ON EXTREME OPERATORS

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Introduction. The problem to characterize the extreme points of $U(E, F)$, the unit ball in the space of linear operators from a Banach space $E$ to a Banach space $F$, was investigated by several authors (see, e.g., [1], [3], [4], [7], the literature cited there, and the survey article [2]). The most desirable characterization seems to be that an operator $T$ is extreme iff its adjoint $T^*$ maps a dense subset of the extreme points of the unit ball in $F^*$ into the extreme points of the unit ball in $E^*$. (It is easy to see that an operator satisfying this condition is extreme.)

A theorem of that type, extending a result of Blumenthal et al. [1], is stated by Fakhoury in [4]. However, the proof given there appears to be incomplete (see Remark 2 below) and it is the aim of this note to provide a complete proof of that theorem.

Notation. For a compact Hausdorff space $X$ we let $C(X)$ be the Banach space (under the supremum norm) of continuous real-valued functions on $X$ and identify its dual with the space $M(X)$ of real Radon measures on $X$ which will always carry the weak* (or vague) topology. If $\sigma: X \to X$ is an involution (i.e. $\sigma$ composed with itself is the identity on $X$) which is continuous, we let

$$C_\sigma(X) := \{f \in C(X) : f \circ \sigma = -f\}$$

be the Banach space of odd continuous functions on $X$. Its dual may be identified with the set

$$M_\sigma(X) := \{\mu \in M(X) : \mu \circ \sigma = -\mu\}$$

of odd measures on $X$, where $\mu \circ \sigma(f) := \mu(f \circ \sigma)$ for $f$ in $C(X)$. An identification of the fixed points of $\sigma$ yields another compact Hausdorff space $X'$ and another continuous involution $\sigma': X' \to X'$ such that $C_{\sigma'}(X')$ is isometric to $C_\sigma(X)$. Therefore, since we are only interested in the spaces $C_\sigma(X)$, we may and do always assume that $\sigma$ has at most one fixed point.
Then the map 
\[ \omega : X \to M^1_\sigma(X) := \{ \mu \in M_\sigma(X) : \|\mu\| \leq 1 \} \]
defined by 
\[ x \to \omega_x := \frac{1}{2} (\varepsilon_x - \varepsilon_{\sigma(x)}) \]
is a homeomorphic embedding (\( \varepsilon_x \) denotes the point mass concentrated at \( x \)). Further, the set \( \text{ex}M^1_\sigma(X) \) of extreme points in \( M^1_\sigma(X) \) is exactly \( \{ \omega_x : x \in X, x \neq \sigma(x) \} \). Therefore, its closure is either \( \text{ex}M^1_\sigma(X) \cup \{0\} \) or \( \text{ex}M^1_\sigma(X) \) itself, depending on whether \( \sigma \) has a non-isolated fixed point or not.

**The Theorem.** We now state Fakhoury's theorem in a version slightly different from that given in [4].

**Theorem.** Let \( X \) and \( Y \) be compact Hausdorff spaces, let \( \sigma : X \to X \) and \( \tau : Y \to Y \) be continuous involutions, and suppose that \( X \) is a metrizable space. Let \( U \) be the convex set of linear operators from \( C_\sigma(X) \) into \( C_\tau(Y) \) with norm not greater than 1.

Then for \( T \in U \) the following statements are equivalent:

1. \( T \) is an extreme point of \( U \).

2. There exists a dense open set \( D \subseteq \text{ex}M^1_\sigma(Y) \) such that \( T^*(D) \subseteq \text{ex}M^1_\sigma(X) \).

We have already mentioned that (2) implies (1). To prove the converse we proceed in three steps, each step represented by a lemma.

**The Lemmata.** Our first lemma is a part of the proof of Theorem 1 in [1] and stated without proof.

**Lemma 1.** Let \( K \) be a compact Hausdorff space. Then the set-valued maps \( \Phi^+ \) and \( \Phi^- \) from the unit ball \( M^1(K) \) of \( M(K) \) into its non-empty compact convex subset, defined by
\[ \Phi^+(\mu) := \{ \nu \in M^1(K) : 0 \leq \nu \leq \mu^+ \}, \quad \Phi^-(\mu) := \{ \nu \in M^1(K) : 0 \leq \nu \leq \mu^- \}, \]
are lower semicontinuous in the sense of Michael [5].

**Lemma 2.** Let \( X \) and \( \sigma \) be as in the statement of the Theorem. Suppose that \( \lambda \in M^1_\sigma(X) \) is such that the support of \( \lambda^+ \) contains at least two points. Then there exists a continuous map \( g : M^1_\sigma(X) \to M^1_\sigma(X) \) such that

(a) \( g(-\mu) = -g(\mu) \) for all \( \mu \),

(b) \( \|\mu \pm g(\mu)\| \leq 1 \) for all \( \mu \),

(c) \( g(\lambda) \neq 0 \).

**Proof.** By our assumption there exists a compact subset \( F \) of \( X \) such that \( 0 < \lambda^+(F) < \lambda^+(X) \). Let \( \nu \in M^1(X) \) be defined by \( \nu(h) := \lambda^+(h|F) \) for \( h \) in \( C(X) \). From Lemma 1 and from Michael's continuous selection
theorem [6] we infer that there exist continuous maps $f^+$ and $f^-$ from $M^1_0(X)$ into $M^1(X)$ such that

$$0 \leq f^+(\mu) \leq \mu^+ \quad \text{and} \quad 0 \leq f^-(\mu) \leq \mu^- \quad \text{for all} \ \mu,$$

$$f^+(\lambda) = \nu, \quad f^-(\lambda) = \lambda^-, \quad f^+(\lambda) = f^-(\lambda) = 0.$$

Define $f: M^1_0(X) \to M^1(X)$ by

$$f(\mu) := f^-(\mu) \|f^+(\mu)\| + f^+(\mu)\|f^-(\mu)\|.$$

Since the norm is weak* continuous on the cone of positive measures, $f$ is continuous. Further, $\|\mu \pm f(\mu)\| \leq 1$ holds for all $\mu$ (see, e.g., [1], p. 751).

Finally, let

$$g(\mu) := \frac{1}{2}[f(\mu) - f(\mu) \circ \sigma + f(-\mu) \circ \sigma - f(-\mu)].$$

Then one easily verifies that $g$ maps $M^1_0(X)$ into itself, is continuous, and satisfies (a) and (b). It remains to show that $g(\lambda) \neq 0$ holds. If not, then

$$0 = \lambda^- \|\nu\| + \nu \|\lambda^-\| - (\lambda^- \circ \sigma) \|\nu\| - (\nu \circ \sigma) \|\lambda^-\|,$$

hence $\|\nu\|(\lambda^- \circ \sigma - \lambda^-) = \|\lambda^-\|(\nu - \nu \circ \sigma)$. Now $\lambda^- \circ \sigma = \lambda^+$ because $\lambda$ is odd, and we get $0 \leq \nu \circ \sigma \leq \lambda^+ \circ \sigma = \lambda^-$, which implies that $\nu$ and $\nu \circ \sigma$ are mutually singular. Therefore $\lambda^+ = \alpha \nu$ must hold. But this fact and the choice of $\nu$ let us end up with the absurd inequality

$$0 < \lambda^+(X \setminus F) = \alpha \nu(X \setminus F) = a\lambda^+((X \setminus F) \cap F) = 0.$$

The next lemma is modelled after a similar lemma due to Sharir [7], Theorem 1.

**Lemma 3.** Let $Y$ and $\tau$ be as in the statement of the Theorem, $E$ a Banach space, and $S: E \to C_\tau(Y)$ an extreme point of $U(E, C_\tau(Y))$. Then the set

$$D := \{\mu \in \text{ex}M^1_0(Y): \|S^*(\mu)\| = 1\}$$

is dense in $\text{ex}M^1_0(Y)$.

**Proof.** We first observe that $\text{ex}M^1_0(Y)$ is open in its closure, and therefore a Baire space. The map $e^* \to \|e^*\|$ is weak* lower semicontinuous on $E^*$, which implies that the set

$$D_n := \{\mu \in \text{ex}M^1_0(Y): \|S^*(\mu)\| > 1 - 1/n\}$$

is open for each positive integer $n$. Hence, by Baire's category theorem, the lemma is proved if we can show that each of the sets $D_n$ is dense.
Suppose this is false for some $n$. Choose $\mu_0$ in $\text{ex} M^1_n(Y)$ not in the closure of $D_n$. Then $\mu_0 \neq 0$ and $\mu_0 \neq -\mu_0$; hence there exists a continuous real-valued function $k$ on the closure of $\text{ex} M^1_n(Y)$ such that $0 \leq k \leq 1$, $k(\mu_0) = 1$, $k(-\mu_0) = 0$, and $k(\mu) = 0$ for all $\mu$ in $D_n$. Then the function $\hat{k}$ on $Y$ defined by
\[ \hat{k}(y) := \frac{1}{2} [k(\omega_y) - k(-\omega_y)] \]
is in $C_\tau(Y)$ and $0 < \|\hat{k}\| \leq 1$. Choose $e^*$ in $E^*$ with $\|e^*\| = 1$ and define $H: E \to C_\tau(Y)$ by
\[ [H(e)](y) := (1/n) \hat{k}(y) e^*(e) \]
for $e$ in $E$ and $y$ in $Y$. Then $H$ is linear, $\|H\| \leq 1$, $H \neq 0$, and it is easily checked that $\|S \pm H\| \leq 1$ holds. Consequently, $S$ would not be extreme and we obtain a contradiction.

**Proof of the Theorem.** Let $T$ be an element of $\text{ex} U$. We first show that $T^*(\text{ex} M^1_n(Y))$ is contained in $Z := \{a\omega_x: 0 \leq a \leq 1, x \in X\}$. Suppose that $T^*(\omega_a) = \lambda$ is not in $Z$ for some $a$ in $Y$ with $a \neq \tau(a)$. Then the support of $\lambda^+$ contains more than one point. Let $g: M^1_n(X) \to M^1_o(X)$ be a map as described in Lemma 2 and define a linear operator $G: C_o(X) \to C_\tau(Y)$ by
\[ [G(h)](y) := \langle g(T^*(\omega_y)), h \rangle \]
for $y$ in $Y$ and $h$ in $C_o(X)$. Then $G \neq 0$ and $\|T \pm G\| \leq 1$, which is impossible because $T$ is an extreme point of $U$.

Next, we infer from Lemma 3 that $\|T^*(\nu)\| = 1$ for all $\nu$ in a dense subset $D$ of $\text{ex} M^1_n(Y)$. Therefore, the image $T^*(D)$ is contained in $\{\omega_x: x \in X, x \neq \sigma(x)\}$ which is $\text{ex} M^1_o(X)$. We infer also that $T^*(\text{ex} M^1_n(Y))$ is contained in the closure of $\text{ex} M^1_o(X)$ which in turn is contained in $\text{ex} M^1_o(X) \cup \{0\}$. Consequently, $D$ is in fact equal to $\text{ex} M^1_o(Y) \setminus [(T^*)^{-1}(0)]$, and therefore open in $\text{ex} M^1_n(Y)$.

This completes the proof of the Theorem.

**Remarks.** 1. Denoting, as we already did, the homeomorphic embeddings of $X$ and $Y$ into $\text{ex} M^1_o(X)$ and $M^1_\tau(Y)$ by the same symbol $\omega$ and defining $\varphi: Y \to X$ by $\varphi(y) := \omega^{-1} T^* \omega_y$, we see that, under the hypotheses of the Theorem, an operator $T$ in $\text{ex} U$ is induced in the following sense:

There exists a continuous map $\varphi: Y \to X$ such that
\[ \varphi(\tau(y)) = \sigma(\varphi(y)) \text{ for all } y \in Y, \]
\[ [T(f)](y) = f(\varphi(y)) \text{ for all } y \in Y \text{ and } f \in C_o(X), \]

$\varphi^{-1}(x_o) \cap (Y \setminus \{y_o\})$ has an empty interior, where $x_o$ and $y_o$ are the fixed points — if there are any — of $\sigma$ and $\tau$, respectively.

Since, conversely, an induced operator is always extreme in $U$ and since $C_o(X)$ is separable iff $X$ is metrizable, we also get Fakhoury's theorem as in [4].
2. The gap in the proof of Theorem 14 in [4] occurs on page 2017, where it is implicitly assumed that for a non-zero non-extreme element υ of $M_1^*(X)$ the set

$$\{υ' ∈ M_1^*(X) : υ' = fυ, f ∈ L^1(υ), 1 ≤ f ≤ 2\}$$

contains elements different from υ. But this can be true only if $||υ|| < 1$. This means that from the arguments in [4] one can only conclude that for every μ in $exM_1^*(Y)$ the norm of $T^*(μ)$ is either 0 or 1. Obviously, this fact combined with Lemma 2 would be sufficient to prove the Theorem; however, since Lemma 3 does not depend on any special property of $E$, it seemed justified to include it and use it in lieu of Fakhoury’s result.

REFERENCES


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