The radius of $p$-valent starlikeness for certain classes of analytic functions*

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1. Introduction. Let $E = \{z: |z| < 1\}$. Suppose $p$ is a positive integer and let $S^*(p, a)$ denote the class of functions $f(z) = z^p + \sum_{k=0}^{\infty} a_k z^k$ which are regular in $E$ and satisfy $\Re\{sf^*(z)/f(z)\} > a, z \in E, 0 \leq a < p$. The members of $S^*(p, a)$ are $p$-valent and starlike in $E \ [2]$. Let $n > p$ be a positive integer and suppose $g_n(z) = \sum_{k=0}^{\infty} b_k z^k$, $b_n \neq 0$, is regular in $E$.

We consider the class of functions $h_n(z) = f(z) + g_n(z)$, where $f(z) \in S^*(p, a)$ and $g_n(z)$ satisfies $\Re\{g_n(z)/f(z)\} > -1, z \in E$. In the first part of this paper we determine the radius of $p$-valent starlikeness for this class and also for the subclass consisting of those functions $h_n(z) = f(z) + g_n(z)$ for which $|g_n(z)| \leq |f(z)|, z \in E$.

Let $CS^*(p, a)$ denote the class of functions $h(z) = z^p + \sum_{k=0}^{\infty} c_k z^k$ which are regular in $E$ and satisfy $\Re\{h(z)/f(z)\} > 0, z \in E$, for some $f(z) \in S^*(p, a)$. When $p = 1$, $a = 0$, this definition gives the class of close-to-star functions introduced by Reade \[6\]. If $h(z) \in CS^*(p, a)$, then $h(z) = f(z) + [h(z) - f(z)] = f(z) + g_n(z)$, where $f(z) \in S^*(p, a)$ and $g_n(z) = \sum b_k z^k$ ($n > p$) is regular and satisfies $\Re\{g_n(z)/f(z)\} > -1, z \in E$. Similarly, if $|h(z)/f(z) - 1| < 1, z \in E$, then $h(z) = f(z) + g_n(z)$, where $|g_n(z)| \leq |f(z)|, z \in E$. Thus, the results mentioned above yield the radius of $p$-valent starlikeness for the class $CS^*(p, a)$ and that of the subclass of $CS^*(p, a)$ consisting of those functions $h(z)$ which satisfy $|h(z)/f(z) - 1| < 1, z \in E$, for some $f(z) \in S^*(p, a)$.

Suppose $0 < \beta \leq 1$. In the last section we give the radius of $p$-valent starlikeness for the two subclasses of $CS^*(p, a)$ consisting of the functions $h(z)$ which satisfy respectively $\Re\{h(z)/f(z)\}^{1/\beta} > 0$, and $|\{h(z)/f(z)\}^{1/\beta} - 1| < 1, z \in E$, for some $f(z) \in S^*(p, a)$.

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The results in the paper are extensions of some similar work done by MacGregor [3] and [4].

2. Preliminaries. We shall make frequent use of the fact that for a function \( h(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k \) which is regular in \( |z| < r \), the condition \( \text{Re}\{zh'(z)/h(z)\} > 0 \), \( |z| < r \), is necessary and sufficient for \( h(z) \) to be \( p \)-valent and starlike for \( |z| < r \) [2].

The following lemma is well-known for the case \( p = 1, a = 0 \) ([5], p. 173, problem 11). The general result is easily obtained from this special case.

**Lemma 1.** If \( P(z) = p + \sum_{k=1}^{\infty} p_k z^k \) is regular in \( E \) and satisfies \( \text{Re}\{P(z)\} > \alpha \), \( 0 \leq \alpha < p \), then

\[
\text{Re}\{P(z)\} \geq \frac{p - (p-2\alpha)|z|}{1+|z|}, \quad z \in E.
\]

We shall also need the following extension of Schwartz's lemma ([1], p. 290).

**Lemma 2.** If \( \varphi(z) = d_0 + \sum_{k=1}^{\infty} d_k z^k \), \( m \geq 1 \), is regular and bounded by 1 in \( E \), then

\[
|\varphi'(z)| \leq \frac{m|z|^{m-1}}{1-|z|^{2m}} \left(1 - |\varphi(z)|^2\right), \quad z \in E.
\]

If \( d_0 = 0 \), then

\[
|\varphi(z)| \leq |z|^m, \quad z \in E.
\]

3. Main results.

**Theorem 1.** If \( f(z) \in S^*(p, a) \) and \( \text{Re}\{g_n(z)/f(z)\} > -1 \), \( z \in E \), then \( h_n(z) = f(z) + g_n(z) \) is \( p \)-valent and starlike for \( |z| < r(p, a, n) \), where \( r(p, a, n) \) is the smallest positive root of

\[
\lambda(p, a, n; x) = p - (p-2a)x - 2(n-p)x^{n-p} - 2(n-p)x^{n-p+1} - px^{2(n-p)} + (p-2a)x^{2(n-p)+1} = 0.
\]

**Proof.** The function \( k(z) = -2z/(1+z) \) maps \( E \) onto the half plane \( \text{Re}\{w\} > -1 \), and by hypothesis, \( g_n(z)/f(z) \) is subordinate to \( k(z) \). Thus, there is a function \( \varphi(z) \) which is regular and bounded by 1 in \( E \) such that

\[
\frac{g_n(z)}{f(z)} = \frac{-2\varphi(z)}{1+\varphi(z)}.
\]

Furthermore, \( \varphi(z) \) has a zero of order \( n-p \) at \( z = 0 \). It follows that

\[
h_n(z) = f(z) \left\{ \frac{1-\varphi(z)}{1+\varphi(z)} \right\}.
\]
and a computation yields

\[ \frac{zh_n'(z)}{h_n(z)} = \frac{zf'(z)}{f(z)} - \frac{2zp'(z)}{1 - \varphi^2(z)}. \]

The functions \(zf'(z)/f(z)\) satisfies the hypotheses of Lemma 1, so from (1) we obtain

\[ \Re \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{2|z|\varphi'(z)}{|1 - \varphi^2(z)|}. \]

Applying (2) with \(m = n - p\) yields

\[ \frac{|z|\varphi'(z)}{|1 - \varphi^2(z)|} \leq \frac{(n - p)|z|^{n-p}(1 - |\varphi(z)|^2)}{(1 - |z|^{2(n-p)})(1 - |\varphi(z)|^2)} = \frac{(n - p)|z|^{n-p}}{1 - |z|^{2(n-p)}}, \]

and thus

\[ \Re \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{2(n - p)|z|^{n-p}}{1 - |z|^{2(n-p)}}. \]

The last expression is positive for \(|z| < r(p, a, n)\), and so \(h_n(z)\) is \(p\)-valent and starlike for \(|z| < r(p, a, n)\).

If \(f(z) = z^p/(1 + z)^{(n-p)}\) and \(g_n(z) = -2z^{n-p}f(z)/(1 + z^{n-p})\), then \(\Re \{zh_n'(z)/h_n(z)\} = 0\) for \(z = r(p, a, n)\). Thus, for this choice of \(f(z)\) and \(g_n(z)\) the function \(h_n(z)\) is not \(p\)-valent and starlike in \(|z| < r\) for any \(r > r(p, a, n)\).

**Corollary.** \(\Re \{g_n(z)/z^p\} > -1, z \in E\), then \(h_n(z) = z^p + g_n(z)\) is \(p\)-valent and starlike for

\[ |z| < \left\{ \frac{p - n + \sqrt{(n-p)^2 + p^2}}{p} \right\}^{1/(n-p)}. \]

**Proof.** Letting \(f(z) = z^p\) in Theorem 1 yields

\[ \Re \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - 2(n - p)|z|^{n-p}}{1 - |z|^{2(n-p)}} = \frac{p - 2(n - p)|z|^{n-p} - p|z|^{2(n-p)}}{1 - |z|^{2(n-p)}}, \]

so \(\Re \{zh_n'(z)/h_n(z)\} > 0\) for

\[ |z| < \left\{ \frac{n - p - \sqrt{(n-p)^2 + p^2}}{-p} \right\}^{1/(n-p)}. \]

The radius is exact for the choice \(g_n(z) = -2z^n/(1 + z^{n-p})\).
THEOREM 2. If \( f(z) \in S^+(p, a) \) and \( |g_n(z)| \leq |f(z)|, z \in E \), then \( h_n(z) = f(z) + g_n(z) \) is \( p \)-valent and starlike for \( |z| < R(p, a, n) \), where \( R(p, a, n) \) is the smallest positive root of

\[
\mu(p, a, n; x) = p - (p - 2a)x - nx^{n-p} - (n + 2a - 2p)x^{n-p+1} = 0.
\]

Proof. Let \( \varphi(z) = g_n(z)/f(z) = \sum_{n-p}^\infty d_k z^k \). Then \( \varphi(z) \) is regular and bounded by 1 in \( E \), and

\[
h_n(z) = f(z) \{1 + \varphi(z)\}.
\]

A computation yields

\[
\frac{zh_n'(z)}{h_n(z)} = \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{1 + \varphi(z)},
\]

and so

\[
\text{Re} \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{|z||\varphi'(z)|}{|1 + \varphi(z)|}.
\]

Applying (2) and (3) with \( m = n-p \) we get

\[
\frac{|z||\varphi'(z)|}{|1 + \varphi(z)|} \leq \frac{(n - p)|z|^{n-p}(1 - |\varphi(z)|^4)}{(1 - |z|^{2n-p})(1 - |\varphi(z)|^4)} \leq \frac{(n - p)|z|^{n-p}(1 + |z|^{n-p})}{1 - |z|^{2n-p}}.
\]

Thus,

\[
\text{Re} \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{(n - p)|z|^{n-p}}{1 - |z|^{n-p}} = \frac{\mu(p, a, n; |z|)}{(1 + |z|)(1 - |z|^{n-p})},
\]

and the last expression is positive for \( |z| < R(p, a, n) \).

To see that the result is sharp let \( f(z) = z^p/(1 + z)^{2(n-p)} \) and \( g_n(z) = -z^{n-p}f(z) \), in which case, \( \text{Re} \{zh_n'(z)/h_n(z)\} = 0 \) for \( z = R(p, a, n) \).

COROLLARY. If \( |g_n(z)| \leq |z|^p, z \in E \), then \( h_n(z) = z^p + g_n(z) \) is \( p \)-valent and starlike for \( |z| < (p/n)^{(n-p)} \).

Proof. Letting \( f(z) = z^p \) in Theorem 2 yields

\[
\text{Re} \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - (n - p)|z|^{n-p}}{1 - |z|^{n-p}} = \frac{p - n|z|^{n-p}}{1 - |z|^{n-p}},
\]

and the result follows. The radius \( (p/n)^{(n-p)} \) is exact for the choice \( g_n(z) = -z^n \).

4. Throughout this section \( h(z) \) denotes a function of the form

\[
h(z) = z^p + \sum_{p+1}^\infty c_k z^k
\]

which is regular in \( E \) and vanishes only at \( z = 0 \). We assume \( 0 < \beta \leq 1 \).
Theorem 3. If \( f(z) \in S^*(p, a) \) and \( \text{Re}\{h(z)/f(z)\}^{1/\beta} > 0, z \in E \), then \( h(z) \) is \( p \)-valent and starlike for

\[
|z| < \sigma(p, a, \beta) = \frac{(p + \beta - a) - \sqrt{(p + \beta - a)^2 - p(p - 2a)}}{p - 2a},
\]

where the expression above is defined by its limit when \( a = p/2 \).

Proof. With the appropriate choice of the branch, \( \{h(z)/f(z)\}^{1/\beta} \) takes the value 1 at \( z = 0 \) and is subordinate to \((1 - z)/(1 + z)\). Thus

\[
h(z) = f(z) \left( \frac{1 - \varphi(z)}{1 + \varphi(z)} \right)^\beta,
\]

where \( \varphi(z) \) is regular and bounded by 1 in \( E \), \( \varphi(0) = 0 \). A computation yields

\[
\frac{zh_n'(z)}{h_n(z)} = \frac{zf'(z)}{f(z)} - \frac{2\beta z\varphi'(z)}{1 - \varphi^2(z)},
\]

and from (2) with \( m = 1 \) we get

\[
\text{Re} \left\{ \frac{zh_n'(z)}{h_n(z)} \right\} \geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{2\beta |z| |\varphi'(z)|}{1 - |\varphi(z)|^2} \geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{2\beta |z|}{1 - |z|^2} = \frac{p - 2(p + \beta - a)|z| + (p - 2a)|z|^2}{1 - |z|^2}.
\]

The last expression is positive for \( |z| < \sigma(p, a, \beta) \), and so \( h_n(z) \) is \( p \)-valent and starlike for \( |z| < \sigma(p, a, \beta) \).

The radius \( \sigma(p, a, \beta) \) is exact for the choice \( f(z) = z^p/(1 + z)^{(p - a)} \) and \( h(z) = f(z)\{((1 - z)/(1 + z))^{1/\beta}\} \).

Corollary. If \( \text{Re}\{h(z)/z^p\}^{1/\beta} > 0, z \in E \), then \( h(z) \) is \( p \)-valent and starlike for

\[
|z| < \frac{-\beta + \sqrt{\beta^2 + p^2}}{p}.
\]

Proof. If \( f(z) = z^p \) in Theorem 3, then

\[
\text{Re} \left\{ \frac{zh'(z)}{h(z)} \right\} \geq p - \frac{2\beta |z|}{1 - |z|^2} \geq \frac{p - 2\beta |z| - p |z|^2}{1 - |z|^2},
\]

and the result follows.

The radius is exact for the choice \( h(z) = z^p\{((1 - z)/(1 + z))^{1/\beta}\} \).
Theorem 4. If \( f(z) \epsilon S^*(p, a) \) and \( \{|h(z)/f(z)|^{1/\beta} - 1| < 1, z \epsilon E \), then \( h(z) \) is \( p \)-valent and starlike for

\[
|z| < \Sigma(p, a, \beta) = \frac{(2p + \beta - 2a - \sqrt{(2p + \beta - 2a)^2 - 4p(p - 2a - \beta)}}{2(p - 2a - \beta)},
\]

where the expression above is defined by its limit when \( a = (p - \beta)/2 \).

Proof. With the appropriate choice of the branch, \( \{|h(z)/f(z)|^{1/\beta} \) takes the value 1 at \( z = 0 \) and is subordinate to \( 1 + z \). Thus,

\[
h(z) = f(z)(1 + \varphi(z))^\beta,
\]

where \( \varphi(z) \) is regular and bounded by 1 in \( E \), \( \varphi(0) = 0 \). It follows that

\[
\frac{zh'(z)}{h(z)} = \frac{zf'(z)}{f(z)} + \frac{\beta z\varphi'(z)}{1 + \varphi(z)},
\]

and from (2) and (3) with \( m = 1 \) we get

\[
\text{Re} \begin{Bmatrix} \frac{zh'(z)}{h(z)} \end{Bmatrix} \geq \frac{p - (1 - 2a)|z|}{1 + |z|} - \frac{\beta |z| |\varphi'(z)|}{1 - |\varphi(z)|}
\]

\[
\geq \frac{p - (p - 2a)|z|}{1 + |z|} - \frac{\beta |z|}{1 - |z|}
\]

\[
= \frac{p - (2p + \beta - 2a)|z| + (p - 2a - \beta)|z|^2}{1 - |z|^2}.
\]

The last expression is positive for \( |z| < \Sigma(p, a, \beta) \).

The radius \( \Sigma(p, a, \beta) \) is exact for the choice \( f(z) = z^p/(1 + z)^{(p-a)} \) and \( h(z) = f(z)(1 - z)^\beta \).

Corollary. If \( \{|h(z)/z^p|^{1/\beta} - 1| < 1, z \epsilon E \), then \( h(z) \) is \( p \)-valent and starlike for \( |z| < p/(p + \beta) \).

Proof. Letting \( f(z) = z^p \) in Theorem 4 yields

\[
\text{Re} \begin{Bmatrix} \frac{zh'(z)}{h(z)} \end{Bmatrix} \geq \frac{p - \beta |z|}{1 - |z|} = \frac{p - (p + \beta)|z|}{1 - |z|},
\]

and so \( \text{Re} (zh'(z)/h(z)) > 0 \) for \( |z| < p/(p + \beta) \).

The radius \( p/(p + \beta) \) is exact for the choice \( h(z) = z^p(1 - z)^\beta \).

References


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