ON BASES, COMPACTNESS AND WEAK CONVERGENCE
IN THE BANACH SPACE $A_p$

BY

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1. Introduction. Let $A_p$ ($1 \leq p < \infty$) be the Banach space of all holomorphic functions $f(z)$ in the unit disc $D = \{z \mid |z| < 1\}$ such that
$$
\int_D |f(z)|^p d\mu(z) < \infty,
$$
with norm
$$
\|f\| = \left[ \int_D |f(z)|^p d\mu(z) \right]^{1/p},
$$
where $\mu$ is the planar Lebesgue measure in $D$. It turns out that $A_p$ is a closed linear subspace of the Banach space $L_p(\mu)$ of the set of all equivalence classes of $p$-th-power integrable complex functions on $D$. The usual Hardy spaces $H_p$ [3] are the Banach spaces of all elements of $A_p$ for which the norm
$$
\|f\| = \sup \left\{ \left( (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} \mid 0 \leq r < 1 \right\}
$$
is finite. $A_p$, as a set is distinct of $H_p$, i.e. for $1 \leq p < \infty$, the functions in $H_p$ form a proper subset of $A_p$. This is easily seen by choosing, for example, a suitable branch of the function $(1-z)^{-1/p}$, which is in $A_p$ but not in $H_p$. Next, a sequence $\{x_j\}$ in a Banach space $X$ is called a basis for $X$ if every $x$ in $X$ has the unique series expansion
$$
\lim_{n \to \infty} \sum_{j=0}^{n} a_j x_j
$$
with scalar coefficients $a_j$, where the convergence is in the strong topology of $X$. It is known that the sequence $\{x_j\}$ defined by
$$
x_j(z) = \left( \frac{(j+1)/\pi}{2^j} \right)^{1/2} z^j,
$$
forms an orthonormal basis for the Hilbert space $A_2$, and that the same sequence constitutes a basis for $H_p$ ($1 < p < \infty$) [3].

In this paper we state the result that $\{x_j\}$ is a basis for $A_p$ ($1 \leq p < \infty$). This gives an affirmative answer to the question, whether the Taylor series expansion at $z = 0$ for each function $f$ in $A_p$ converges to $f$ in the topology of $A_p$. However, it remains an open problem whether $\{x_j\}$ forms a basis for $A_1$. Next, as in the case of the spaces $C(S)$ and $L_p$, it is possible
to specify weak convergence and conditionally compact sets in \( A_p \). Indeed, necessary and sufficient conditions can be given for the weak convergence of a sequence in \( A_p \) and also for the elements of conditionally compact sets in \( A_p \), both relating the abstract concepts with the special form of the elements as functions of the complex variable \( z \). Finally, it is shown that the shift operator \( T \) in \( A_p \), as usual defined by \((Tf)(z) = zf(z), z \in D\), has the following interesting spectral properties: The spectrum is \( \bar{D} \), the point spectrum is empty, the residual spectrum is \( D \) and the continuous spectrum is the unit circle.

2. A basis for the Banach space \( A_p \). The proof that \( A_p \) \((1 \leq p < \infty)\) is a Banach space is, in principle, based on the following estimate. It is easy to see that every \( f \) in \( A_p \) satisfies the mean value equation

\[
f(z) = \frac{1}{\pi(1-|z|)^2} \int_{|\lambda-z|<1} f(\lambda) d\mu(\lambda), \quad z \in D,
\]

from which one obtains

\[
|f(z)| \leq \left[ \pi(1-|z|)^2 \right]^{-1/p} ||f||, \quad z \in D.
\] (1)

We omit this proof, since it follows directly the lines of that for the case \( p = 2 \) given in [2], where Hölder’s inequality is used instead of Schwarz’s.

The following lemma is substantial for proving the theorem on the existence of the basis \( \{x_j\} \) for \( A_p \):

**Lemma 1.** If \( 1 \leq p < \infty \), the sequence \( \{x_j\} \) is total in \( A_p \).

**Proof.** If \( f \) is any element of \( A_p \) with Taylor series \( \sum_{j=0}^{\infty} a_j z^j \) in \( D \), we define the function \( f_t, 0 < t < 1, \) on \( D \) by \( f_t(z) = f(tz), z \in D \). Since the Taylor series for \( f(z) \) converges absolutely and uniformly for \( |z| \leq t \), the Taylor series \( \sum_{j=0}^{\infty} a_j t^j z^j \) for \( f_t(z) \) converges absolutely and uniformly in \( D \). Hence \( f_t \) is an element of \( \text{sp}\{x_j\} \) in \( A_p \) for each \( t \in (0, 1) \). We show that, given \( \varepsilon > 0 \), there exists a \( t \) in \( (0, 1) \) for which \( ||f-f_t|| < \varepsilon \).

Since \( f \in L_1(\mu) \), there is a \( \delta \in (0, \frac{1}{2}) \) such that

\[
\left[ \int_{1-\delta<|z|<1} |f(z)|^p d\mu(z) \right]^{1/p} < \varepsilon/5.
\]

Taking \( r = 1 - \delta/2 \), we thus have

\[
\left[ \int_{r<|z|<1} |f_t(z)|^p d\mu(z) \right]^{1/p} = \left[ t^{-2} \int_{t r<|z|<t} |f(z)|^p d\mu(z) \right]^{1/p} < 2 \varepsilon/5
\]
for $t \varepsilon(r, 1)$. The uniform and absolute convergence of the Taylor series for $f(z)$ for $|z| \leq r$ implies the existence of an integer $n$ for which

$$
\sum_{j=n+1}^{\infty} |a_j(1-t^j)x^j| < \pi^{-1/p} \varepsilon/5,
$$

$|z| \leq r$ and $t \varepsilon(r, 1)$. Furthermore, there is a fixed $t \varepsilon(r, 1)$, depending on $n$, such that

$$
\sum_{j=0}^{n} |a_j(1-t^j)x^j| < \pi^{-1/p} \varepsilon/5, \quad |z| \leq r.
$$

Therefore, one has

$$
\|f-f_t\| \leq \left[ \int_{0<|z|<r} |f(z) - f_t(z)|^p \, d\mu(z) \right]^{1/p} + \left[ \int_{r<|z|<1} |f(z)|^p \, d\mu(z) \right]^{1/p} +
$$

$$
+ \left[ \int_{r<|z|<1} |f_t(z)|^p \, d\mu(z) \right]^{1/p} + \sum_{j=0}^{n} |a_j(1-t^j)x^j|^p \, d\mu(z) \right]^{1/p} +
$$

$$
+ \left[ \int_{0<|z|<r} \sum_{j=n+1}^{\infty} |a_j(1-t^j)x^j|^p \, d\mu(z) \right]^{1/p} + \varepsilon/5 + 2\varepsilon/5 < \varepsilon,
$$

which is the desired result.

There is a more general concept than that of a basis [5]: A biorthogonal system $\{x_j, x_j^*\}$ is called a Markušević basis for $X$, if $\{x_j\}$ is total in $X$ and $\{x_j^*\}$ in $X^*$ is such that for every $x \in X$, $x_j^*(x) = 0$, all $j$, implies $x = 0$. It is clear that the functionals $x_j^* \in A_p^*$, defined by

$$
x_j^*(f) = \int_{D} f(z) \overline{x_j(z)} \, d\mu(z),
$$

are biorthogonal to the $x_j$'s used in the above lemma. Also, $x_j^*(f)$ is proportional to the $j$-th coefficient of the Taylor series for $f$ at $z = 0$. From this it follows that $x_j^*(f) = 0$, $f \in A_p$, $j = 0, 1, \ldots$, implies $f = 0$. Thus, as an immediate consequence of Lemma 1, one obtains

**Corollary 2.** If $1 \leq p < \infty$, $\{x_j\}$ is a Markušević basis for $A_p$.

One of the most important theorems in the theory of bases is the theorem of Grinblyum-Nikol’skii [6] which states that a sequence $\{x_j\}$ in $X^*$ is a basis for $\overline{span}\{x_j\}$ if and only if there is a constant $K \geq 1$ such that

$$
\left\| \sum_{j=m}^{n} a_j x_j \right\| \leq K \left\| \sum_{j=n}^{\infty} a_j x_j \right\|
$$
for every pair of integers $m, n$ with $m \leq n$ and any scalars $a_j$. Now, using the fact that for $1 < p < \infty$ the trigonometrical system forms [7] a basis for $L_p[0, 2\pi]$ and by a twofold application of the Grinblyum-Nikol’skii theorem, it is possible to prove 

**Theorem 3.** If $1 < p < \infty$, the sequence $\{x_j\}$ is a basis for $A_p$ and the associated biorthogonal set to $\{x_j\}$ is $\{x_j^*\}$.

3. **Compact sets and weak convergence in $A_p$.** It is of special interest to investigate the conditionally compact sets in $A_p$; as has been done for many other important spaces, such as $O(S)$ (theorem of Arzelà-Ascoli) or $L_p$ (cf. [1]).

**Theorem 4.** Let $1 \leq p < \infty$. A set $K$ in $A_p$ is conditionally compact if and only if

(i) $K$ is bounded,

(ii) the functions in $K$ are equicontinuous on each compact subset of $D$ and

(iii) $\lim_{r \to 1} \int_{|z| < 1} |f(z)|^p d\mu(z) = 0$ uniformly for all $f$ in $K$.

**Proof.** Let $S$ be any compact set in $D$ and let $\delta = 1 - \sup \{|z| : z \in S\}$. If $K$ is conditionally compact, then $K$ is bounded and for every $\varepsilon > 0$ there exist functions $f_1, \ldots, f_n$ in $K$ such that

$$\inf_{i \leq n} \|f_i - f_i^*\| < (\pi \delta^2)^{1/p} \varepsilon / 3$$

for each $f$ in $K$ (i.e. $K$ is totally bounded). Then, by estimate (1) we have on $S$, $|f(z)| \leq (\pi \delta^2)^{-1/p} \|f\|$. Given any $z_0$ in $S$ we can choose a neighborhood $N$ of $z_0$ in $D$ such that

$$\sup_{i \leq n} |f_i(z) - f_i(z_0)| < \varepsilon / 3, \quad z \in N.$$

Thus for $f \in K$ and some $i \leq n$ we have on $N \cap S$

$$|f(z) - f(z_0)| \leq |f(z) - f_i(z)| + |f_i(z) - f_i(z_0)| + |f_i(z_0) - f(z_0)|$$

$$< 2(\pi \delta^2)^{-1/p} \|f - f_i\| + \varepsilon / 3 < \varepsilon,$$

showing that the functions in $K$ are equicontinuous on $S$.

In a similar manner we can prove the last statement of the theorem. Let $r < 1$ be such that $\sup \{\|\chi_r f_i\| : i \leq n\} < \varepsilon / 2$, where $\chi_r$ is the characteristic function of the set $\{z \in D : |z| > r\}$. By Minkowski's inequality one obtains for some $i \leq n$,

$$\|\chi_r f\| \leq \|\chi_r (f - f_i)\| + \|\chi_r f_i\| < \|f - f_i\| + \varepsilon / 2 < 2\varepsilon$$

and so $\lim_{r \to 1} \|\chi_r f\| = 0$ uniformly for all $f$ in $K$.

To prove the converse we assume (i), (ii) and (iii) to be true. Let $r < 1$. By (ii) the subset $K_r = \{(1 - \chi_r)f : f \in K\}$ of the Banach space
$C(D_r), D_r = \{z \in D \mid |z| \leq r\}$ (the norm in $D_r$ given by $\|g\|_\infty = \sup_{z \in D_r} |g(z)|, g \in C(D_r)$) is equicontinuous and by (i),

$$\sup_{r \leq n} \{\|g\|_\infty \mid g \in K_r\} \leq [\pi(1-r)^2]^{-1/p} \sup \{\|f\|_i \mid f \in K\} < \infty.$$  

Thus the theorem of Arzelà-Ascoli ([1], p. 266) applies and $K_r$ is conditionally compact, hence totally bounded in $C(D_r)$. Therefore, given $\varepsilon > 0$, there exist functions $f_1, \ldots, f_n$ in $K$ such that

$$\inf_{i \leq n} \|(1 - \chi_r)(f - f_i)\|_\infty < \pi^{-1/p} \varepsilon / 3$$

for every $f \in K$. According to (iii) we now take $r$ such that $\sup \{\|\chi_r f\|_i \mid f \in K\} < \varepsilon / 3$ and have for every $f$ in $K$ and some $i \leq n$,

$$\|f - f_i\| \leq \|(1 - \chi_r)(f - f_i)\| + \|\chi_r(f - f_i)\|$$

$$< \pi^{1/p} \|(1 - \chi_r)(f - f_i)\| + 2\varepsilon / 3 < \varepsilon.$$  

Consequently, $K$ is totally bounded in $A_p$. Since $\overline{K}$ is complete, it follows that $K$ is a conditionally compact subset of $A_p$ (or equivalently, since $A_p$ is a metric space, $K$ is sequentially compact).

In the case $1 < p < \infty$, $A_p$ is reflexive and it is possible to characterize weak convergence in $A_p$. One observes that from estimate (1) it immediately follows that for any fixed $z$ in $D$, the functional $x^*_z$ on $A_p$ defined by $x^*_z(f) = f(z), f \in A_p$, belongs to $A^*_p$ so that the weak convergence of a sequence $\{f_n\}$ in $A_p$ implies simple convergence of $f_n(z)$ in $D$.

**Theorem 5.** A sequence $\{f_n\}$ in $A_p, 1 < p < \infty$, converges weakly to zero if and only if it is bounded and for each $j \geq 0$ the coefficient $a_{nj}$ of the Taylor series $\sum_{j=0}^{\infty} a_{nj}z^j$ for $f_n(z)$ converges to zero with $n$.

**Proof.** If $\{f_n\}$ converges weakly to zero, $\{f_n\}$ must be bounded. Moreover, since

$$a_{nj} = ((j+1)/\pi)^{1/2} x^*_j(f_n),$$

where $\{x^*_j\}$ is the biorthogonal sequence in $A^*_p$ which belongs to the basis $\{x_j\}$ of Theorem 3, it follows that

$$\lim_{n} a_{nj} = ((j+1)/\pi)^{1/2} \lim_{n} x^*_j(f_n) = 0, \quad j = 0, 1, \ldots$$

This is the necessary condition.

Conversely, suppose that $\lim_{n} a_{nj} = 0$ for all $j$. Since $A_p$ is reflexive, the basis $\{x_j\}$ for $A_p$ has, by a known theorem of James ([4], p. 519), the following property:
For any \( x^* \in A_p^* \) one has
\[
\limsup_m \left\{ |x^*(x)| \left| x \in \text{sp} \{x_m, x_{m+1}, \ldots \}, \|x\| = 1 \right\} = 0
\]
(i.e. the basis is shrinking). Now, due to the principle of uniform boundedness, there is a constant \( K \geq 1 \) such that
\[
\sup_m \sup \left\{ \left\| \sum_{j=m}^{\infty} x_j^* (x) x_j \right\| \left| \|x\| \leq 1 \right\} \leq K.
\]
Hence
\[
\sup \left\{ \left| x^* \left( \sum_{j=m}^{\infty} x_j^* (x) x_j \right) \right| \left| x \in A_p, \|x\| \leq 1 \right\} \leq \sup \left\{ |x^*(x)| \left| x \in \text{sp} \{x_m, x_{m+1}, \ldots \}, \|x\| \leq K \right\}.
\]
Thus
\[
\limsup_m \left\{ |x^* \left( \sum_{j=m}^{\infty} x_j^* (x) x_j \right) \right\| \left| x \in A_p, \|x\| \leq 1 \right\} = 0
\]
and we have for every \( \varepsilon > 0 \) an \( m \) such that
\[
\sup_n \left| x^* \left( f_n - \sum_{j=m}^{\infty} x_j^* (f_n) x_j \right) \right| < \varepsilon/2,
\]
where without loss of generality one may take \( x^* \) and all \( f_n \)'s of norm one. Because there is an \( n_\varepsilon \), depending on \( m \), for which
\[
\left\| \sum_{j=m}^{n} x_j^* (f_n) x_j \right\| = \left\| \sum_{j=m}^{n} (\pi/(j+1))^{1/2} a_{nj} x_j \right\| < \varepsilon/2, \quad n \geq n_\varepsilon,
\]
one gets
\[
|x^* (f_n)| \leq |x^* \left( f_n - \sum_{j=m}^{\infty} x_j^* (f_n) x_j \right) | + \left\| \sum_{j=m}^{n} x_j^* (f_n) x_j \right\| < \varepsilon, \quad n \geq n_\varepsilon,
\]
and the theorem follows.

4. The spectrum of the shift operator. Let \( T: A_p \to A_p \) be the linear operator defined by \((Tf)(z) = zf(z), z \in D\). It is immediate that \( T \), usually called shift operator, is bounded, with norm \( \|T\| \leq 1 \). Therefore the spectrum \( \sigma(T) \) of \( T \) must be contained in the closed unit disc \( \overline{D} \). If \( \sigma_p(T), \sigma_r(T) \) and \( \sigma_c(T) \) denote the point spectrum, residual spectrum and continuous spectrum respectively, and if \( \delta D \) denotes the unit circle \( \overline{D} - D \), we can determine the partition of \( \overline{D} \) into the mutually exclusive sets \( \sigma_p(T), \sigma_r(T) \) and \( \sigma_c(T) \).

**Theorem 6.** The shift operator \( T \) has the following spectral properties:
(i) \( \sigma(T) = \overline{D}, \)
(ii) \( \sigma_p(T) \) is empty,
(iii) $\sigma_r(T) = D$
(iv) $\sigma_c(T) = \delta D$.

Proof. Assume $\lambda \in D$ but $\lambda \notin \sigma(T)$. Then $(\lambda I - T)^{-1}$ would exist and would be bounded in $A_p$. Thus $(\lambda I - T)^{-1}f$ would be in $A_p$ for any $f$ in $A_p$ which is a contradiction since $(\lambda - z)^{-1}f(z)$ is not holomorphic in $D$. This shows that $D \subset \sigma(T)$ and, since $\sigma(T)$ is closed, (i) follows. Next, because $(\lambda I - T)f = 0$, $f \in A_p$, implies $(\lambda - z)f(z) = 0$ and thus $f(z) = 0$, $z \not= \lambda$, it is clear that $\sigma_p(T)$ is empty.

To determine whether a point $\lambda$ of $\sigma(T)$ is in $\sigma_r(T)$ or in $\sigma_c(T)$, we look at the range of the operator $\lambda I - T$. Let first $\lambda$ be in $D$. Suppose that the range of $\lambda I - T$ is dense in $A_p$. Then, given $f$ in $A_p$ with $f(\lambda) \neq 0$, there would exist a sequence $\{g_n\}$ in $A_p$ such that $\lim_{n}(\lambda I - T)g_n = f$. But by estimate (1) this would imply that

$$\lim_{n}(\lambda - z)g_n(z) = f(z), \quad z \in D,$$

and hence that $f(\lambda) = 0$ which is impossible. Thus $D \subset \sigma_r(T)$. On the other hand, if $\lambda$ is of modulus 1, then the range of $\lambda I - T$ is dense in $A_p$ (it is always assumed that $1 \leq p < \infty$), i.e. for every $f \in A_p$ there exists a sequence $\{g_n\}$ in $A_p$ with

$$\lim_{n}\|f - (\lambda I - T)g_n\| = 0.$$

Let $\{g_n\}$ be defined by

$$g_n(z) = f(z) \sum_{j=0}^{n}(z^j/\lambda^{j+1}), \quad z \in D.$$

Clearly, $g_n \in A_p$. By

$$(\lambda - z)g_n(z) = f(z)\left[\sum_{j=0}^{n}(z/\lambda)^j - \sum_{j=1}^{n+1}(z/\lambda)^j\right] = f(z)\left[1 - (z/\lambda)^{n+1}\right],$$

it is apparent that

$$\|f - (\lambda I - T)g_n\| = \left[\int_{D} |f(z)z^{n+1}|^p d\mu(z)\right]^{1/p} \leq r^{n+1}\left[\int_{0<|z|<r} |f(z)|^p d\mu(z)\right]^{1/p} + \left[\int_{r<|z|<1} |f(z)|^p d\mu(z)\right]^{1/p}, \quad 0 < r < 1.$$

Given $\epsilon > 0$, it is now possible to choose $r$ such that the last term on the right-hand side of the above inequality is smaller than $\epsilon/2$. On the other hand, the first term on the right is dominated by $r^{n+1}\|f\|$ so that $\|f - (\lambda I - T)g_n\| < \epsilon$ for $n$ large enough. Hence $(\lambda I - T)(A_p) = A_p$,
implying that $\delta D \subset \sigma_c(T)$. Since the sets $\sigma_r(T)$ and $\sigma_c(T)$ are disjoint, one obtains the results (iii) and (iv).

**Corollary 7.** $T$ is not compact.

The result is an immediate consequence of the fact that the non-zero points in the residual spectrum of a compact linear operator are isolated.

**REFERENCES**


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