On a subclass of univalent functions I

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Abstract. Let $S^*(\lambda)$ denote the class of holomorphic functions $f$ in the unit disc $E$, with $f(0) = 0 = f'(0) - 1$, $f(z)f'(z)/z \neq 0$ for $z$ in $E$ and satisfying the condition
\[
||\frac{z f'(z)}{f(z)} - 1|| \left/ \frac{||zf'(z)/f(z)|| + 1}{||zf'(z)/f(z)|| - 1}\right. < \lambda, \quad 0 < \lambda \leq 1,
\]
$z \in E$. In this paper the class $M(\alpha, \lambda)$ consisting of functions $f$ satisfying in $E$ the condition
\[
||J(\alpha, f) - 1||/||J(\alpha, f) + 1|| < \lambda, \quad 0 < \lambda \leq 1,
\]
where $J(\alpha, f) = \alpha \{1 + zf''(z)/f'(z)\} + (1 - \alpha)zf'(z)/f(z)$, $\alpha > 0$, is introduced and its properties are investigated. It is proved that $M(\alpha, \lambda) \subset S^*(\lambda)$ and the sharp radius $r_0$, such that $f \in S^*(\lambda)$ also satisfies the condition
\[
||J(\alpha, f) - 1||/||J(\alpha, f) + 1|| < \lambda, \quad 0 < \lambda \leq 1, \text{ for } |z| < r_0,
\]
is determined. Further, a representation formula for $f \in M(\alpha, \lambda)$ and an inequality relating the coefficients of functions in $M(\alpha, \lambda)$ are obtained.

1. Introduction. Let $f$ be analytic in the unit disc $E$, with $f(0) = 0$, $f'(0) = 1$, $f(z)f'(z)/z \neq 0$ in $E$. Denote by $V$ the class of these functions.

Let $S^*(\lambda)$ denote the class of functions $f \in V$ satisfying in $E$ the condition
\[
||z f'(z) \left/ \frac{f'(z)}{f(z)} \right. - 1|| \left/ \frac{zf'(z)}{f(z)} + 1\right. < \lambda, \quad 0 < \lambda \leq 1.
\]
This class was introduced by the first author in [4]. For $\lambda = 1$, the class $S^*(\lambda)$ coincides with the well-known class of starlike functions.

Let $K(\lambda)$ denote the class of functions $f \in V$ satisfying in $E$ the condition
\[
||zf'''(z)\left/ \frac{f''(z)}{f'(z)} \right. + 2zf''(z)\left/ \frac{f'(z)}{f(z)} \right. < \lambda, \quad 0 < \lambda \leq 1.
\]
For $\lambda = 1$, the class $K(\lambda)$ coincides with the class of convex functions.

We now introduce the class $M(\alpha, \lambda)$ of functions $f \in V$ satisfying in $E$ the condition
\[
||J(\alpha, f) - 1||/||J(\alpha, f) + 1|| < \lambda, \quad 0 < \lambda \leq 1,
\]
where $J(\alpha, f) = \alpha \{1 + zf''(z)/f'(z)\} + (1 - \alpha)zf'(z)/f(z)$ and $\alpha$ is any positive
real number. For \( \lambda = 1 \), the class \( M(\alpha, \lambda) \) coincides with the class of \( \alpha \)-convex functions.

In this paper we investigate a few properties of the class \( M(\alpha, \lambda) \).

2. It is well known that all \( \alpha \)-convex functions are starlike [3]. We now prove an analogous theorem for the class \( M(\alpha, \lambda) \).

**THEOREM 1.** Let \( f \in M(\alpha, \lambda) \), \( \alpha > 0 \). Then \( f \in S^*(\lambda) \).

**Proof.** Let

\[
\frac{zf'(z)}{f(z)} = \frac{1 - \lambda w(z)}{1 + \lambda w(z)}.
\]

Evidently, \( w(0) = 0 \) and \( 1 + \lambda w(z) \neq 0 \). We shall show that \( |w(z)| < 1 \) for \( z \) in \( E \). For if not, by Jack's lemma [2], there exists \( z_0, z_0 \in E \) such that \( |w(z_0)| = 1 \) and \( z_0w'(z_0) = kw(z_0), k \geq 1 \),

\[
J(\alpha, f(z_0)) = 1 - \lambda w(z_0) \frac{2\alpha k w(z_0)}{1 + \lambda w(z_0)} \\
J(\alpha, f(z_0) + 1) = \lambda \frac{1 + \alpha k - \lambda w(z_0)}{1 - (1 + \alpha k) \lambda w(z_0)}.
\]

Now \( |J(\alpha, f - 1) - 1)/J(\alpha, f + 1) + 1| \leq \lambda \), according as

\[
|1 + \alpha k - \lambda w(z_0)|^2 \leq |1 - (1 + \alpha k) \lambda w(z_0)|^2 \quad \text{or} \quad (2\alpha k + \alpha^2 k^2)(1 - \lambda^2) \leq 0.
\]

Since \( \alpha \) and \( k \) are positive and \( 0 < \lambda \leq 1 \), this last expression is positive. This means that \( f(z) \not\in M(\alpha, \lambda) \), a contradiction. Thus the proof is complete.

**THEOREM 2.** For \( 0 \leq \beta < \alpha \), \( M(\alpha, \lambda) \subset M(\beta, \lambda) \).

**Proof.** If \( \beta = 0 \), then \( M(\alpha, \lambda) \subset M(0, \lambda) \), by Theorem 1. Assume therefore that \( \beta \neq 0 \) and \( f \in M(\alpha, \lambda) \). Then there exist functions \( w_i, i = 1, 2 \), analytic in \( E \) with \( w_i(0) = 0 \) and \( |w_i(z)| < 1 \), \( i = 1, 2 \), such that

\[
\frac{zf'(z)}{f(z)} = \frac{1 - \lambda w_1(z)}{1 + \lambda w_1(z)} \quad \text{and} \quad J(\alpha, f) = \frac{1 - \lambda w_2(z)}{1 + \lambda w_2(z)}.
\]

Now,

\[
J(\beta, f) = \frac{\beta}{\alpha} J(\alpha, f) + \frac{1 - \beta/\alpha}{zf'(z)}.
\]

Since \( \beta < \alpha \), one can show that

\[
J(\beta, f) = \frac{1 - \lambda w(z)}{1 + \lambda w(z)},
\]

for some \( w \) analytic in \( E \), \( w(0) = 0 \) and \( |w(z)| < 1 \).

**Corollary.** For \( \alpha \geq 1 \), \( M(\alpha, \lambda) \subset K(\lambda) \).
Remark. For $\lambda = 1$, the above corollary yields the established result that $\alpha$-convex functions are convex for $\alpha \geq 1$ [3].

3. In view of Theorem 2, given a function in $S^*(\lambda)$ we can find the largest possible value of $\alpha$ such that $f \in M(\alpha, \lambda)$, $\alpha \geq 0$.

Definition. Let $f \in S^*(\lambda)$ and

$$\alpha = \alpha(f) = \text{l.u.b.} \{\beta| f \in M(\beta, \lambda), \beta \geq 0\}.$$ 

Then we say that $f$ is starlike of order $\lambda$ and type $\alpha$ and we write $f \in M^*(\alpha, \lambda)$. Clearly $\alpha$ is non-negative and may be infinite.

If $f \in M^*(\alpha, \lambda)$, then $f \in M(\beta, \lambda)$ for all $\beta$, $0 \leq \beta \leq \alpha$. That is,

$$J(\beta, f) = \frac{1 - \lambda w(z)}{1 + \lambda w(z)}, \quad 0 \leq \beta \leq \alpha,$$

where $w$ is analytic in $E$, $w(0) = 0$ and $|w(z)| < 1$ in $E$. Allow $\beta \prec \alpha$. Then

$$J(\alpha, f) = \frac{(1 - \lambda w(z))/(1 + \lambda w(z))}{f \in M(\alpha, \lambda)}.$$ 

Hence $f \in M^*(\alpha, \lambda)$ for $\alpha < \infty$ if and only if $f \in M(\beta, \lambda)$ for $0 \leq \beta \leq \alpha$ and $f \notin M(\beta, \lambda)$ for $\beta > \alpha$. Thus we can write $S^*(\lambda)$ as a disjoint union

$$S^*(\lambda) = \bigcup_{\alpha > 0} M^*(\alpha, \lambda).$$

Theorem 3. Let $f \in M^*(\alpha, \lambda)$, $\alpha > 0$. For $0 < \beta < \alpha$, choose the branch of

$$\left(zf'(z)/f(z)\right)^\beta$$

which takes the value 1 at the origin. Then the function

$$F_\beta(z) = f(z) \left(zf'(z)/f(z)\right)^\beta$$

belongs to $S^*(\lambda)$.

Proof. $f \in M^*(\alpha, \lambda)$ implies that $f \in M(\beta, \lambda)$ for all $\beta < \alpha$. The result immediately follows from the relation

$$zf'(z)/F_\beta(z) = J(\beta, f).$$

Conversely, assume that $F \in S^*(\lambda)$ and $\alpha > 0$. Define $f$ by the differential equation

$$F(z) = f(z) \left(zf'(z)/f(z)\right)^\alpha.$$  \hspace{1cm} (3.1)

Obviously,

$$f(z) = \left\{\frac{1}{\alpha} \int_0^1 \frac{F^{1/\alpha}(t)}{t} dt\right\}^\alpha$$  \hspace{1cm} (3.2)

is a solution of the differential equation (3.1) with the initial condition $f(0) = 0$. We now show that this formal solution is indeed a function in $M(\alpha, \lambda)$.

Theorem 4. Let $F \in S^*(\lambda)$ and $\alpha > 0$. Then $f$ defined by (3.2) belongs to $M(\alpha, \lambda)$.

Proof. Let $\gamma$ be a path in $E$ connecting 0 and $z$. We assign a value to
\[ \lim \arg t \text{ as } t \to 0 \text{ on } \gamma \text{ and the same value to } \lim \arg F(t) \text{ as } t \to 0 \text{ on } \gamma. \] Now define \( t^{1/a} \) and \([F(t)]^{1/a} \) by continuation. Since \( F(t) = t + A_2 t^2 + \ldots = t(1 + A_2 t + \ldots) \) belongs to \( S^*(\lambda) \), the bracketed series has no zeros in \( E \). Hence

\[
[F(t)]^{1/a} = t^{1/a}(1 + A_2 t + \ldots)^{1/a} = t^{1/a}(1 + b_1 t + \ldots),
\]

where \((1 + b_1 t + \ldots)\) is the branch of \((1 + A_2 t + \ldots)^{1/a}\) which equals 1 when \( t = 0 \),

\[
\int_0^z F^{1/a}(t) t^{-1} dt = \alpha z^{1/a} \left[ 1 + \frac{b_1}{\alpha + 1} z + \ldots \right].
\]

Let

\[
g(z) = \frac{1}{z^{1/a}} \int_0^z F^{1/a}(t) t^{-1} dt = \alpha \left[ 1 + \frac{b_1}{\alpha + 1} z + \ldots \right].
\]

We now show that \( g(z) \) has no zeros in \( E \).

Let \( t = H(u) \) be the inverse of \( u = F(t) \) and let \( p = F(z) \), \( z = H(p) \). Further let \( \Gamma \) be a line segment joining 0 and \( p \). Then \( \Gamma \) lies in the image of \( E \) under \( F \), since \( F \in S^*(\lambda) \) is also starlike. Let \( \gamma \) denote the preimage of \( \Gamma \), in \( E \). Then

\[
g(z) = \frac{1}{z^{1/a}} \int \gamma F(t)^{1/a} t^{-1} dt = \frac{1}{z^{1/a}} \int_0^1 F(t)^{1/a} \frac{H'(u)}{H(u)} du.
\]

Let \( u = pe^{i\theta} \) and \( p = Re^{i\theta} \). Then

\[
|g(z)| = \frac{1}{|z|^{1/a}} \left| \int_0^R \frac{1}{p^{1/a}} \frac{uH'(u)}{H(u)} dp \right| \geq \frac{1}{|z|^{1/a}} \int_0^R \frac{1}{p^{1/a}} \Re \left\{ \frac{uH'(u)}{H(u)} \right\} dp.
\]

Since \( F \in S^*(\lambda) \), there exist constants \( M, N > 0 \) such that

\[
\Re \left\{ \frac{tF'(t)}{F(t)} \right\} \geq M \quad \text{and} \quad \left| \frac{tF'(t)}{F(t)} \right| \leq N
\]

on \( \gamma \). Hence on \( \Gamma \),

\[
\Re \left\{ \frac{uH'(u)}{H(u)} \right\} = \Re \left\{ \frac{1}{tF'(t)/F(t)} \right\} \geq \frac{M}{N^2}.
\]

Therefore

\[
|g(z)| \geq \frac{1}{|z|^{1/a}} \frac{M}{N^2} \alpha R^{1/a} = \alpha \frac{M}{N^2} \left| \frac{F(z)^{1/a}}{z} \right| > 0.
\]

Now choose the branch of \( \left[ \frac{1}{z} g(z) \right]^a \) which takes the value 1 at the origin.
Then \( f(z) = \left[ \frac{1}{z} g(z) \right]^z \) is regular, has its only zero at the origin and \( f'(0) = 1 \). Since \( f(z) \) is a solution of the differential equation (3.1), \( f'(z_0) = 0 \) for some \( z_0, 0 < |z_0| < 1 \), would imply that \( F(z_0) = 0 \), which is impossible. Thus \( f'(z) \neq 0 \) for \( z \in E \). Also from (3.1), \( J(\alpha, f) = z F'(z)/F(z) \). This completes the proof.

Remark. Theorems 3 and 4 yield a representation formula (3.2) for functions in \( M(\alpha, \lambda) \).

If we denote by \( B(\alpha, \lambda) \) the subclass of Bazilevič functions \( f \) defined by

\[
f(z) = \left\{ \alpha \int_0^z F(t) t^{-1} \, dt \right\}^{1/\alpha},
\]

where \( F \in S^*(\lambda) \) and \( \alpha > 0 \), then it can be easily seen that

\[
B\left( \frac{1}{\alpha}, \lambda \right) = M(\alpha, \lambda).
\]

4. In this section we obtain an inequality for the coefficients of functions in \( M(\alpha, \lambda) \).

**Theorem 5.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(\alpha, \lambda) \) and let \( s_1 = 0, s_m = (1 - \alpha) (\beta_m - \alpha_m) + \alpha \gamma_{m-1}, \) \( t_1 = 2, t_m = (1 - \alpha) \beta_m + (1 + \alpha) \alpha_m + \alpha \gamma_{m-1}, \) \( m = 2, 3, \ldots, \) where \( \alpha_m, \beta_m \) and \( \gamma_m \) are defined by

\[
\alpha_m = \sum_{k=1}^{m} (m-k+1) a_k a_{m-k-1}, \\
\beta_m = \sum_{k=1}^{m} k (m-k+1) a_k a_{m-k+1}, \\
\gamma_m = \sum_{k=1}^{m} k (k+1) a_{k+1} a_{m-k+1}.
\]

Then the coefficients \( a_n \) satisfy the following inequality:

\[
\sum_{m=1}^{n} |s_m|^2 \leq \lambda^2 \sum_{m=1}^{n-1} |t_m|^2, \quad n = 2, 3, \ldots
\]

Equality holds for the function

\[
f_\alpha(z) = \left\{ \frac{z}{\alpha} \int_0^z t^{1/\alpha} (1 + \varepsilon \lambda t)^{-2/\alpha} \, dt \right\}^z, \quad |\varepsilon| = 1.
\]

**Proof.** Since \( f \in M(\alpha, \lambda) \),

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \frac{zf'(z)}{f(z)} = \frac{1 - \lambda w(z)}{1 + \lambda w(z)},
\]

where \( w(z) \) is the solution of the differential equation (3.1). The inequality follows from the properties of \( M(\alpha, \lambda) \) and the representation formula (3.2).
where \( w \) is analytic in \( E \), \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( E \). This gives

\[
(1 - \alpha) \left[ z \left( f'(z) \right)^2 - f(z) f'(z) \right] + \alpha z f'(z) f''(z) = -\lambda w(z) \left[ (1 - \alpha) z \left( f'(z) \right)^2 + (1 + \alpha) f(z) f'(z) + \alpha z f'(z) f''(z) \right].
\]

Given \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), we note that

\[
f(z)f'(z) = \sum_{m=1}^{\infty} \alpha_m z^m, \quad (f'(z))^2 = \sum_{m=1}^{\infty} \beta_m z^{m-1}, \quad f(z)f''(z) = \sum_{m=1}^{\infty} \gamma_m z^m,
\]

where \( \alpha_m, \beta_m \) and \( \gamma_m \) are defined in (4.1). Thus (4.2) becomes

\[
(1 - \alpha) \sum_{m=1}^{\infty} (\beta_m - \alpha_m) z^m + \alpha \sum_{m=1}^{\infty} \gamma_m z^{m+1}
\]

\[
= -\lambda w(z) \left[ (1 - \alpha) \sum_{m=1}^{\infty} \beta_m z^m + (1 + \alpha) \sum_{m=1}^{\infty} \alpha_m z^m + \alpha \sum_{m=1}^{\infty} \gamma_m z^{m+1} \right]
\]

which simplifies to

\[
\sum_{m=1}^{\infty} s_m z^m = -\lambda w(z) \left[ \sum_{m=1}^{\infty} t_m z^m \right].
\]

Now

\[
\left| \sum_{m=1}^{n} s_m z^m + \sum_{m=n+1}^{\infty} h_m z^m \right| \leq \lambda \left| \sum_{m=1}^{n-1} t_m z^m \right|
\]

where \( h_m \)'s are some complex numbers. This yields

\[
\sum_{m=1}^{n} |s_m|^2 + \sum_{m=n+1}^{\infty} |h_m|^2 \leq \lambda^2 \sum_{m=1}^{n-1} |t_m|^2
\]

or

\[
\sum_{m=1}^{n} |s_m|^2 \leq \lambda^2 \sum_{m=1}^{n-1} |t_m|^2.
\]

5. We now determine the radius of the largest disc where the converse of Theorem 1 holds.

THEOREM 6. Let \( f \in S^*(\lambda) \). Then \( f \) satisfies condition (1.1) for \( |z| < r_0 \), where \( r_0 \) is the smallest positive root of the equation

\[
1 - (1 + \alpha)(1 + \lambda)r + \lambda r^2 = 0.
\]

The bound \( r_0 \) is sharp.
Proof. Since \( f \in S^*(\lambda) \),

\[
\frac{zf'(z)}{f(z)} = \frac{1 - \lambda w(z)}{1 + \lambda w(z)},
\]

where \( w \) is analytic in \( E \), \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( E \),

\[
J(x, f) = \frac{1 - \lambda w(z)}{1 + \lambda w(z)} - \frac{2\lambda z w'(z)}{(1 - \lambda w(z))(1 + \lambda w(z))},
\]

\[
\left| \frac{J(x, f) - 1}{J(x, f) + 1} \right| = \lambda \left| \frac{w(z) + \alpha zw'(z)(1 - \lambda w(z))}{1 - \alpha zw'(z)(1 - \lambda w(z))} \right| < \lambda
\]

provided

\[
|w(z) + \alpha zw'(z)(1 - \lambda w(z))| < |1 - \alpha zw'(z)(1 - \lambda w(z))|.
\]

This, in turn, is true if

\[
|w(z) + \alpha z| |w'(z)|/(1 - \lambda |w(z)|) < 1 - \alpha |z| |w'(z)|/(1 - \lambda |w(z)|).
\]

Using the following well-known estimate

\[
|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2},
\]

inequality (5.2) reduces to

\[
t^2 \frac{(1 + \lambda)(1 - r^2)}{2 [\lambda + \alpha (1 + \lambda)r - \lambda r^2]} - t(1 + \lambda)(1 - r^2) + 1 - r^2 - \alpha r - \alpha \lambda r \
\geq 0,
\]

where \( |w(z)| = t \) and \( |z| = r \). Denoting the left-hand member of (5.3) by \( E(t) \), we see that \( E(t) \) vanishes when

\[
t = t_1 = \frac{(1 + \lambda)(1 - r^2)}{2 [\lambda + \alpha (1 + \lambda)r - \lambda r^2]}.
\]

Evidently \( t_1 \) is positive. Also \( E'(t) \) is positive. Now \( t_1 \leq r \) according as \( Q(r) \equiv 1 + \lambda - 2\lambda r - (1 + \lambda)(1 + 2\alpha)r^2 + 2\alpha r^3 \leq 0 \). The equation \( Q(r) = 0 \) has at least one root in \((0, 1)\). Call the smallest positive root \( r_1 \). Thus for \( 0 \leq r < r_1, Q(r) > 0 \). This means that for \( 0 \leq r < r_1, t_1 > r \) and \( E(t) \) attains its minimum at \( t = r \) for \( 0 \leq t < r < r_1 \). Also \( E(r) > 0 \) would imply \( E(t) > 0, 0 \leq t < r \). This condition becomes

\[
P(r) \equiv 1 - (1 + \alpha)(1 + \lambda)r + \lambda r^2 > 0.
\]

The equation \( P(r) = 0 \) has at least one root in \((0, 1)\) and let \( r_0 \) be the smallest positive root. Hence for \( 0 \leq r < r_0, P(r) > 0 \). Also \( P(r_1) < 0 \), implying that \( r_0 < r_1 \). Thus

\[
\left| \frac{J(x, f) - 1}{J(x, f) + 1} \right| < \lambda \quad \text{for } |z| < r_0.
\]
If \(|z| = r_0\), then for the function \(f\) corresponding to \(w(z) = z\) in (5.1), we see that

\[
\left| \frac{f'(az) - 1}{f'(az) + 1} \right| = \lambda.
\]

This shows that the bound \(r_0\) is sharp.

**Remark.** For \(\lambda = 1\), Theorem 6 gives the radius of \(\alpha\)-convexity for starlike functions [1].

**References**


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*Reçu par la Rédaction le 16.4.1979*