently, if we write $\mu_j$ for $-\lambda_j^{-1}$, then in the classical sense

$$\frac{1}{2} \Delta \varphi_j = \mu_j \varphi_j, \quad \varphi_j \in D(\Delta). \tag{4.4}$$

Since $\{\varphi_j\}$ is a basis it follows that $D(\Delta)$ is dense in $L^2(U)$. Using (2.9) we get $(Af, g) = (f, A\varphi_j)$ for $f, g \in D(\Delta)$. This and (4.4) show that the operator $\frac{1}{2} \Delta$ with the domain $D(\Delta)$ is symmetric, it has a discrete spectrum and an orthonormal complete set of eigenvectors $\{\varphi_j\}$ to which there correspond the eigenvalues $\{\mu_j\}$, $\mu_j = -\lambda_j^{-1}$.

5. The resolvent equation. It is known that for $\lambda > 0$ the operator $(\lambda I - \Delta)^{-1}$ maps $D(\Delta)$ onto $B_0(U)$ in one-to-one way and that it is equal to the resolvent operator $R_\lambda$ (cf. [6], p. 65) defined in (3.9). Thus for $g \in B_0(U)$ and $f = R_\lambda g$ we have

$$\lambda f - Af = g.$$  

Applying to both sides the Green operator $G$ we obtain by (3.11)

$$\lambda Gf + f = Gg.$$  

Introducing $g_\lambda = \lambda R_\lambda g$ we get

$$\lambda Gg_\lambda + g_\lambda = \lambda Gg.$$  

This and inequalities (4.1) and (4.2) give for some $m$

$$g_\lambda(x) = \sum_{i=1}^{m} (-\lambda)^i G^i g(x) + (-\lambda)^{m+1} \sum_{j=1}^{\infty} \frac{\lambda_j^{m+1}}{1 + \lambda \lambda_j} (g, \varphi_j) \varphi_j(x),$$

and the series converges uniformly and absolutely on $U$. 

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It is seen from (3.9) that the last equality can be written by means of the Laplace transform. The uniqueness theorem on Laplace transform and a simple additional argument lead to the formula

\[ Q_t g(x) = \sum_{j=1}^{\infty} e^{-t\lambda_j} (g, \varphi_j) \varphi_j(x), \quad x \in U, \ t > 0, \ g \in B_0(U). \]

Now, \( C(U) \subset B_0(U) \) and therefore (5.1) gives

\[ q(t, x, y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \varphi_j(x) \varphi_j(y), \]

where \( t > 0 \) and \( x, y \in U \).

For \( U \) with smooth boundary formula (5.2) is well known.

6. Theorems of H. Weyl and T. Carleman. To formulate the final results it is more natural to consider the semigroup \( \{Q_t\} \) and its potential \( G \) as operators acting in \( L^2(U) \).

Denoting by \( \|Q_t\|_2 \) the norm of \( Q_t \) in \( L^2(U) \) we check easily with the help of (5.2) that \( \|Q_t\|_2 \leq 1 \).

Let \( Q \) denote the \( L^2(U) \) infinitesimal operator of the semigroup \( \{Q_t\} \) and let \( D(Q) \) be its domain. We are going to show that

\[ D(Q) = \{ g = Gf : f \in L^2(U) \}. \]

Let \( D \) denote the right-hand side of (6.1). The positiveness of \( G \) implies that the mapping \( G : L^2(U) \to D \) is one-to-one. It should be clear that formula (5.1) gives

\[ Q_t g = \sum_{j=1}^{\infty} e^{-t\lambda_j} (g, \varphi_j) \varphi_j \quad \text{for } g \in L^2(U) \]

with the right-hand side convergent absolutely and uniformly on \( U \).

Suppose that \( g \in D \), i.e. \( g = Gf, f \in L^2(U) \). The completeness of \( \{\varphi_j\} \) and (6.2) give

\[ \left\| \frac{Q_t g - g}{t} + f \right\|_2^2 = \sum_{j=1}^{\infty} \left[ \lambda_j e^{-t\lambda_j} - \frac{1}{t} + 1 \right]^2 (f, \varphi_j)^2 \to 0 \quad \text{as } t \to 0^+. \]

Thus, \( g \in D(Q) \) and, moreover,

\[ QGf = -f \quad \text{for } f \in L^2(U). \]

Conversely, let \( g \in D(Q) \). Then

\[ \left\| \frac{G Q_t g - g}{t} + g \right\|_2^2 = \sum_{j=1}^{\infty} \left[ \lambda_j e^{-t\lambda_j} - \frac{1}{t} + 1 \right]^2 (g, \varphi_j)^2 \to 0 \quad \text{as } t \to 0^+, \]
and therefore
\begin{equation}
GQg = -g \quad \text{for } g \in D(Q),
\end{equation}
whence \( g \in D \).

We conclude from (6.3) and (6.4) that \( Q \) is the inverse to \(-G\). Since \( G \) is self-adjoint it follows that \( Q \) is self-adjoint too.

The Lebesgue dominated convergence theorem implies that \( D(A) \subset D(Q) \) hence by (3.12) \( D(A) \subset D(Q) \). Now, (2.9) and (6.3) give \( Qf = \frac{1}{2} \Delta f \) for \( f \in D(A) \).

Recapitulating this discussion we can make the following statement:
The symmetric operator \( \frac{1}{2} \Delta \) given on \( D(A) \) has \( Q = G^{-1} \) as its self-adjoint extension.

It remains to prove asymptotic formulas (1.1) and (1.2).

For the proof of (1.1) let \( U_\delta = \{ x \in U : |x - z| \geq \delta \text{ for } z \notin U \} \). It is clear that \( |U_\delta| |U|^{-1} \rightarrow 1 \) with \( \delta \rightarrow 0^+ \). Property 8 of Section 3 gives for \( x \in U_\delta \) and \( t \leq \delta^2/k \)
\begin{equation}
0 \leq (2\pi t)^{-ik} - q(t, x, x) \leq (2\pi t)^{-ik} \exp \left( -\frac{\delta^2}{2t} \right).
\end{equation}

Since \( 0 \leq q(t, x, y) \leq p(t, x, y) \) we have for \( x \in U \setminus U_\delta \) and \( t \leq \delta^2/k \)
\begin{equation}
0 \leq (2\pi t)^{-ik} - q(t, x, x) \leq (2\pi t)^{-ik}.
\end{equation}

Integrating (6.5) over \( U_\delta \) and (6.6) over \( U \setminus U_\delta \) we obtain
\begin{align*}
0 & \leq |U|(2\pi t)^{-ik} - \int_U q(t, x, x) \, dx \\
& < |U_\delta|(2\pi t)^{-ik} \exp \left( -\frac{\delta^2}{2t} \right) + |U \setminus U_\delta|(2\pi t)^{-ik},
\end{align*}

hence for \( t \leq \delta^2/k \)
\begin{equation}
0 \leq 1 - \frac{(2\pi t)^{ik}}{|U|} \int_U q(t, x, x) \, dx \leq \exp \left( -\frac{\delta^2}{2t} \right) + \left( 1 - \frac{|U_\delta|}{|U|} \right).
\end{equation}

Now let \( t = \delta^3 \); then, for small \( t \), \( t = \delta^3 \leq \delta^2/k \) and therefore
\begin{equation}
\int_U q(t, x, x) \, dx \approx \frac{|U|}{(2\pi t)^{ik}} \quad \text{as } t \to 0^+.
\end{equation}

This and (5.2) give
\begin{equation}
\sum_{j=1}^{\infty} e^{-t/\delta_j} \approx \frac{|U|}{(2\pi t)^{ik}} \quad \text{for } t \to 0^+.
\end{equation}
The Tauberian theorem ([14], p. 192) gives for \( \lambda \to \infty \)

\[
\sum_{-\mu_j < \lambda} 1 \simeq \frac{|U| \lambda^k}{(2\pi)^k \Gamma(\frac{1}{2} k + 1)}.
\]

Substituting \( \lambda = -\mu_n \) we obtain (1.1).

The proof of (1.2) is now easy. Let \( x \in U \) be fixed and let \( \delta \) be such that \( x \in U_\delta \); then by (6.5) we obtain

\[
q(t, x, x) \simeq (2\pi t)^{-k}, \quad t \to 0_+,
\]

whence again by (5.2)

\[
\sum_{j=1}^\infty e^{-t/4} q_j^2(x) \simeq (2\pi t)^{-k}.
\]

The same Tauberian theorem gives for \( \lambda \to \infty \)

\[
\sum_{-\mu_j > \lambda} q_j^2(x) \simeq \frac{\lambda^k}{(2\pi)^k \Gamma(\frac{1}{2} k + 1)}.
\]

Substituting \( \lambda = -\mu_n \) and then applying (1.1) we obtain (1.2).

References