A NON-DEGENERATE $\sigma$-DISCRETE MOORE SPACE
WHICH IS CONNECTED

BY

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A space is $\sigma$-discrete if it is the union of a countable number of sets none of which has a limit point in the space. It is well known that no non-degenerate connected Moore space can have only countably many points and it might seem a natural generalization that no such space could be $\sigma$-discrete. This note* provides an example to show that such a generalization is not valid.

First we define a Moore space $S'$ which is $\sigma$-discrete but not connected. The construction follows closely that of Bing (1). We then enlarge $S'$ to the desired space $S$.

Let $V_1, V_2, \ldots$ be distinct vertical lines in the plane whose union is dense in the plane. The points of $S'$ are the points in this union. For each point $p$ in $S'$ and for each positive integer $n$, the region of $S'$ centred at $p$ of size $n$, denoted by $R_n(p)$, is the set containing only $p$ and those points of $S'$ which lie in the interior of the largest circle in the plane which

1. is tangent at $p$ to the vertical line in the plane containing $p$,
2. lies, except for $p$, to the right of this vertical line,
3. has diameter less than or equal to one and contains in its interior no point in any one of $V_1, \ldots, V_n$.

There is a development $G_1', G_2', \ldots$ for $S'$ such that, for each positive integer $n$, $G_n'$ is the set of all regions of $S'$ which are centred at a point of $S'$ and which have integral size greater than or equal to $n$.

For each point $p$ of $S'$ define $A(p)$ to be a point set conumerous with the real numbers and such that, if $p$ and $q$ are two points of $S'$, then $A(p)$ intersects neither $A(q)$ nor $S'$. When convenient, $A(p)$ is referred to as

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the set of points above \( p \). The union of all the point sets just defined is denoted by \( S'' \). The points of \( S \) are the points in the union of \( S' \), \( S'' \) and \( \{ \omega \} \), where \( \omega \) is some point which is neither in \( S' \) nor in \( S'' \), and the topology of \( S \) is determined by the development \( G_1, G_2, \ldots \) described in the sequel.

For each positive integer \( i \) let \( C_i \) be a set of circles in the plane such that

1. each circle in \( C_i \) lies to the right of \( V_i \) and does not intersect \( V_i \),
2. no two circles in \( C_i \) intersect or have interiors which intersect,
3. if \( R' \) is a region in \( S' \) centred at a point in \( V_i \), then there is a circle in \( C_i \) the common part of whose interior with \( S' \) lies entirely in \( R' \).

Note that \( C_i \) is countable. Further, for each positive integer \( n \) define \( C^n_i \) to be the set to which \( X \) belongs only if, for some circle \( X' \) in \( C_i \), with radius \( r \), \( X \) is the circle concentric with \( X' \) of radius \( r/n \). If \( X_1, X_2, \ldots \) is a sequence in \( C_i \) converging in the plane to a point \( p \) in \( V_i \), then we denote by \( X^n_1, X^n_2, \ldots \) a definite sequence in \( C^n_i \) such that

1. there is a positive integer \( j \) such that, for each integer \( k \) greater than \( j \), \( X^n_{k-j} \) is concentric with \( X_k \),
2. each of \( X^n_1, X^n_2, \ldots \) lies in the interior of a circle in the plane which is centred at \( p \), has radius less than \( 1/n \) and does not intersect any one of the verticals \( V_1, \ldots, V_n \) different from \( V_i \).

In addition, let \( \bar{X}^n \) denote the set to which a point belongs only if it is in \( S' \) and it or its reflexion in a horizontal line through \( p \) lies in the interior of one of \( X^n_1, X^n_2, \ldots \). For each \( p \) in \( V_i \) let \( M(p) \) be a set of sequences in \( C_i \) such that

1. each sequence in \( M(p) \) converges in the plane to \( p \),
2. no term of any sequence in \( M(p) \) intersects \( R_i(p) \) or contains a point with the second coordinate greater than the second coordinate of \( p \),
3. no two sequences in \( M(p) \) share more than a finite number of terms,
4. if \( X_1, X_2, \ldots \) is any sequence in \( C_i \) converging in the plane to \( p \) no term of which intersects \( R_i(p) \) or has a point with the second coordinate greater than the second coordinate of \( p \), then there is a sequence in \( M(p) \) which has infinitely many terms in common with \( X_1, X_2, \ldots \).

5. \( M(p) \) is encomorous with \( A(p) \).

Let \( W \) be a one-to-one function with domain \( S'' \) such that, for each \( p \) in \( S' \), \( W(A(p)) \) is \( M(p) \). Finally, \( G_n \) is defined to be the set to which \( R \) belongs if and only if one of the following holds:

1. \( R \) is the set of all points \( x \) in \( S \) such that there is a region \( R' \) in \( G_n \) centred at \( p \), and \( x \) is either in \( R' \) or above some point in \( R' \) different from \( p \);
2. \( R \) is the set of all points \( x \) in \( S \) such that there are a point \( q \) in \( S'' \) and a positive integer \( m \) greater than or equal to \( n \), and \( x \) is \( q \) or \( x \) is in \( (W(q)^m)^- \), or \( x \) is above some point in \( (W(q)^m)^- \);
(3) \( R \) is the set of all points \( x \) in \( S \) such that \( x \) is \( \omega \) or \( x \) is in \( S' \) and the second coordinate of \( x \) is greater than \( n \), or \( x \) is above some point in \( S' \) whose second coordinate is greater than \( n \).

For each ordered pair of positive integers \((m, n)\) let \( S(m, n) \) denote the set of all \( x \) in \( S \) such that either \( x \) is \( \omega \) or \( x \) is in \( V_m \) and has the second coordinate less than \( n \), or \( x \) is above some point in \( V_m \) whose second coordinate is less than \( n \). No point of \( S \) is a limit point of \( S(m, n) \) and each point of \( S \) is in some \( S(m, n) \). It follows that \( S \) is \( \sigma \)-discrete.

To see that \( S \) is a connected space we suppose that the points of \( S \) lie in two mutually separated point sets \( H \) and \( K \) with \( \omega \) in \( H \) and we show that this leads to a contradiction. Since each point of \( S \) which is not in \( S' \) is a limit point of \( S' \), there is a region \( R' \) of \( S' \) which lies in \( K \) and, consequently, there is a vertical straight line \( V_t \), a segment of which lies entirely in \( K \). Since \( \omega \) is in \( H \), there is a point \( p \) in \( V_t \) such that

(1) \( p \) is a limit point in the plane of those points of \( K \) in \( V_t \) which have the second coordinate smaller than the second coordinate of \( p \).

(2) \( p \) is not a limit point in the plane of those points of \( K \) in \( V_t \) which have the second coordinate greater than the second coordinate of \( p \).

Thus we may choose a sequence of points \( t_1, t_2, \ldots \) common to \( K \) and \( V_t \) which converges in the plane to \( p \) and such that each of its terms has the second coordinate less than the second coordinate of \( p \). We may choose also a sequence of points \( s_1, s_2, \ldots \) common to \( H \) and \( V_t \) such that, for each positive integer \( n, s_n \) is the reflexion of \( t_n \) in the horizontal line through \( p \). Let \( R'(s_1), R'(s_2), \ldots \) be regions of \( S' \) centred at \( s_1, s_2, \ldots \), respectively, and lying in \( H - R_1(p) \). Let \( R'(t_1), R'(t_2), \ldots \) be regions of \( S' \) centred at \( t_1, t_2, \ldots \), respectively, and lying in \( K - R_1(p) \). By construction of \( C_t \) there is a sequence \( X_1, X_2, \ldots \) in \( C_t \), converging in the plane to \( p \), such that for each positive integer \( n \) the common part of \( S' \) with the interior of \( X_n \) is a subset of \( R'(t_n) \), and the common part of \( S' \) with the interior of the reflexion of \( X_n \) in the horizontal line through \( p \) is a subset of \( R'(s_n) \).

By construction of \( M(p) \) there is a sequence \( Y_1, Y_2, \ldots \) in \( M(p) \) which shares infinitely many terms with the sequence \( X_1, X_2, \ldots \) Let \( q \) be the point above \( p \) such that \( W(q) \) is \( Y_1, Y_2, \ldots \) If \( m \) is a positive integer, then \( (W(q)^m)^- \) intersects both the union of \( R'(s_1), R'(s_2), \ldots \), which is a subset of \( H \), and the union of \( R'(t_1), R'(t_2), \ldots \), which is a subset of \( K \). It follows that \( q \) is a limit point in \( S \) both of \( H \) and \( K \). Consequently, \( H \) and \( K \) are not mutually separated. This is a contradiction.

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