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REMARKS ON THE TIME TRANSPORTATION PROBLEM

1. Introduction. Suppose we are given the system \((T, M)\), where \(T = (t_{ij})\) is an \((m \times n)\)-matrix with real numbers \(t_{ij}\), \(M = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)\) and \(a_i, b_j\) are positive real numbers such that

\[
\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.
\]

Any \((m \times n)\)-matrix \(X = (x_{ij})\) which satisfies the conditions

\[
\begin{align*}
\sum_{j=1}^{n} x_{ij} &= a_i, & i &= 1, 2, \ldots, m, \\
\sum_{i=1}^{m} x_{ij} &= b_j, & j &= 1, 2, \ldots, n,
\end{align*}
\]

will be called a solution of \((T, M)\). The problem is to find an optimal solution, i.e. a solution \(X\) of \((T, M)\) which minimizes the function

\[
t(X) = \max_{(i,j)\in\Theta} t_{ij},
\]

where \(\Theta = \{(i, j) : x_{ij} > 0, 1 \leq i \leq m, 1 \leq j \leq n\}\).

Methods of solving time transportation problems are described in [1], [2]. They are not adapted for computer calculations. Therefore a new version of the method for solving TTP, being a modification of that presented in [2], is presented in this paper. This method may be successfully used in computer calculations even for great \(m\) and \(n\).

2. Notation and definitions. Let \(\Phi = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}\). Any subset \(\Omega\) of \(\Phi\) we call a set of nodes. By a route \((p, q) - (r, s)\) connecting in \(\Omega\) nodes \((p, q)\epsilon\Omega, (r, s)\epsilon\Omega\) we mean the smallest sequence \(\{(i_k, j_k)\} (k = 1, 2, \ldots, l)\) of different nodes from \(\Omega\) satisfying the conditions

\[
(i_1, j_1) = (p, q), \quad (i_l, j_l) = (r, s),
\]

for \(k = 1, 2, \ldots, l-1\) either \(i_k = i_{k+1}\) or \(j_k = j_{k+1}\).
If there is in $\Omega$ a pair of nodes $(p, q), (r, s)$ for which two different routes $(p, q) - (r, s)$ exist in $\Omega$, then we say that $\Omega$ contains a cycle. In that case we mean by a cycle the sum of the routes $(p, q) - (r, s)$. A subset $B$ of $\Phi$ consisting of $m + n - 1$ nodes which contains no cycle is called a basis. To each basis $B$ there exist not more than one matrix $X = (x_{ij})$ whose elements satisfy (1) and also the conditions $x_{ij} = 0$ for all $(i, j) \in \Phi - B$. Every matrix $X$ which satisfies these conditions will be called a basic solution of $(T, M)$ and denoted by $X(B) = (x_{ij}^B)$. A basis $B$ for which exists a basic solution $X(B)$ of $(T, M)$ will be called a feasible basis.

Let

$$\Theta(B) = \{(i, j) \in B: x_{ij}^B > 0\}, \quad V(B) = \{(i, j) \in B: x_{ij}^B = 0\},$$

$$W(B) = \{(i, j) \in B: t_{ij} > t(X(B))\}$$

for any basic solution $X(B)$.

We shall say that a basic solution of $(T, M)$ is degenerate if the set $V(B)$ is not empty. The number of nodes in the set $W(B)$ will be called degree of degeneration of $X(B)$ and denoted by $\text{dg } X(B)$.

Any node $(k, l) \in B$ which satisfies the condition

$$t_{kl} = \max_{(i, j) \in \Theta(B)} t_{ij} = t(X(B))$$

will be called a central node of basis $B$. The node $(i, j) \in \Phi - B$ which satisfies the conditions

(i) every route $(i, j) - (k, l)$ in the set $B + (i, j)$ contains an even number of nodes,

(ii) $(k, l)$ belongs to the cycle contained in $B + (i, j)$ will be called a neighbouring node to a central node $(k, l)$.

By the neighbouring set to $(k, l)$ we mean the set of all neighbouring nodes to the central node $(k, l)$. We shall denote it by $\Psi_{kl}(B)$ or shortly by $\Psi_{kl}$.

3. Method of solving TTP. We propose the following method of solving a time transportation problem (TTP).

1. Find an initial basic solution $X(B_1)$ by any of the known methods. (Suppose that $t_{k_0l_0} \neq t(X(B_1)).$)

   Now for $h = 1, 2, \ldots$ do the following:

2. Find the central nodes $(k, l)$ of the basic solution $X(B_h)$ and fix as $(k_h, l_h)$ any of them such that if $(k_{h-1}, l_{h-1})$ is a central node of $B_h$, then $(k_h, l_h) = (k_{h-1}, l_{h-1}).$

3. Find the set $\Psi_{kh,lh}$ neighbouring to $(k_h, l_h)$. 

4. Choose an arbitrary node \((p_h, q_h)\) belonging to the set 
\[
\{(p, q): t_{pq} = \min_{(i,j), (p, q) \in \mathcal{W}_{k_h l_h}} t_{ij}, (p, q) \in \mathcal{W}_{k_h l_h}\}.
\]

There exist two possibilities (a) and (b).

(a) \(t_{k_h l_h} = t(X(B_h))\).

Then stop because the basic solution \(X(B_h)\) is an optimal solution. This follows from theorem 2 in section 5.

(b) \(t_{k_h l_h} < t_{k_h l_h}\).

The set \(B_h + (p_h, q_h)\) contains exactly one cycle, say \(G_h\). Divide \(G_h\) into two subsets \(G_h', G_h''\) assigning to \(G_h'\) nodes \((i, j)\) for which the route \((i, j) - (p_h, q_h)\) in \(G_h\) contains an odd number of nodes. Subset \(G_h''\) contains the remaining nodes. Of course, \((p_h, q_h) \in G_h', (k_h, l_h) \in G_h''\).

5. Find

\[
\min_{(i,j) \in G_h'} x_{ij}^{B_h} = \bar{x}_h.
\]

Determine a set \(F_h = \{(r, s) \in G_h'': x_{rs}^{B_h} = \bar{x}_h\}\) and fix as \((r_h, s_h)\) any node from it such that if \((k_h, l_h) \in F_h\), then \((r_h, s_h) = (k_h, l_h)\). Determine a new basis \(B_{h+1} = B_h + (p_h, q_h) - (r_h, s_h)\) and a new basic solution \(X(B_{h+1})\) defined by the formulae

\[
x_{ij}^{B_{h+1}} = \begin{cases} 
  x_{ij}^{B_h} + \bar{x}_h & \text{if } (i, j) \in G_h', \\
  x_{ij}^{B_h} - \bar{x}_h & \text{if } (i, j) \in G_h'', \\
  x_{ij}^{B_h} & \text{if } (i, j) \notin G_h.
\end{cases}
\]

Repeat steps 2, 3, 4, and 5.

It will be proved that for any TTP only a finite number of iterations is needed.

4. Commentary. In comparison with the method described in [2] the algorithm presented in this paper includes 3 corrections. Steps 6 and 7 have been omitted. This is very important in computer calculations. It is practically impossible to verify whether a sequence of basic solutions \(X(B_1), X(B_2), \ldots, X(B_h)\) contains two identical solutions. Just this difficulty causes that the method described in [2] could not be used in computer calculations. The modifications done in steps 2 and 5 of the algorithm allow to leave out steps 6 and 7 from our algorithm. This will be illustrated by two examples.
Suppose we have following TTP:

\[
\begin{array}{cccc}
8 & 6 & 3 & 1 \\
7 & 9 & 7 & 5 \\
4 & 5 & 4 & 8 \\
3 & 2 & 3 & 9 \\
4 & 8 & 4 & 8 \\
\end{array}
\]

(The numbers \(a_i\) and \(b_j\) are on the right and below the matrix \(T = (t_{ij})\), respectively.)

The initial solution is given in Fig. 1.

Leaving out the modification done in step 2 we would obtain the sequence of basic solutions given in Figs. 2 and 3. We see that \(X(B_1) = X(B_3)\) and \(B_1 = B_3\). Using the algorithm presented in this paper and leading from the initial solution \(X(B_1)\) we obtain after six iterations the optimal basic solution given in Fig. 4.
Suppose we have the TTP

\[
\begin{array}{cccc}
5 & 4 & 1 & 4 \\
6 & 8 & 3 & 3 \\
2 & 3 & 4 & 4 \\
3 & 3 & 5 & \\
\end{array}
\]

with the initial basic solution \(X(B_1)\) presented in Fig. 5.

Leaving out the modification done in step 5 we would obtain the sequence of basic solutions given in Figs. 6 and 7. We see that again \(X(B_1) = X(B_3)\) and \(B_1 = B_3\), while the exact result is that of Fig. 8. (Elements \((k_h, l_h), (r_h, s_h), (p_h, q_h)\) and \((i, j)\in\Psi_{k_h l_h}\) have been denoted in figures by a square, a crossed square, a crossed circle, and crosses, respectively.)

5. Theorems. First, let us state and prove a series of properties of the sequence \(X(B_1), X(B_2), \ldots, X(B_h), \ldots\) of basic solutions of \((T, M)\) obtained by using the method given in steps 1-5.

Let \(z(X(B)) = \max_{(i, j)\in B} t_{ij}\).

Remark 1. If \((i, j)\in W(B)\), then \(x_{ij}^B = 0\).

Remark 2. \(z(X(B)) \geq t(X(B))\).

Remark 3. \(z(X(B)) = t(X(B))\) if and only if \(dg X(B) = 0\).

Proof. \(z(X(B)) = t(X(B)) \iff W(B) = \emptyset \iff dg X(B) = 0\).

Corollary 1. \(t(X(B_h)) \geq t(X(B_{h+1}))\).

Proof. \(B_{h+1} = B_h - \{r_h, s_h\} + \{p_h, q_h\}\) and \(t_{p_h q_h} < t(X(B_h))\). It follows from Remark 2 that \(t(X(B_h)) \leq z(X(B_h))\) and thus \(t_{p_h q_h} < z(X(B_h))\).

Corollary 2. \(t(X(B_h)) \leq z(X(B_1)) \iff \) \((h = 1, 2, \ldots, )\).

Proof. \(t(X(B_h)) \leq z(X(B_h))\).

Corollary 3. If \(t_{r_s} = z(X(B_1))\), then there exist no basic solution \(X(B_h)\) \((h > 1)\) such that \((r_s, s)\in B_h\).

Proof. \((r_s, s)\notin B_1, B_h = B_2 - B_h + \{p_h, q_h\} + \{p_h, q_h\} + \{p_h, q_h\} + \cdots + \{p_h, q_h\}\),

where \(1 \leq l \leq m+n-1\), \(t_{r_s} = z(X(B_1)) \geq z(X(B_h))\) and thus \(t_{p_h q_h} > t_{r_s}\) or \((r_s, s) \neq (p_h, q_h)\). This means that \((r_s, s)\notin B_h\).

Corollary 4. If \(dg X(B_1) = 0\) and if \((k_1, l_1)\) is a central node of \(B_1\), then there exist no solution \(X(B_h)\) \((h > 1)\) for which \(x_{k_1 l_1}^{B_1} = 0\).

Proof. Let \(i > 1\) be the smallest number such that \((k_1, l_1)\notin B_i\) and \(x_{k_1 l_1}^{B_i} = 0\). \(t_{k_1 l_1} = z(X(B_1)) \geq z(X(B_h)) \geq z(X(B_i)) \geq t_{k_1 l_1}\) and thus from Corollary 3 it follows that \((k_1, l_1)\in B_h\) and, for \(1 \leq h < i\), \(x_{k_1 l_1}^{B_h} > 0\). In particular, \(x_{k_1 l_1}^{B_{i-1}} > 0\), and thus \((k_1, l_1)\) is a central node of \(B_{i-1}\). But \(x_{k_1 l_1}^{B_i} = x_{k_1 l_1}^{B_{i-1}} - x_{r_{i-1} l_{i-1}}^{B_{i-1}} = 0\). This means that \(x_{k_1 l_1}^{B_{i-1}} = x_{r_{i-1} l_{i-1}}^{B_{i-1}}\) and because of the modification introduced in step 5 of the algorithm \((k_1, l_1)\notin B_i\) must
hold because \((r_{i-1}, s_{i-1}) = (k_1, l_1)\). This contradiction completes the proof.

**Theorem 1.** The sequence \(X(B_1), X(B_2), \ldots, X(B_h)\) of basic solutions of the TTP \((T, M)\) constructed by the method given in steps 1—5 contains no two identical solutions \(X(B_i), X(B_j)\) for \(i \neq j\).

**Proof.** Suppose that in the sequence \(X(B_1), X(B_2), \ldots, X(B_e)\), \((e > 1)\) of basic solutions of \((T, M)\) obtained by using the discussed method there is \(X(B_1) = X(B_e), B_1 = B_e\). From corollary 1 it follows that \(z(X(B_1)) = z(X(B_2)) = \ldots = z(X(B_e))\).

Let \(Z(B_h) = \{(i, j) \in B_h : t_{ij} = z(X(B_i))\}\). Of course, always \(Z(B_h) \neq \emptyset\). Take then \(Z(B_h), Z(B_{h+1}) (1 \leq h \leq e-1)\). We have \(B_{h+1} = B_h - (r_h, s_h) + + (p_h, q_h), t_{phq_h} < t(X(B_h)) \leq z(X(B_h)) = z(X(B_{h+1})), (p_h, q_h) \notin Z(B_{h+1})\), and thus \(Z(B_{h+1}) \subset Z(B_h)\).

Suppose that \((i, j) \notin Z(B_h)\) and \((i, j) \notin Z(B_{h+1})\). Corollary 3 allows us to state that \((i, j) \notin B_e\). On the other hand, we have

\[
B_h = B_1 \cdot B_h + (p_{hi}, q_{hj}) + \ldots + (p_{hi}, q_{hj})
\]

\[
(1 \leq h_i \leq h, 1 \leq l \leq \min\{h, m+n-1\})
\]

\[
t_{phq_h} < z(X(B_h)) = t_{ij}
\]

and thus \((i, j) \notin B_1 \cdot B_h\). This means that \((i, j) \notin B_1\). However, \(B_1 = B_e\). This allows us to assert that \(Z(B_h) \subset Z(B_{h+1})\). Finally,

\[
Z(B_1) = Z(B_2) = \ldots = Z(B_e) = Z.
\]

Now the proof splits in two parts.

(a) Suppose that \(dg(X(B_1)) = d > 0\). This allows us to assert that \(x_{ij}^{B_1} = 0\) for \((i, j) \notin Z\). If \(t(X(B_h)) = z(X(B_h))\) for some \(1 < h < e\), then \(dg(X(B_h)) = 0\) (remark 3), the central node \((k_h, l_h)\) of \(B_h\) belongs to \(Z\) and from corollary 4 it follows that there exists no solution \(X(B_i) (i > h)\) for which \(x_{k_hl_h}^{B_i} = 0\). This means that \(x_{k_hl_h}^{B_i} \neq 0\). But we know that \(X(B_i) = X(B_e)\) and thus \(x_{k_hl_h}^{B_i} = 0\). This contradiction shows that \(t(X(B_h)) < z(X(B_h))\) or, otherwise, for all \((i, j) \notin Z\) and \(h = 1, 2, \ldots, e\) we have \(Z \subset W(B_h) \subset B_h\), \(x_{ij}^{B_h} = 0\). Thus, we see that

\[
X^1(B_1), X^1(B_2), \ldots, X^1(B_e) (X^1(B_i) = X(B_i)\) for \(i = 1, 2, \ldots, e)\)
\]
is a sequence of basic solutions to TTP \((T^1, M)\), where \(t_{ij} = t_{ij}\) for \((i, j) \notin Z\), \(t_{ij}^1 = c\) for \((i, j) \notin Z\), \(c\) is an arbitrary number such that \(c < t(X(B_i))\) and \(T^1 = (t_{ij}^1)\), and it could have been obtained by using steps 1—5, because \((k_h, l_h) \notin Z, (r_h, s_h) \notin Z, (p_h, q_h) \notin Z\) for \(h = 1, 2, \ldots, e\). Of course, \(t(X(B_i)) = t(X^1(B_i))\) and \(z(X(B_i)) > z(X^1(B_i))\).

If we repeat our argumentation, then after \(p (p \leq d)\) steps we come to the conclusion that there exist the \((m \times n)\)-matrix \(T^p\) and the sequence
$X^p(B_1), X^p(B_2), \ldots, X^p(B_e)$ of basic solutions of $(T^p, M)$ which can be obtained by using the discussed method such that $t(X^p(B_1)) = z(X^p(B_1))$. This means that $dg X^p(B_1) = 0$.

All that allows us to assume that $dg X(B_1) = 0$ in the sequence $X(B_1), X(B_2), \ldots, X(B_e)$ of basic solutions of $(T, M)$.

(b) Suppose now that $X(B_1)$ is a degenerate basic solution of $(T, M)$, $(B_1 - \Theta(B_1) \neq \emptyset)$ and that $dg X(B_1) = 0$. We know (see (3)) that $z(X(B_1)) = \ldots = z(X(B_e))$ and $Z(B_1) = \ldots = Z(B_e) = Z$. We assumed $dg X(B_1) = 0$ and thus the central node $(k_1, l_1)$ of $B_1$ belongs to $Z$. From corollary 4 it follows that for any $1 \leq h \leq e$ the condition $x_{k_1 l_1}^B = 0$ is not true. The modification done in step 2 of the algorithm allowed us to assert that $(k_1, l_1)$ is a central node of all $B_h$ ($1 \leq h \leq e$). Now, $x_{k_1 l_1}^B = x_{k_1 l_1}^B - \bar{x}_h$, $\bar{x}_h \neq 0$, and thus $x_{k_1 l_1}^B = x_{k_1 l_1}^B$. But $x_{k_1 l_1}^B = x_{k_1 l_1}^B$, and thus $x_{k_1 l_1}^B = x_{k_1 l_1}^B$ or

$$x_{k_1 l_1}^B = 0 \quad \text{for} \quad h = 1, 2, \ldots, e.$$ 

It is clear now that $x_{i j k}^{B'} = x_{i j k}^{B''}$ for $1 \leq h', h'' \leq e$ and $\Theta(B_i) = \Theta(B_1) = \ldots = \Theta(B_e) = \emptyset$.

Let us divide the subset $U$ of all nodes $(i, j) \in B_1 - \Theta$ for which there exists the route $(i, j)(k_1, l_1)$ in the set $\Theta + (i, j)$ into two subsets

$$U_1 = \{(i, j) \in U: (i, j)(k_1, l_1) \text{ contains an even number of nodes}\},$$

$$U_2 = \{(i, j) \in U: (i, j)(k_1, l_1) \text{ contains an odd number of nodes}\}.$$

Of course, $U_1 + U_2 = U \neq \emptyset$ because $B_1 - \Theta(B_1) \neq \emptyset$.

Let $U_1 \neq \emptyset$ and $(i, j) \in U_1$. $(k_1, l_1)$ is a central node of all $B_h$ ($1 \leq h \leq e$) and thus $(r_h, s_h) = (i, j)$ cannot be true. This means that $(i, j) \in B_h$ ($1 \leq h \leq e$).

Let $U_2 \neq \emptyset$ and $(i, j) \in U_2$. Suppose that $(r_h, s_h) = (i, j)$ for $1 \leq h \leq e$. Then $(i, j) \notin B_{h+1}$. The route $(i, j)(k_1, l_1)$ in the set $B_{h'} + (i, j)$ ($h < h' \leq e$) contains an odd number of nodes. But $B_1 = B_e$ and thus there exists a number $h' (h < h' \leq e)$ such that $(i, j) \notin \Gamma_{k_1 l_1}$. This is, however, impossible because the route $(i, j)(k_1, l_1)$ contains an even number of nodes would belong to $\Gamma_{k_1 l_1}$. Finally, $U \subset B_h$ for $h = 1, 2, \ldots, e$.

Let us now introduce the TTP $(T, M^1)$, where $M^1 = (a_1^1, \ldots, a_m^1, \ldots, b_n^1)$ is defined by the following formulae:

- if $(i, j) \in U$, then $a_i^1 = a_i + 1$ and $b_j^1 = b_j + 1$,
- if $(i, j) \notin U$, then $a_i^1 = a_i$ and $b_j^1 = b_j$. 
$B_1$ is a feasible basis for $(T, M^1)$ and a basic solution $X^1(B_1)$ can be defined as follows:

$$x^1_{ij} = \begin{cases} x^1_{ij} + 1 & \text{for } (i, j) \in U, \\ x^1_{ij} & \text{for } (i, j) \notin U. \end{cases}$$

The $(m \times n)$-matrix $X$ such that $x_{ij} = x^1_{ij}$ for $(i, j) \notin U + \Theta$ and $x_{ij} = 0$ for $(i, j) \notin U + \Theta$ is of course a solution to $(T, M^1)$. We know that $U + \Theta \subset B_h$ and thus $B_h$ is a feasible basis of $(T, M^1)$ and $X^1(B_1)$ is a basic solution of $(T, M^1)$ for $h = 1, 2, \ldots, e$. Therefore the sequence $X^1(B_1), X^1(B_2), \ldots, X^1(B_e)$ is a sequence of basic solutions of $(T, M^1)$ obtained by using steps 1—5. Also, $t(X^1(B_1)) = t(X(B_1)) = z(X(B_1)) = z(X^1(B_1))$ and thus $dgX^1(B_1) = 0$. Moreover, $\Theta^1(B_1) = \Theta(B_1) + U$ and $U \neq \emptyset$.

Bearing this in mind we can assert that if we repeat our argumentation given in part (b) of the proof, then after $r$ steps we will come to the conclusion that there exists a TTP $(T, M^r)$ such that the sequence $X^1(B_1), X^1(B_2), \ldots, X^1(B_e)$ is a sequence of basic solutions of $(T, M^r)$ obtained by using the discussed method and for which $\Theta^r(B_1) = B_1$. This means that the basic solution $X^r(B_1)$ of $(T, M^r)$ is not degenerate. All which was said in parts (a) and (b) of our proof allows us to assume that

(i) in the sequence $X(B_1), X(B_2), \ldots, X(B_e)$ of basic solutions of $(T, M)$ holds $X(B_1) = X(B_2), B_1 = B_e (e > 1)$,

(ii) $dgX(B_1) = 0$ and $X(B_1)$ is an undegenerate basic solution of $(T, M)$.

We know (see (4)) that in this case $x^e_{rh} = \bar{x}_h = 0$ for $h = 1, 2, \ldots, e$.

On the other hand, $X(B_1)$ is a non-degenerate basic solution of $(T, M)$ and thus $x^1_{ij} > 0$ for $(i, j) \in B$ and $x^2_{11} = x_1 > 0$.

This contradiction completes the proof.

Theorem 2. If $(k, l)$ is a central node of a basis $B$, $\Psi_{kl}(B)$ is a set neighbouring to $(k, l)$ and if $\Psi_{kl}(B) \subset \Pi$, then there exists no solution $X = (x_{ij})$ such that $x_{ij} = 0$ for all $(i, j) \notin \Pi + (k, l)$.

The proof of this theorem can be found in [2].

Theorem 3. If TTP is solved by the method given in steps 1—5, then the number of iterations leading from $X(B_1)$ to the optimal solution is finite.

Proof. The number of all basic solutions of $(T, M)$ is finite. The sequence $X(B_1), X(B_2), \ldots, X(B_h), \ldots$ of basic solutions of $(T, M)$ obtained by using the method given in steps 1—5 consists of different elements (see theorem 1) and, therefore, is a finite sequence. It means that there exists a number $h \geq 1$ such that the basic solution $X(B_h)$ of $(T, M)$ satisfies the condition formulated in step 4, (a) of our algorithm. Theorem 2 shows that $X(B_h)$ is an optimal solution of $(T, M)$. The proof is completed.
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UWAGI O ZAGADNIENIU TRANSPORTOWYM Z KRYTERIUM CZASU

STRESZCZENIE

Zagadnienie transportowe z kryterium czasu można sformułować w następująco sposób: Dany jest układ \((T, M)\), gdzie \(T = (t_{ij})\) jest macierzą typu \(m \times n\) o elementach rzeczywistych, a \(M = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)\) jest układem \(m + n\) liczb dodatnich \(a_i, b_j\), przy czym

\[
\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.
\]

Problem polega na znalezieniu minimum funkcji

\[
t(X) = \max_{(i,j) \in \Theta} t_{ij},
\]

gdzie \(\Theta = \{(i, j) : x_{ij} > 0, 1 \leq i \leq m, 1 \leq j \leq n\}\)

określonej na zbiorze macierzy \(X = (x_{ij})\) typu \(m \times n\) spełniających dla \(i = 1, 2, \ldots, m\) i \(j = 1, 2, \ldots, n\) następujące warunki:

\[
x_{ij} > 0, \quad \sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{m} x_{ij} = b_j.
\]

Przedstawiona w niniejszej pracy metoda rozwiązania zagadnienia transportowego z kryterium czasu jest metodą iteracyjną. Przy pomocy dowolnej ze znanych metod konstruuje się rozwiązanie początkowe \(X(B_1)\), a następnie wyznacza się skończony ciąg \(X(B_1), X(B_2), \ldots, X(B_k)\) rozwiązań bazowych, stosując do tego celu wzory zawarte w punktach 2.5 algorytmu opisanego w rozdziale 3. Po wykonaniu skończonej ilości iteracji otrzymujemy szukane rozwiązanie optymalne \(X(B_k)\). Odpowiednie twierdzenia można znaleźć w rozdziale 5.

Omawiana tutaj metoda jest zmodyfikowaną i znacznie uproszczoną wersją metody opublikowanej w [2]. Modyfikacje wprowadzono z myślą o adaptacji metody dla obliczeń wykonywanych przy pomocy maszyny cyfrowej. Uzyskaną w ten sposób metodę można z powodzeniem stosować w obliczeniach wykonywanych na maszynach cyfrowych nawet dla dużych \(m\) i \(n\).
ЗАМЕЧАНИЯ О ТРАНСПОРТНОЙ ЗАДАЧЕ С КРИТЕРИЕМ ВРЕМЕНИ

РЕЗЮМЕ

Транспортная задача с критерием времени формулируется следующим образом: задача система \((T, M)\), где \(T = (t_{ij})\) матрица порядка \(m \times n\), элементы которой действительны, а \(M = (a_1, a_2, \ldots, a_m, b_1, b_1, \ldots, b_n)\) — система \(m + n\) положительных чисел \(a_i, b_j\), причем

\[
\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.
\]

Следует найти минимум функции

\[
(t(x) = \max_{(i,j) \in \Theta} t_{ij}, \text{где} \ \Theta = \{(i, j): x_{ij} > 0, 1 \leq i \leq m, 1 \leq j \leq n\}
\]

определенной в совокупности всех матриц \(X = (x_{ij})\) порядка \(m \times n\) удовлетворяющих для \(i = 1, 2, \ldots, m\) и \(j = 1, 2, \ldots, n\) следующим условиям

\[
x_{ij} > 0, \quad \sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{m} x_{ij} = b_j.
\]

В статье приведен итерационный метод решения транспортной задачи с критерием времени.

Сначала любым известным методом находится начальное решение \(X(B_1)\), а потом конечную последовательность \(X(B_1), X(B_2), \ldots, X(B_k)\) базисных решений, применяя к этой цели формулы приведенные в пунктах 2-5 алгоритма описанного в главе 3.

Базисное решение \(X(B_k)\) является искомым оптимальным решением. Соответствующие теоремы можно найти в главе 5.

Метод, приведенный в статье, является модифицированной, и значительно упрощенной версией метода опубликованного в [2]. Modiфикации приведены с целью приспособить метод к вычислениям на электронных вычислительных машинах.