Maria Zofia Banaszynska, Alina Chudzyszka

ON LOCAL ONE-PARAMETER GROUPS OF LOCAL TRANSFORMATIONS IN DIFFERENTIAL SPACES

In the paper there has been considered a problem of mutual correspondence between the vector fields and local one-parameter groups of local transformations in the category of differential spaces generated by a single function.

On a $C^\infty$-manifold, every vector field $X$ corresponds to an equivalence class of local one-parameter groups of local transformations in a $1:1$ manner.

In this paper we consider the problem of a correspondence between the vector fields and local one-parameter groups of local transformations for a differential space $(M,C)$, where $C$ is a differential structure on $M$ generated by a single function $f$.

We wish to thank Prof. W. Waliszewski for suggesting the problem.

Throughout this paper, by a differential space we shall mean a couple $(M,C)$, where $C$ is a differential structure on a set $M$ in the sense of Sikorski (see [1]). If $h$ is a smooth mapping from a differential space $(M,C)$ into a differential space $(N,C')$, we write $h : (M,C) \rightarrow (N,C')$. The set of all $C^\infty$-functions on the set $\mathbb{R}$ of all real numbers is denoted by $C^\infty(\mathbb{R})$. Let $V$ be an open subset of $\mathbb{R}$ with the natural topological structure. By $C^\infty(V)$ we denote the set of all $C^\infty$-functions on $V$. The set of all $C^\infty$-functions on $\mathbb{R} \times \mathbb{R}$ is denoted by $C^\infty(\mathbb{R}^2)$. $J_E$ designates the differential space $(I_E,$
$C^0(I, \mathcal{C})$, where $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. $C(A)$ denotes the set of all functions $f|A$, where $f \in C$ and $A$ is an open set of the topology of $(M, C)$.

Let $(M, C)$ be a differential space, and let $U$ and $V$ be two open sets of the topology of $(M, C)$. A diffeomorphism $\psi : (U, C(U)) \to (V, C(V))$ is called a local transformation of $(M, C)$. $U$ is called the domain of $\psi$.

A local one-parameter group of local transformations of $(M, C)$ is a set $\{U_a, 0 < a < \varepsilon\}$, where $U_a$ is an open set of the topology of $(M, C)$, $0 < a < \varepsilon$ a positive number, and $\psi_t(a)$ a local transformation of $(M, C)$ for each $t$, $|t| < \varepsilon$, satisfying the following conditions:

1) $\{U_a\}_{a \in A}$ is an open cover of $(M, C)$,
2) the domain of $\psi_t(a)$, $|t| < \varepsilon$, contains $U_a$, and $\psi_0(a)$ is the identity transformation on $U_a$, the map $(t, p) \mapsto \psi_t(a)(p)$ is a smooth map from $J \times U_a \times C(U_a)$ into $(M, C)$,
3) if $0 < s < t < \varepsilon$, then $\psi_t(a) \circ \psi_s(a)$ is defined, its domain contains $U_a$ and $(\psi_t(a) \circ \psi_s(a))(q) = \psi_{t-s}(a)(q)$ for $q \in U_a$,
4) if $U_a \cap U_B \neq \emptyset$, then for each point $p \in U_a \cap U_B$ one can choose $\varepsilon < \min(\varepsilon_a, \varepsilon_B)$ such that, for $|t| < \varepsilon$, $\psi_t(a)$ and $\psi_t(b)$ agree on a sufficiently small neighbourhood of $p$.

Let $G = \{U_a, 0 < a < \varepsilon\}$ and $G' = \{V_y, 0 < y < \varepsilon\}$ be two local one-parameter groups of local transformations. We say that $G$ and $G'$ are equivalent and write $G \sim G'$ if the following conditions is satisfied: if $U_a \cap V_y \neq \emptyset$, then for each point $p \in U_a \cap V_y$, there is a number $\delta > 0$, $\delta < \min(\varepsilon_a, \varepsilon_y)$, such that, for $|t| < \delta$, $\psi_t(a)$ and $\psi_t(y)$ agree on a sufficiently small neighbourhood of $p$.

To $G$, we can associate a vector field $X$ on $(M, C)$ as follows. For $p \in M$ and $f \in C$ we define the value $X(p)(f)$ of the vector field $X$ at $p$ by

\[ X(p)(f) = \frac{d}{dt} (f \circ \psi_t(a))(p) \bigg|_{t=0}, \quad p \in U_a. \]
The vector field \( \mathbf{X} \) is called the infinitesimal transformation of the local one-parameter groups \( G_t \) of local transformations.

**Lemma.** Let \( C \) be the smallest differential structure on \( M \) containing a function \( f \), and let \( \mathbf{X} \) be the infinitesimal transformation of two local one-parameter groups of local transformations \( G_t = \{ U_\alpha, \xi_\alpha, \varphi_t^{(\alpha)} \}_{\alpha \in \Lambda} \) and \( G_{t_1} = \{ V_{Y_1}, \eta_{Y_1}, \psi_{t_1}^{(Y_1)} \}_{Y_1 \in Y} \) of \( (M, C) \). Then, for each point \( p_0 \in U_\alpha \cap V_{Y_1} \), there exist a neighbourhood \( U \), \( U \subset U_\alpha \cap V_{Y_1} \), of \( p_0 \) and \( 0 < \varepsilon < \min(\varepsilon_\alpha, \eta_{Y_1}) \), such that

\[
(f \circ \varphi_t^{(\alpha)})(p) = (f \circ \psi_{t_1}^{(Y_1)})(p) \quad \text{for } p \in U, t \in I_t.
\]

**Proof.** Suppose \( G_t = \{ U_\alpha, \xi_\alpha, \varphi_t^{(\alpha)} \}_{\alpha \in \Lambda} \) and \( G_{t_1} = \{ V_{Y_1}, \eta_{Y_1}, \psi_{t_1}^{(Y_1)} \}_{Y_1 \in Y} \) have as their infinitesimal transformation the same vector field \( \mathbf{X} \). Let \( p_0 \) be a point of \( U_\alpha \cap V_{Y_1} \). \( \mathbf{X}(f) \) is a smooth function on \( (M, C) \). Thus there exist a neighbourhood \( U \) of \( p_0 \) contained in \( U_\alpha \cap V_{Y_1} \) and a function \( g \in C^\infty(M) \), such that

\[
\mathbf{X}(f)(p) = (g \circ f)(p) \quad \text{for } p \in U.
\]

Denote by the symbols \( \varphi_t^{(\alpha)} \) and \( \psi_{t_1}^{(Y_1)} \) the smooth mappings, \( (t, p) \mapsto \varphi_t^{(\alpha)}(p) \) from \( (U_\alpha, C(U_\alpha)) \times J_{\alpha} \) into \( (M, C) \) and \( (t, p) \mapsto \psi_{t_1}^{(Y_1)}(p) \) from \( (V_{Y_1}, C(V_{Y_1})) \times J_{Y_1} \) into \( (M, C) \), respectively. Because of the continuity of the mappings \( \varphi_t^{(\alpha)} \) and \( \psi_{t_1}^{(Y_1)} \) it follows that there are neighbourhoods \( U_\alpha \) of \( p_0 \) contained in \( U_\alpha \) and a positive number \( \varepsilon_0 < \min(\varepsilon_\alpha, \eta_{Y_1}) \) such that \( \varphi_t^{(\alpha)}(U_\alpha) \cup \psi_{t_1}^{(Y_1)}(U_\alpha) \subset C U_\alpha \) for \( |t| < \varepsilon_0 \).

From condition (3) of the definition of a local one-parameter group of local transformations, one sees that, for \( p \in U_\alpha \cap V_{Y_1} \), \( |t| < \varepsilon_0 \),

\[
\mathbf{X}(\varphi_t^{(\alpha)}(p))(f) = \frac{d}{dt} (f \circ \varphi_t^{(\alpha)})(p),
\]

(4)

\[
\mathbf{X}(\psi_{t_1}^{(Y_1)}(p))(f) = \frac{d}{dt} (f \circ \psi_{t_1}^{(Y_1)})(p).
\]

Next from (3) we have
The mapping \( f \circ \varphi(\alpha) \) belongs to \( C(U_a) \times C^\infty(I\xi_a) \). The mapping \( f \circ \varphi(\gamma) \) belongs to \( C(V_\gamma) \times C^\infty(I\eta_\gamma) \). Therefore, there exist a neighbourhood \( U_2 \) of \( p_0 \) contained in \( U_1 \), a positive number \( \varepsilon < \varepsilon_0 \) and mappings \( F_\varphi \) and \( F_\psi \in C^\infty(\mathbb{R}^2) \), such that

\[
(f \circ \varphi(\alpha)(p,t) - F_\varphi(f(p),t),
(6) \\
(f \circ \varphi(\gamma)(p,t) = F_\psi(f(p),t).
\]

Thus, if \( p \in U_2 = U \), \( |t| < \varepsilon_1 \) from (4), (5) and (6) we get

\[
\frac{d}{dt} F_\varphi(f(p),t) = (g \circ F_\varphi)(f(p),t), \quad F_\varphi(f(p),0) = f(p),
\]

\[
\frac{d}{dt} F_\psi(f(p),t) = (g \circ F_\psi)(f(p),t), \quad F_\psi(f(p),0) = \dot{f}(p).
\]

From the uniqueness theorem for solutions of differential equations we have

\[
F_\varphi(f(p),t) = F_\psi(f(p),t) \quad \text{for} \quad p \in U, \quad |t| < \varepsilon.
\]

So, by (6) we obtain (2).

Theorem. Let \( C \) be the smallest differential structure on a set \( M \) which contains a function \( f \) and let the topology \( (M, C) \) satisfy Kolmogorov's separation axiom \( T_\infty \). If two local one-parameter groups of local transformations have the same vector field as their infinitesimal transformation, then they are equivalent.

Proof. Since the topology of \( (M, C) \) satisfies Kolmogoroff's separation axiom \( T_\infty \), then \( f \) is one-to-one. Suppose \( G_1 = \{ \alpha, \varphi(\alpha) \} \) and \( G_2 = \{ \gamma, \psi(\gamma) \} \) have, as their
infinitesimal transformations, the same vector field $X$. We shall show that $G_1 \sim G'_1$. From Lemma, for $p \in U \cap V$, there exist a neighbourhood $U$ of $p$ contained in $U \cap V$, and $0 < \varepsilon < \min \{C, \varepsilon_p, \varepsilon_q\}$, such that

$$(f \circ \psi_{\varepsilon}^a)(p) = (f \circ \psi_{\varepsilon}^a)(p)$$

for $p \in U$, $|t| < \varepsilon$. Hence, $\psi_{\varepsilon}^a$ and $\psi_{\varepsilon}^q$ coincide on $U$ for $|t| < \varepsilon$, and this completes the proof.

If the topology of a differential space does not satisfy the separation axiom $T_0$, then the 1:1 correspondence between the vector fields and the equivalence classes of one-parameter groups of local transformations can fail. Now we shall give suitable examples.

**Example 1.** Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projection $(\pi_1(p_1, p_2) = p_1)$ and let $\mathcal{C}$ be the smallest structure on $\mathbb{R}^2$ containing $\pi_1$. Let us put

$$\psi_t^a(p_1, p_2) = (p_1 + t, p_2 + t),$$

$$\psi_t^q(p_1, p_2) = (p_1 + t, p_2 - t).$$

$\{\mathbb{R}^2, \varepsilon, \psi_t^a\}$ and $\{\mathbb{R}^2, \varepsilon, \psi_t^q\}$, $\varepsilon > 0$, are two different, not equivalent, local one-parameter groups of local transformations which have the same vector field as their infinitesimal transformations.

Professor Z. Moszner in his paper (which will appear in Tensor) considered the function $\psi_1 : \mathbb{R}^2 \to \mathbb{R}$ given by the formula

$$\psi_1(p, t) = \begin{cases} 
0 & \text{for } p = 0, t \in \mathbb{R}, \\
\text{sgn}(\ln|p| + t) |p|e^t & \text{for } p < 0, t \in (-1, 0), t \in \mathbb{R}, \\
-\text{sgn}(\ln|p| + t) |p|e^t & \text{for } p \in (0, 1) \cup (-\infty, -1), t \in \mathbb{R}.
\end{cases}$$

This function will be used in the following example.

**Example 2.** Let $r$ be a nonzero number and let $t \in \mathbb{R}$. Put
Let \( C \) be the smallest differential structure on \( R \) which contains the function \( f \). Consider the differential space \((R,C)\). For each \( \varepsilon \neq 0\), \( G^\varepsilon = \{R, \varepsilon, \phi^\varepsilon_t\} \) and \( G^1 = \{R, \varepsilon, \phi^1_t\}, \varepsilon > 0\), are two different, not equivalent, local one-parameter groups of local transformations which have the same vector field as their infinitesimal transformations.

REFERENCES


Institute of Mathematics
University of Łódź

Maria Zofia Banaszczuk, Alina Chądzyńska

O LOKALNYCH JEDNOPARAMETRowych GRUPACH PRZEKszTAŁCEŃ NA PRZESTRZENiACH RÓŻNICZKOWYCH

W pracy rozważany jest problem wizualnej odpowiedniości pomiędzy polami wektorowymi i lokalnymi jednoparametrowymi grupami przekształceń lokalnych w kategorii przestrzeni różniczkowych generowanych przez jedną funkcję.