On the continuous dependence of local analytic solutions of a functional equation on given functions

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We consider the problem of the continuous dependence on the given functions, for local analytic solutions of the equation

\[ \Phi(z) = H_0(z, \Phi[f_0(z)]), \]

where \( H_0 \) and \( f_0 \) are given functions and \( \Phi \) is unknown.

This problem was investigated in [2] for the equation

\[ \Phi[f_0(z)] + g_0(z) \Phi(z) = h_0(z). \]

Together with equation (1) we consider the sequence of equations

\[ \Phi(z) = H_n(z, \Phi[f_n(z)]) \quad (n = 1, 2, \ldots). \]

We shall assume that for \( n = 0, 1, 2, \ldots: \)

(I) \( f_n \) is analytic in the disc \( |z| \leq r_0, f_n(0) = 0, \) and \( |f_n'(0)| \leq \delta < 1. \)

(II) \( H_n \) is an analytic function of two complex variables \((z, w)\) for \( |z| \leq r_0; |w| \leq R_0 \) and \( H_n(0, 0) = 0, \)

and

(III) \( f_n \to f_0, H_n \to H_0 \) uniformly for \( |z| \leq r_0, |w| \leq R_0. \)

By (I) and (III) there exists a positive integer \( p \) such that

\[ \left| \frac{\partial H_n}{\partial w}(0, 0) \right| < 1 \quad (n = 0, 1, 2, \ldots). \]

Further, we suppose that for \( n = 0, 1, 2, \ldots \)

(V) \( [f_n'(0)]^k \frac{\partial H_n}{\partial w}(0, 0) \neq 1 \quad (k = 1, 2, \ldots, p-1). \)

It follows from W. Smajdor's theorem [4] (cf. also [1], p. 188, and [3]) that for every \( n = 0, 1, 2, \ldots \) there exists exactly one solution \( \Phi_n \) of equation (2) analytic in a neighbourhood of \( z = 0. \)
In this paper we give the proof of the following

Theorem. If hypotheses (I)-(IV) are fulfilled, then solutions $\Phi_n$ exist in a common neighbourhood of $z = 0$ and $\Phi_n$ tends to $\Phi_0$ uniformly in this neighbourhood.

First we prove two lemmas.

Lemma 1. Let $\Phi_n$ be a solution of equation (2) and let hypotheses (I)-(IV) be fulfilled. Then $\Phi_n^{(k)}(0)$ tends to $\Phi_0^{(k)}(0)$ as $n \to \infty$ for $k = 1, 2, \ldots$

Proof. We define the functions $H_{n,k}(z, w, w_1, \ldots, w_k)$ by the recurrent relations

$$H_{n,1}(z, w, w_1) = \frac{\partial H_n}{\partial z} + f'_n(z) \frac{\partial H_n}{\partial w} w_1,$$

(4)

$$H_{n,k+1}(z, w, w_1, \ldots, w_{k+1}) = \frac{\partial H_{n,k}}{\partial z} + f'_n(z) \left( \frac{\partial H_{n,k}}{\partial w} w_1 + \ldots + \frac{\partial H_{n,k}}{\partial w_k} w_k + 1 \right)$$

($k = 1, 2, \ldots$).

$H_{n,k}$ are analytic functions of variables $z, w, w_1, \ldots, w_k$ in the domain $D_k = \{z, w, w_1, \ldots, w_k): |z| \leq r_0; |w| \leq R_0; w_i \leq C, i = 1, \ldots, k\}$, where $C$ is a complex plane. Moreover, we have (4)

$$H_{n,k}(z, w, \ldots, w_k) = G_{n,k}(z, w, \ldots, w_{k-1}) + \frac{\partial H_n}{\partial w} [f'_n(z)]^k w_k,$$

where $G_{n,k}$ is analytic in $D_{k-1}$, and

(6) $\Phi_n^{(k)}(0) = H_{n,k}(0, 0, \Phi_n^{(0)}(0), \ldots, \Phi_n^{(k-1)}(0))$.

Hence and from (5), (3), (IV) we get

(7) $\Phi_n^{(k)}(0) = \frac{G_{n,k}(0, 0, \Phi_n^{(0)}(0), \ldots, \Phi_n^{(k-1)}(0))}{1 - [f'_n(0)]^k \frac{\partial H_n}{\partial w}(0, 0)}$.

Since

$$\Phi_n'(0) = \frac{\frac{\partial H_n}{\partial z}(0, 0)}{1 - \frac{\partial H_n}{\partial w}(0, 0)}$$

it follows from (III) and (IV) that $\Phi_n'(0)$ tends to

$$\Phi_0'(0) = \frac{\frac{\partial H_0}{\partial z}(0, 0)}{1 - \frac{\partial H_0}{\partial w}(0, 0)}.$$

Thus Lemma 1 is true for $k = 1$. 
It follows by induction from (III) that \( H_{n,k} \) converges to \( H_{0,k} \) and \( G_{n,k} \) converges to \( G_{0,k} \) uniformly on every compact \( K \subset D_k \). The proof of Lemma 1 results hence by induction in view of (7).

**Lemma 2.** Suppose that

1. \( A \) is a compact metric space,
2. \( T_n \) is a continuous transformation of \( A \) into itself,
3. \( T_n \) converges to \( T_0 \) uniformly in \( A \),
4. there exists exactly one fixed point \( x_n \) of \( T_n \) in \( A \) for \( n = 0, 1, 2, \ldots \).

Then \( x_n \) converges to \( x_0 \).

**Proof.** Suppose that Lemma is false. Then exists a subsequence \( x_{k_n} \) such that \( \lim_{n \to \infty} x_{k_n} = y \) and \( y \neq x_0 \). From \( 2^o \) and \( 3^o \) we have

\[
y = \lim_{n \to \infty} x_{k_n} = \lim_{n \to \infty} T_{k_n}[x_{k_n}] = T_0[y].
\]

It is a contradiction with \( 4^o \) and the lemma is proved.

**Proof of the theorem.** Let \( \Phi_n \) be the analytic solution of equation (2). Evidently, we may write

\[
\Phi_n(z) = P_n(z) + z^p \varphi_n(z);
\]

\( \varphi_n \) is analytic at \( z = 0 \) and \( \varphi_n(0) = 0 \).

According to Lemma 1, for the proof our theorem it is enough to show that \( \varphi_n \) converges uniformly to \( \varphi_0 \) in a neighbourhood of \( z = 0 \).

Let us define the functions

\[
h_n(z, v) = \frac{H_n(z, P_n[f_n(z)] + [f_n(z)]^p v) - P_n(z)}{z^p}.
\]

By (I) and (II) the partial derivative

\[
\frac{\partial h_n}{\partial v}(z, v) = \frac{\partial H_n}{\partial w}(z, w) \left[ \frac{f_n(z)}{z} \right]^p,
\]

\( w = P_n[f_n(z)] + [f_n(z)]^p v \)

is an analytic function at \((z, v) = (0, 0)\). Next we put

\[
g(z) = H_n(z, P_n[f_n(z)]) - P_n(z).
\]

We shall show that

\[
g^{(p)}(z) = H_{n,p}(z, P_n[f_n(z)], P'_n[f_n(z)], \ldots, P^{(p)}_n[f_n(z)]) - P^{(p)}_n(z).
\]

In fact, we have by (4)

\[
g'(z) = \frac{\partial H_n}{\partial z}(z, P_n[f_n(z)]) + \frac{\partial H_n}{\partial w}(z, P_n[f_n(z)]) f'_n(z) P'_n[f_n(z)] - P'_n(z)
\]

\[
= H_{n,1}(z, P_n[f_n(z)], P'_n[f_n(z)]) - P'_n(z).
\]
Thus (11) is true for $s = 1$. We assume that (11) holds for an $s \geq 1$. Hence we get
\[
g^{(s+1)}(z) = \frac{\partial H_{n,s}}{\partial z} + f_n(z) \left[ \frac{\partial H_{n,s}}{\partial w} P_n'[f_n(z)] + \cdots + \frac{\partial H_{n,s}}{\partial w_k} P_{n(s+1)}[f_n(z)] \right] - P_{n(s+1)}^{(s+1)}(z)
\]
\[
= H_{n,s+1}(z, P_n[f_n(z)], \ldots, P_{n(s+1)}^{(s+1)}[f_n(z)]) - P_{n(s+1)}^{(s+1)}(z)
\]
and (11) is proved.

From (8) we obtain $P_n^{(s)}(0) = \Phi_n^{(s)}(0), s = 1, 2, \ldots, p$. Now, putting $z = 0$ in (11), we have by (6)
\[
g^{(0)}(0) = H_{n,0}(0, \Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(p)}(0)) - \Phi_n^{(p)}(0) = 0, \quad s = 1, \ldots, p
\]
so $h_n(z, 0)$ is an analytic function of $z$ at the point $z = 0$. Hence and from (10) we conclude that $h_n$ is analytic at $(z, v) = (0, 0)$. Moreover,
\[
h_n(0, 0) = 0
\]
and $\varphi_n$ defined by relation (8) satisfies the equation
\[
\varphi(z) = h_n(z, \varphi[f_n(z)]).
\]

Let us take an arbitrary $R_1 > 0$ and let $|v| \leq R_1$. Since $P_n(0) = f_n(0) = 0$, there is a $\sigma_1 > 0, \sigma_1 \leq r_0$, such that
\[
|P_n[f_n(z)]| < \frac{R_0}{2} \quad \text{and} \quad |f_n(z)|^p < \frac{R_0}{2R_1} \quad \text{for} \quad |z| \leq \sigma_1.
\]

By Lemma 1 and (III) there is a positive integer $N$ such that
\[
|P_n[f_n(z)]| \leq \frac{R_0}{2} \quad \text{and} \quad |f_n(z)|^p \leq \frac{R_0}{2R_1} \quad \text{for} \quad n \geq N \text{ and } |z| \leq \sigma_1.
\]

From the continuity of $P_n[f_n(z)]$ and $f_n(z)$ there exists a $\sigma_2 > 0$ such that these inequalities are valid for $n = 1, \ldots, N-1$ and $|z| \leq \sigma_2$.

Taking $r_1 = \min(\sigma_1, \sigma_2)$ we have
\[
|P_n[f_n(z)] + [f_n(z)]^p v| \leq \frac{R_0}{2} + \frac{R_0}{2R_1} \cdot R_1 = R_0
\]
for $(n = 1, 2, \ldots)$, and $|z| \leq r_1$.

Thus we see that $h_n$ is analytic for $|z| \leq r_1, |v| \leq R_1, n = 0, 1, 2, \ldots$, and, moreover,
\[
h_n \rightarrow h_0 \quad \text{uniformly for} \quad |z| \leq r_1, |v| \leq R_1.
\]

It follows from (3) that there exists a $\mu < 1$ such that
\[
|f_n'(0)|^p \frac{\partial H_n}{\partial v}(0, 0) \leq \mu \quad (n = 0, 1, 2, \ldots).
\]
By (10) we have
\[
\frac{\partial h_n}{\partial v}(0,0) = f'_n(0)^p \frac{\partial H_n}{\partial v}(0,0).
\]

Now, by (14) and (15) there exist numbers \(r_2 > 0\) and \(R_2 > 0\) such that
\[
|h_n(x, v_i) - h_n(x, v_2)| \leq \mu |v_1 - v_2|, \quad |z| \leq r_2, \quad |v_i| \leq R_2,
\]
i = 1, 2 (n = 0, 1, 2, ...).

Let us fix a \(K, 0 < K \leq R_2\). It follows from (12), (14) and from the continuity of \(h_0\) that there exists an \(r_3 > 0\) such that
\[
|h_n(x, 0)| \leq (1 - \mu)K \quad \text{for } |z| \leq r_3 \quad (n = 0, 1, 2, ...).
\]

Moreover, by (I) and (III) there exists an \(r_4 > 0\) such that
\[
|f_n(z)| \leq |z| \quad \text{for } |z| \leq r_4 \quad (n = 0, 1, 2, ...).
\]

Let us choose \(r = \min(r_1, r_2, r_3, r_4)\). Define \(A\) as the set of analytic functions \(\varphi\) in the disc \(|z| \leq r\) fulfilling the following condition
\[
|\varphi(z)| \leq K \quad \text{for } |z| \leq r \text{ and } \varphi(0) = 0.
\]

Next, define the transformation \(\psi = T_n[\varphi]\) by formula
\[
\psi(z) = h_n(z, \varphi[f_n(z)]).
\]

We shall prove that the space \(A\) with the metric
\[
\rho(\varphi_1, \varphi_2) = \sup_{|z| \leq r} |\varphi_1(z) - \varphi_2(z)|
\]
and the transformation \(T_n\) fulfills the conditions of Lemma 2.

1° By Vitali’s theorem \(A\) is a compact metric space.

2° By (16) \(T_n\) is continuous. From (20), (18), (19), (16) and (17) we have
\[
|\psi(z)| \leq |h_n(z, \varphi[f_n(z)]) - h_n(z, 0)| + |h_n(z, 0)|
\leq \mu |\varphi[f_n(z)]| + (1 - \mu)K \leq K.
\]

Since \(h_n(0, 0) = 0\), we have \(\psi(0) = 0\). Thus \(\psi \in A\) and this completes the proof of 2°.

3° Let us take an \(\varepsilon > 0\). It follows from (14) that there exists an \(n_1\) such that
\[
|h_n(x, v) - h_0(x, v)| \leq \varepsilon(1 - \mu) \quad \text{for } |z| \leq r, \quad |v| \leq R_2, \quad n \geq n_1.
\]

There is an \(n_2\) such that for \(n \geq n_2\), \(|z| \leq r\) and every \(\varphi \in A\)
\[
|\varphi[f_n(z)] - \varphi[f_0(z)]| \leq \varepsilon.
\]
Indeed
\[ |\varphi[f_n(z)] - \varphi[f_0(z)]| \leq c|f_n(z) - f_0(z)|. \]
where \( c \overset{df}{=} \sup_{\varphi \in A} \sup_{|t| \leq r} |\varphi'(t)| \). The number \( c \) must be finite, for in the opposite case \( A \) cannot be compact.

Now from (22), (16), and (21) we get, for \( n \geq \max(n_1, n_2) \),
\[
|h_n(z, \varphi[f_n(z)]) - h_0(z, \varphi[f_0(z)])| \\
\leq |h_n(z, \varphi[f_n(z)]) - h_0(z, \varphi[f_n(z)])| + |h_0(z, \varphi[f_n(z)]) - h_0(z, \varphi[f_0(z)])| \\
\leq \varepsilon(1 - \mu) + \mu|\varphi[f_n(z)] - \varphi[f_0(z)]| \leq \varepsilon(1 - \mu) + \mu \varepsilon = \varepsilon.
\]

Taking supremum on the left-hand side we get
\[ \varrho(T_n[\varphi], T_0[\varphi]) \leq \varepsilon \quad \text{for } n \geq \max(n_1, n_2) \text{ and } \varphi \in A. \]
This proves 3°.

Let \( \varphi_1, \varphi_2 \in A, \psi_1 = T_n[\varphi_1], \psi_2 = T_n[\varphi_2]. \) From (16) and (18) we get
\[
\varrho(\psi_1, \psi_2) = \sup_{|t| \leq r} |h_n(z, \varphi_1[f_n(z)]) - h_n(z, \varphi_2[f_n(z)])| \\
\leq \mu \sup_{|t| \leq r} |\varphi_1[f_n(z)] - \varphi_2[f_n(z)]| \leq \mu \varrho(\varphi_1, \varphi_2).
\]

Since \( \mu < 1 \), \( T_n \) is a contraction and 4° follows from Banach’s principle.

Now Lemma 2 completes the proof.

References


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