On the equation $x''(t) = F(t, x(t))$
in the Sobolev space $H^1(R)$

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Abstract. The existence of solutions of the nonlinear equation $x''(t) = F(t, x(t))$ in the Sobolev space $H^1(R)$ is established.

1. Introduction. We study the existence of solutions of the nonlinear equation $x''(t) = F(t, x(t))$ in the Sobolev space $H^1(R)$. We make assumptions concerning $F$ under which the function $F(\cdot, x(\cdot))$ is locally integrable for any $x \in H^1(R)$. In this way, we may understand the above equation in the sense of distributions.

Other assumptions concerning $F$ give an a priori bound for solutions. Assumptions of this kind may be found in papers [1], [2] concerning equations on a bounded interval, and in paper [5] treating equations on the half-line.

2. Notation. By $H^s(R)$, for integer $s \geq 0$, we denote the Sobolev space

$$\{x \in L^2(R): x^{(i)} \in L^2(R), \ 0 \leq i \leq s\}$$

normed in the standard way:

$$\|x\|_s^2 = \sum_{i=0}^{s} \|x^{(i)}\|^2,$$

where $\|\cdot\|$ stands for the norm in $L^2(R)$.

We denote by $H^s_0 (H^0_{loc}(R) = L^2_{loc}(R))$ the local Sobolev space (see for instance [3]) and treat it as a Fréchet space with the topology defined by the system of semi-norms

$$p_n^2(x) = \sum_{i=0}^{s} \int_{-n}^{n} |x^{(i)}(t)|^2 dt \quad \text{for} \ n = 1, 2, \ldots$$

We denote by $C^\infty_0(R)$ the space of $C^\infty$-functions on the line with compact support and by $\mathcal{D}'(R)$ the space of distributions on the line.

3. Existence of solutions of the equation $x''(t) = F(t, x(t))$.

Theorem 1. Let $F: R^2 \rightarrow R$ have the form

$$F(t, y) = F_1(t, y) + F_2(t),$$

(1)
where $F_2 \in L^2_{\text{loc}}(\mathbb{R})$ and $F_1$ is continuous on the set $\bigcup_{i \in \mathbb{Z}} (t_i, t_{i+1}] \times \mathbb{R}$ and has continuous extensions to every product $[t_i, t_{i+1}] \times \mathbb{R}$ ($i \in \mathbb{Z}$). Here, $\{t_i: i \in \mathbb{Z}\}$ is a division of the line such that $t_i < t_{i+1}$, $t_i \to +\infty$ as $i \to +\infty$ and $t_i \to -\infty$ as $i \to -\infty$.

Suppose that there exist positive constants $a, C$ and a nonnegative function $f \in L^2(\mathbb{R})$ such that

$$y(F(t, y) - a^2 y) \geq 0 \quad \text{for } |y| \geq f(t)$$

almost everywhere with respect to $t$ (a.e. $t$), and

$$|F(t, y)| \leq |F(t, 0)| + C|y| \quad \text{for } |y| \leq f(t) \text{ a.e. } t.$$

Suppose finally that

$$F(\cdot, 0) \in L^2(\mathbb{R}).$$

Then the equation

$$x''(t) = F(t, x(t))$$

has a solution $x$ in $H^1(\mathbb{R})$ for which $\|x\|_1 \leq M$, where

$$M = \left(\min(1, \alpha)^{-1}(\|F(\cdot, 0)\| + \|x\| + (C + a^2) \|f\|^2)^{1/2}. $$

The proof is based on several lemmas.

**Lemma 1.** If $x \in H^1(\mathbb{R})$ then $x$ is a continuous function tending to 0 at $\pm \infty$ and

$$\sup_{t \in \mathbb{R}} |x(t)| \leq 2^{-1/2} \|x\|_1.$$

**Proof.** See [3], Corollary 7.9.4. We prove only (7):

$$x'(t) = \int_{-\infty}^{t} x(\tau)x'(\tau)d\tau - \int_{-\infty}^{+\infty} x(t)x'(t)dt \leq \int_{-\infty}^{+\infty} \|x\| \|x'\|dt \leq 2^{-1}(\|x\|^2 + \|x'\|^2) = 2^{-1} \|x\|_1^2,$$

and (7) follows.

Write equation (5) in the form

$$x''(t) - a^2 x(t) = G(t, x(t)),$$

where $G(t, y) = F(t, y) - a^2 y$. Observe that a.e. $t$

$$|G(t, y)| \leq |F(t, 0)| + (C + a^2)|y| \quad \text{for } |y| \leq f(t).$$

Let

$$G_n(t, y) = \begin{cases} G(t, y) & \text{for } |t| \leq n, \\ 0 & \text{for } |t| > n, \end{cases} \quad n = 1, 2, \ldots$$
Lemma 2. $G_n$ has the following properties:

(i) If $x_k \to x$ in $H^1(\mathbb{R})$, then
$$G_n(t, x_k(t)) \to G_n(t, x(t)) \quad \text{as} \quad k \to \infty$$
uniformly outside a set of measure zero.

(ii) If $\|x\|_1 \leq N$, then
$$|G_n(t, x(t))| \leq K + |F_2(t)|$$
a.e. $t$ for some constant $K = K(N, n)$.

(iii) $G_n(\cdot, x(\cdot)) \in L^2(\mathbb{R})$ for $x \in H^1(\mathbb{R})$.

Proof. (i) Let $x_k \to x$ in $H^1(\mathbb{R})$ and $\|x_k\|_1, \|x\|_1 \leq N$. Then (7) implies $|x_k(t)|, |x(t)| \leq 2^{-1/2}N$ for $t \in \mathbb{R}$. From (1), $G - F_2$ is uniformly continuous on any set of the form $]t_i, t_{i+1}[ \times [-2^{-1/2}N, 2^{-1/2}N]$, because it has a continuous extension to the compact set $[t_i, t_{i+1}[ \times [-2^{-1/2}N, 2^{-1/2}N]$. Then $G_n(t, x_k(t)) \to G_n(t, x(t))$ as $k \to \infty$, uniformly for $t \in ]t_i, t_{i+1}[\cap [-n, n] \neq \emptyset$ we get the assertion.

We prove (ii) likewise using the boundedness of a continuous function on a compact set.

(iii) $G_n(\cdot, x(\cdot))$ is measurable and vanishes outside a compact set, thus it belongs to $L^2(\mathbb{R})$ by (ii).

Now, consider the equation
$$x''(t) - a^2x(t) = \lambda G_n(t, x(t))$$
with the parameter $\lambda \in [0, 1]$, and compute an a priori estimate of the norm of its solutions:

Lemma 3. If $x = x_{\lambda,n} \in H^1(\mathbb{R})$ is a solution of (11), then $\|x\|_1 \leq M$, where $M$ is defined by (6).

Proof. Observe that Lemma 2(iii) implies that $x \in H^2(\mathbb{R})$. Multiply (11) by $x(t)$ and integrate over $\mathbb{R}$:

$$\int_{\mathbb{R}} x(t)x'(t)dt - a^2 \int_{\mathbb{R}} x^2(t)dt = \lambda \int_{\mathbb{R}} x(t)G_n(t, x(t))dt.$$

We integrate by parts the first integral in (12) making use of $x(\pm \infty) = x'(\pm \infty) = 0$ (Lemma 1), to obtain
$$\|x''\|^2 + a^2\|x\|^2 = -\lambda \int_{\mathbb{R}} x(t)G_n(t, x(t))dt.$$ 

Let $S = \{t \in \mathbb{R}: |x(t)| \leq \int f(t)\}$. Inequalities (2), (3) and (9) imply

$$\min(1, a^2) \|x\|_1^2 \leq \|x''\|^2 + a^2\|x\|^2$$

$$= -\lambda \int_{S} x(t)G_n(t, x(t))dt - \lambda \int_{\mathbb{R}\setminus S} x(t)G_n(t, x(t))dt$$

$$\leq \int_{S} |x(t)G_n(t, x(t))|dt \leq \int_{\mathbb{R}} |f(t)||F(t, 0)| + (C + a^2) \int f(t)dt$$

$$\leq \|f\|_{1} \left( \|F(\cdot, 0)\| + (C + a^2) \|f\| \right).$$

We have used the Schwarz inequality in the last step.
Simple calculations finish the proof.

Inverting the operator \( x \mapsto x'' - a^2 x \), we see that in \( H^1(\mathbb{R}) \) equation (11) is equivalent to

(13) \[ x = \lambda A_n x, \]

where

(14) \[ (A_n x)(t) = -(2a)^{-1} \int_{-n}^{n} e^{-a|t-s|} G(s, x(s)) ds. \]

We have

(15) \[ (A_n x)'(t) = 2^{-1} \int_{-n}^{n} \text{sgn}(t-s)e^{-a|t-s|} G(s, x(s)) ds, \]

(16) \[ (A_n x)''(t) = G_n(t, x(t)) - 2^{-1} a \int_{-n}^{n} e^{-a|t-s|} G(s, x(s)) ds. \]

An important step in the proof of Theorem 1 is:

**Lemma 4.** The embedding \( H^2_{\text{loc}}(\mathbb{R}) \to H^1_{\text{loc}}(\mathbb{R}) \) is continuous and transforms bounded sets into precompact ones.

The proof is in [3], Theorem 10.1.27.

**Lemma 5.** The operator \( A_n : H^1(\mathbb{R}) \to H^1(\mathbb{R}) \) defined by (14) is continuous and transforms bounded sets into precompact ones.

**Proof.** The continuity of \( A_n \) can be obtained from Lemma 2(i), (14) and (15).

Take a bounded sequence \( (x_k) \), \( k = 1, 2, \ldots \), in \( H^1(\mathbb{R}) \). Lemma 2(ii) and (14)--(16) imply the boundedness of the sequence \( (A_n x_k) \), \( k = 1, 2, \ldots \), in \( H^2(\mathbb{R}) \), hence also in \( H^2_{\text{loc}}(\mathbb{R}) \). Using Lemma 5, we take a subsequence \( (A_n x_{k_l}) \) which is convergent to some \( y \) in \( H^1_{\text{loc}}(\mathbb{R}) \).

Let \( \psi \in C_0^\infty(\mathbb{R}) \), \( \psi(t) = 1 \) for \( t \in [-n, n] \). We have

(17) \[ \| \psi A_n x_{k_l} - \psi y \|_1 \to 0 \quad \text{as} \quad l \to \infty. \]

Observe that

(18) \[ (A_n x_{k_l})(t) = e^{a(t-n)}(A_n x_{k_l})(-n) \quad \text{for} \quad t \leq -n, \]

(19) \[ (A_n x_{k_l})(t) = e^{a(t-n)}(A_n x_{k_l})(n) \quad \text{for} \quad t \geq n. \]

Notice that convergence in \( H^1_{\text{loc}}(\mathbb{R}) \) implies pointwise convergence (Lemma 1), hence

(20) \[ y(t) = e^{a(t-n)} y(-n) \quad \text{for} \quad t \leq -n, \]

(21) \[ y(t) = e^{a(t-n)} y(n) \quad \text{for} \quad t \geq n. \]
Now, it is easy to see that (17)–(21) imply that \( A_n x_n \to y \) in \( H^1(\mathbb{R}) \). Lemma 5 is proved.

**Lemma 6.** The equation

\[
x''(t) - a^2 x(t) = G_n(t, x(t))
\]

has a solution in \( H^1(\mathbb{R}) \).

**Proof.** Equation (22), considered in \( H^1(\mathbb{R}) \), is equivalent to (13) for \( \lambda = 1 \). Write (13) in the form

\[
(I - \lambda A_n) x = 0,
\]

where \( I \) stands for the identity mapping. We treat \( I - \lambda A_n \) as a mapping from the ball \( B(0, M + \vartheta) \subset H^1(\mathbb{R}) \) into \( H^1(\mathbb{R}) \) (\( M \) is defined by (6)) and use the Leray–Schauder degree theory (see, for instance, [4]), since \( A_n \) is compact due to Lemma 5. From Lemma 3, we know that \( (I - \lambda A_n) x \neq 0 \) for \( \|x\|_1 = M + \vartheta \), so the Leray–Schauder degree

\[
\deg(I - A_n, B(0, M + \vartheta), 0) = \deg(I, B(0, M + \vartheta), 0) = 1 \neq 0.
\]

Therefore, equation (22) has a solution in \( H^1(\mathbb{R}) \).

Consider the sequence \( (x_n), n = 1, 2, \ldots, \) of solutions of equation (22). Lemma 3 implies that \( (x_n) \) is bounded in \( H^1(\mathbb{R}) \), and, by Lemma 2(ii), (10) and (22), \( (x_n) \) is bounded in \( H^1_{\text{loc}}(\mathbb{R}) \).

Using Lemma 4, we choose a subsequence \( (x_{n_j}) \) which is convergent to some \( x \) in \( H^1_{\text{loc}}(\mathbb{R}) \). But \( \|x_{n_j}\|_1 \leq M \), so \( x \in H^1(\mathbb{R}) \) and \( \|x\|_1 \leq M \).

We shall prove that \( x \) is a solution of (8). We have \( \varphi x_{n_j} \to \varphi x \) in \( H^1(\mathbb{R}) \) for any \( \varphi \in C_0^\infty(\mathbb{R}) \). Therefore, Lemma 2(i) and (10) imply that

\[
G(\cdot, x_{n_j}(\cdot)) \to G(\cdot, x(\cdot)) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).
\]

The convergence \( x_{n_j} \to x \) in \( H^1_{\text{loc}}(\mathbb{R}) \) implies that

\[
x_{n_j} \to x \quad \text{in} \quad \mathcal{D}'(\mathbb{R}),
\]

hence

\[
x_{n_j}' \to x' \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).
\]

(23)–(25) imply that \( x \) is a solution of (8).

The proof of Theorem 1 is complete.

**Theorem 2.** For any solution \( x \) of (5) in \( H^1(\mathbb{R}) \), we have \( \|x\|_1 \leq M \), where \( M \) is defined by (6).

**Proof.** Let \( x \) be a solution of (5) in \( H^1(\mathbb{R}) \). For \( n = 1, 2, \ldots \), \( x|_{[-n,n]} \) is a solution of (22) on \( [-n,n] \). Extending \( x|_{[-n,n]} \) by (14), we get a solution \( x_n \) of (22) on the line. Since \( \|x_n\|_1 \leq M \) (Lemma 3), we have \( \|x\|_1 \leq M \).
References


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