LINEARLY FUNDAMENTALLY ORDERED SEMIGROUPS

BY

B. M. SCHEIN (SARATOV)

In this paper a function on a set $A$ means a transformation of a subset of $A$ into $A$. Superposition $g \circ f$ of two functions on $A$ is defined as a function on $A$ such that, for every $a \in A$, we have $g \circ f(a) = g(f(a))$, where the left-hand side of this equality is meaningful if the right-hand side is.

A function semigroup is a non-empty set of functions on a set $A$, closed with respect to the superposition which is then considered as the semigroup operation. Every function semigroup $F$ is ordered by the inclusion relation: $f \subset g$ means that $g$ is an extension of $f$.

Let $S$ be a semigroup ordered by an order relation $\leq$ and $P$ be a map of $S$ onto a function semigroup $F$. Then $P$ is an order-isomorphism if and only if $s \leq t \iff P(s) \subset P(t)$ and $P(st) = P(t) \circ P(s)$ for all $s, t \in S$. The form of the last equality depends on the fact that in the product $st$ the first factor is $s$, and in the product $P(t) \circ P(s)$ the first factor is $P(s)$. The ordered semigroup $(S, \leq)$ is called fundamentally ordered if it is order-isomorphic with an inclusion-ordered function semigroup. $(S, \leq)$ is strictly fundamentally ordered if it is order-isomorphic with an inclusion-ordered semigroup of univalent (i.e., one-to-one) functions.

The aim of this paper is to give a complete description of all linearly fundamentally ordered semigroups, all linearly fundamentally orderable semigroups and all semigroups on which every fundamental order may be extended to a linear fundamental order. The analogous problem is solved for strict fundamental orders.

An order relation $\leq$ on a semigroup $S$ is stable if

\[
 s_1 \leq s_2 \land t_1 \leq t_2 \rightarrow s_1 t_1 \leq s_2 t_2
\]

(here $\land$ is the sign of conjunction, $\rightarrow$ is the sign of implication). An order relation $\leq$ is called weakly steady if

\[
 z \leq xv \land z \leq wy \land x \leq u \rightarrow z \leq xy
\]

for all $u, x, z \in S$ and all $v, y \in S^1$; $\leq$ is called steady if

\[
 z \leq xv \land z \leq uv \land z \leq wy \rightarrow z \leq xy
\]
for all \( z \in S \) and \( u, v, x, y \in S^1 \) such that \( xv, uv, uy, xy \in S \). Here \( S^1 \) denotes \( S \) in the case where \( S \) contains an identity, otherwise \( S^1 \) is \( S \) with an identity adjoined.

The main tool used in the present paper is the following

**Theorem on fundamental orders**\(^{(1)}\). An ordered semigroup is fundamentally ordered if and only if the order relation on the semigroup is stable and weakly steady; it is strictly fundamentally ordered if and only if the order is stable and steady.

To formulate our main results we need the following constructions:

**Construction I.** Let \( I \) be a non-empty set linearly ordered by an irreflexive order relation \( \preceq \) and let \((A_i)_{i \in I}\) and \((B_i)_{i \in I}\) be two families of non-empty sets indexed by \( I \) and such that

\[
\begin{align*}
1^o & \quad A_i \cap B_j \neq \emptyset \leftrightarrow A_i \cap A_j \neq \emptyset \leftrightarrow B_i \cap B_j \neq \emptyset \leftrightarrow i = j, \\
2^o & \quad A_i \cap B_i \text{ is a one-element set denoted by } \{a_i\}.
\end{align*}
\]

Write \( C_i = A_i \cup B_i, C = \bigcup (C_i)_{i \in I} \) and define a binary multiplication in \( C \) in the following way:

\[
x y = \begin{cases} 
  a_j & \text{if } x \in C_i, y \in C_j, j \preceq i, \\
  a_i & \text{if } x, y \in C_i, x \notin A_i, \\
  x & \text{otherwise}.
\end{cases}
\]

One can verify that \( C \) endowed with this operation is a semigroup, \( C_i \) are subsemigroups of \( C \), \( A = \bigcup (A_i)_{i \in I} \) is the set of all idempotents of \( C \) and \( A \) is a subsemigroup of \( C \).

**Construction II.** Let \( C \) be a semigroup described in Construction I. Let every set \( A_i \) be linearly ordered in such a way that \( a_i \) is the largest element of \( A_i \). If \( i \) is the largest element of \( I \) and \( B_i = \{a_i\} \), then the linear order on \( A_i \) may be arbitrary and \( a_i \) need not be the largest element of \( A_i \). Let \( B_i \setminus A_i \) be linearly ordered in an arbitrary way. There exists a uniquely determined linear order \( \leq \) on \( C \) such that each element of \( C_i \) precedes each element of \( C_j \) for \( i \preceq j \), each element of \( A_i \) precedes each element of \( B_i \setminus A_i \) for all \( i \in I \), and restrictions of \( \leq \) on \( A_i \) and \( B_i \setminus A_i \) coincide with the order relations given on these sets for all \( i \in I \).

Let us show that \( (C, \leq) \) is a fundamentally ordered semigroup. To prove stability of \( \leq \), suppose \( x \in C_i, y \in C_j, z \in C_k \), and \( x \leq y \). Then \( i \prec j \) or \( i = j \). If \( k \preceq i \), then \( xz = a_k = yz \); if \( i = k \prec j \), then \( xz \in A_i \) and \( yz = a_i \), hence, \( xz \leq yz \); if \( i \preceq k \prec j \), then \( xz = x \leq a_k = yz \); if \( i \preceq k \), then \( xz = x \leq y = yz \). Now let \( i = j = k \). If \( y \in A_i \), then \( x \in A_i \) and \( xz = x \leq y = yz \); if \( y \notin A_i \), then \( yz = a_i \) and \( xz \in A_i \), whence \( xz \leq yz \). Therefore, \( x \leq y \rightarrow xz \leq yz \), i.e., \( \leq \) is right regular.

\(^{(1)}\) Б. М. Шайн, *Представление упорядоченных полугрупп*, Математический сборник 65 (1964), p. 188 - 197.
Now if \( k \preceq i \), then \( zx = z = zy \); if \( i \preceq k \preceq j \), then \( zx = a_i \leq z = zy \); if \( i \preceq k = j \), then \( zx = a_i \leq z \leq zy \); if \( j \preceq k \), then \( zx = a_i \leq a_j = zy \).

If \( i = k \) and \( z \in A_i \), then \( zx = z = zy \); if \( i = k \) and \( z \not\in A_i \), then \( zx = a_i \).

If \( zy \) equals either \( z \) or \( a_i \) in both cases \( zx \leq zy \). Thus, \( \preceq \) is right and left regular, which means \( \preceq \) is stable.

To prove the weak steadiness of \( \preceq \), suppose \( z \preceq xv, z \preceq uy \) and \( x \preceq u \) for \( u, x, z \in C \) and \( v, y \in C^1 \). One can easily see that \( xv \preceq x \), whence \( z \preceq x \) and \( zy \preceq xy \). Suppose \( z \in C_i \) and \( y \in C_j \). If either \( j \preceq i \) or \( i = j \) and \( z \in A_i \), then \( uy \preceq a_i \leq z \leq uy \), a contradiction. It follows that either \( i \preceq j \) or \( i = j \) and \( z \in A_i \). In both cases \( z = zy \preceq xy \).

Thus \( (C, \preceq) \) is a linearly fundamentally ordered semigroup. Now we are able to formulate the main results:

**Theorem 1.** A semigroup can be linearly fundamentally ordered if and only if it belongs to the class of semigroups built up in Construction I. These semigroups are precisely those which satisfy the following elementary universal conditions:

I. \( xy^2 = xy \);

II. \( xyz = xzy \);

III. \( x^2y = x^2 \vee y^2 \) \( x = y^2 \);

IV. \( xy = x^2 \wedge yx = y^2 \) \( x = x^2 \vee y = y^2 \vee x^2 = y^2 \);

V. \( zx = z \wedge z^2y = z \) \( x = x^2 \vee y = y^2 \vee xz = yz \);

VI. \( x^2y = x^2 \rightarrow xy = y^2 \vee x^2 = x \rightarrow (xy = x \wedge yx = x^2) \).

Here \( \vee \) is the sign of disjunction; conjunction and disjunction tie up the formulas stronger than implication does: \( a \wedge \beta \rightarrow \gamma \vee \delta \) means \( (a \wedge \beta) \rightarrow (\gamma \vee \delta) \).

**Theorem 2.** The class of all linearly fundamentally ordered semigroups coincides with the class of ordered semigroups built up in Construction II.

**Corollary.** A semigroup \( S \) can be linearly fundamentally ordered in a unique way if and only if \( S \) satisfies conditions I—VI, \( S \) has at most one right identity and every left zero subsemigroup and every zero subsemigroup of \( S \) have at most two elements.

**Theorem 3.** Every fundamentally ordered on a semigroup \( S \) can be extended to a linear fundamental order if and only if \( S \) has the structure described in Construction I and every left zero subsemigroup of \( S \) either is trivial or consists of right identities of \( S \).

**Theorem 4.** An ordered semigroup \( (S, \preceq) \) is a linearly strictly fundamentally ordered semigroup if and only if \( S = A \cup B \), where \( A \) is a linear semilattice and \( B \) is either empty or a zero semigroup. Every element of \( B \) acts as identity on \( A \), the order \( \preceq \) coincides on \( A \) with the natural semilattice order, all the elements of \( A \) are smaller than all the elements of \( B \), the zero of \( B \) is the smallest element of \( B \).
Proofs. Let \((S, \leq)\) be a linearly fundamentally ordered semigroup. We shall prove that \(S\) satisfies conditions I—VI. If \(x \leq xy\), then \(xy \leq x \cdot y\), \(xy \leq xy \cdot 1\), \(x \leq xy\), whence, by the weak steadiness of \(\leq\), \(xy \leq x \cdot 1 = x\) and \(xy = x\). Thus, \(xy \leq x\) for all \(x, y \in S\). Suppose now \(x \leq yz\). Since \(yz \leq y\), we obtain \(x \leq x, x \leq y \cdot z\), and \(x \leq y\). By the weak steadiness of \(\leq\), \(x \leq xz \leq x\). Thus, \(x \leq yz \rightarrow x = xz\).

The just proved two formulas

\[ xy \leq x, \quad x \leq yz \rightarrow x = xz \]

will be used later on without special reference.

Now \(xy \leq xy\) implies \(xy = xyy = xy^2\), \(xyz = x(yz)^2 = xxyz \leq xzy\). In the same way, \(xzy \leq xy\). Thus, \(xzy = xzy\). For all \(x, y\) and \(z\), there is \(x^2 \leq y^2\) or \(y^2 \leq x^2\). It follows \(x^2y = x^2y^2x = y^2\). If \(xy = x^2\) and \(yx = y^2\), then \(x \leq y^2\) implies \(x^2 = x = x, y \leq x^2\) implies \(y^2 = yx = y, y^2 \leq x \cdot x^2\). \(\leq y\) implies \(y^2 = y^2 \leq xy = x^2\) and, in the same way, \(x^2 \leq y^2\), whence \(x^2 = y^2\).

Let \(zx = z^2y = z\). Then \(z = z^2y = z^3y = z^2\). If \(x \leq z\), then \(x \leq z^2\) and \(xz = x\). Therefore, \(x^2 = (ax)^2 = xxx = xz^2x = xz = x\). In the same way, \(y \leq z \rightarrow y = y^2\). One can easily verify that \((xy)^2 = xz\) and \((yz)^2 = yz\).

In the same way as above, \(y \leq xz \rightarrow y = y^2, x \leq yz \rightarrow x = x^2\). If \(ax \leq y\) and \(yz \leq x\), then \(xz = xz^2 \leq yz\) and \(yz = yz^2 \leq xz\). This, \(xz = yz\). Condition V is verified.

Now let \(x^2y = x^2\). If \(y^2 \leq x\), then \(y^2 = y^4 \leq x^2\) and \(y^2 = y^2x = yxy = yxy^2y = yx^2 = yx\); if \(x \leq y\), then \(x = x^2\), if \(yx \leq x \leq y^2\), then \(y = x\) and \(yx = yx^2 \leq x^2 \leq x^2y = x^2 = x^2 = x^2\). Thus, VI is verified.

To prove that \((S, \leq)\) has the structure described in Construction II, define a binary relation \(\rho\) on \(S\): \((x, y) \in \rho \leftrightarrow x^2y = x^2\). By I, \(\rho\) is reflexive, \(x^2y = x^2\) and \(y^2z = y^2\) imply \(x^2 = x^2y \cdot z = x^2y = x^2 = x^2\), thus \(\rho\) is transitive, i.e., \(\rho\) is a quasi-order relation. Let \(\varepsilon\) be the symmetric part of \(\rho\). Then the quotient set \(S/\varepsilon = I\) is linearly (by III) ordered by \(\rho|\varepsilon\). Let \(-\) be the irreflexive part of \(\rho|\varepsilon\) and \(C_i\) be the \(\varepsilon\)-class corresponding to \((I, -)\). Let \(x \leq y \leq z\) and \(x, z \in C_i\). Then \(x^2 \leq x^2y \leq x^2z = x^2\) and \(x^2y = x^2\), and, on the other hand, \(y^2 = y^2y \leq y^2z \leq x^2\) and \(y^2z = y^2\). Therefore, \(y \in C_i\) and \(C_i\) is convex. Now \(y \leq z \rightarrow (y, z) \in \rho\), as we have proved. It follows that \(S\) is the ordinal sum of \((C_i)_{i \in I}\) along \((I, -)\), each \(C_i\) ordered by the restriction of \(\leq\). If \(x, y \in C_{i_1}\), then \(xy = x^2y = x^2y = x^2\), thus \(C_i\) is subsemigroup of \(S\). Let \(x \in C_i\), \(x \in C_j\), \(i \sim j\). Then \(x^2y = x^2\) and \(y^2x \neq y^2\). If \(yx = y^2\), then \(y^2x = y^2x = y^2\), whence \(yx \neq y^2\) and, by VI, \(xy = x\). Let \(A_i\) be the set of all idempotents in \(C_i\). If \(x \in A_i\) and \(y \in C_i\), then \(xy = x^2y = x^2 = x\), if \(y \leq x\), then \(y \leq x^2\) and \(yx = y\). It follows that \(y^2 = yxy = yx^2y = yxy = yy = y\). Thus \(y \in A_i\), \(C_i\) is the ordinal sum of \(A_i\), and \(C_i \setminus A_i\) ordered by the restrictions of \(\leq\). If \(x, y \in C_i \setminus A_i\), then, by IV, \(x^2 = y^2 \in A_i\) and \(x^2 = x = yx^2 = xy\). The
common value of \(x^2, y^2\), and \(xy\) will be denoted as \(a_i\). Let \(B_i = (C_i \setminus A_i) \cup \{a_i\}\). If \(x \in B_i\) and \(y \in A_i\), then \(y = xy\) and \(xy = xyx = x^2y = a_i\), thus \(B_iC_i = B_iB_i \cup B_iA_i = \{a_i\}\); moreover, \(y \leq x\) and \(y = y^2 \leq x^2 = a_i\), i.e. \(a_i\) is the largest element of \(A_i\). Now let \(x \in C_i\), \(y \in C_j\), \(i \prec j\). Then \(x^2y = x^2\) and \(y^2x \neq y^2\). If \(yx = y^2\), then, by I, \(y^2 = yy^2 = yyx \neq y^2\), a contradiction. By VI, \(x \in A_i\) or \(yx = x^2 = a_i\).

Let \(x \in A_i\). Then \((yx)^2x = (yx)^2\) and \(x^2(xy) = x^2y = x^2y = x^2\), and therefore \(yx \in C_i\). Since \((yx)^2 = yxx = yx^2y = yxx = yx\), \(yx \in A_i\). If \(z \in A_i\), then \(z \leq y\) and \(z = zx \leq yx\). It follows that \(yx\) is the largest element of \(A_i\). If the element \(a_i\) has been defined, then \(yx = a_i\), otherwise let us define \(yx = a_i\). Now \(a_i\) is defined for all \(i \in I\) except the case when \(i\) is the largest element of \(I\) and \(A_i = C_i\). This being the case, define \(a_i\) arbitrarily. We have proved that \((S, \leq)\) has the structure described in Construction II and \(\hat{S}\) has the structure described in Construction I. On the other hand, every semigroup described in Construction I is linearly fundamentally orderable (see Construction II) and every semigroup described in Construction II is linearly fundamentally orderable.

To end the proof of theorems 1 and 2 we need to verify that every semigroup satisfying conditions I—VI is linearly fundamentally orderable.

Suppose \(S\) satisfies I—VI. Define \(\rho\) and \(\varepsilon\) as above. Then \(\hat{S} = \bigcup_{i \in I} (C_i \cup I)\) and \(I\) is linearly ordered by \(\prec\). The same argument as above shows that \(C_i = A_i \cup B_i\), where \(B_i = (C_i \setminus A_i) \cup \{a_i\}\), \(a_i\) being a special element of \(A_i\) defined by axioms I—VI. Now we can see that \(S\) is a semigroup built up along the plan exposed in Construction I. Therefore \(S\) is linearly fundamentally orderable.

To prove the corollary note that, as follows from Construction II and Theorem 2, \(S\) can be linearly fundamentally ordered in a unique way if and only if \(|A_i| \leq 2\), \(|B_i| \leq 2\) and if \(i\) is the largest element of \(I\) and \(C_i = A_i\), then \(|A_i| = 1\) (here \(|A|\) denotes the cardinality of \(A\)). These conditions mean that every left zero subsemigroup and every zero subsemigroup (which are included in an \(A_i\) and a \(B_i\), respectively) have at most two elements; the set of right identities of \(S\) is just the set \(A_i\), where \(i\) is the largest element of \(I\) and \(C_i = A_i\). Thus \(S\) contains at most one right identity.

To prove Theorem 3 note that the identical order relation \((x \leq y \leftrightarrow x = y)\) is fundamental, and so every semigroup with extendable fundamental orders can be linearly fundamentally ordered and, by Theorem 1, satisfies I—VI.

Now let \(S\) be a semigroup satisfying I—VI. By Theorem 1, \(S\) has the structure described in Construction I. Let \(\leq\) be a fundamental order on \(S\). By Construction II and Theorem 2, \(\leq\) may be extended to a linear fundamental order if and only if for all \(x \in C_i, y \in C_j, x \leq y\) the following conditions are satisfied:
1) $i \neq j \rightarrow i \not\rightarrow j$;
2) \[ i = j \land y \epsilon A_i \rightarrow x \epsilon A_i; \]
3) $i = j$, $x = a_i$, and $y \epsilon A_i$ is possible only if $i$ is the largest element of $I$ and $A_i = C_i$.

Thus we have to prove conditions 1)–3). If $u \leq uv$, then $uv \leq u \cdot v$, $uv \leq u \cdot 1$, and $u \leq uv$. By the weak steadiness of $\leq$, $uv \leq u \cdot 1 = u$. Therefore, $u \leq uv \rightarrow u = uv$.

1) Let $i \neq j$. Then $x^2 \leq y^2$, and so $x^2 = x^2 x^2 \leq x^2 y^2 = x^2 y$. It follows that $x^2 = x^2 y$ which means that $i \not\rightarrow j$.

2) Let $i = j$ and $y \epsilon A_i$. Then $x \leq x \cdot 1$, $x \leq y \cdot y = y$, and $x \leq y$. By the weak steadiness of $\leq$, $x \leq x \cdot y = xy^2 = xy^2 x = x^2 y^2 = x^2 y = x^2$. Hence, $x = x^2$.

3) Define the following order relation $\leq$ on $S$: $x \leq y$ means either $x = y$ or $x = a_i \land y \epsilon A_i$ for some $i \epsilon I$. Clearly, if $x \leq y$, then $x, y \epsilon C_i$ for some $i$; therefore $xx = x^2 = x^2 y = x y = y x = y^2 x = y^2 y = y^2$. If $x = y$, then $xz = yz$. If $x \neq y$, then $x = a_i$ and $y \epsilon A_i$; one can easily verify that $xx \leq yz$ in this case. Thus, $\leq$ is stable.

To show $\leq$ is weakly steady, suppose $z \leq xv, x \leq uy, x \leq u$. If $x = u$, then $z \leq uy = xy$. Let $x < u$. Then $x = a_i$ and $u \epsilon A_i$. If $y = 1$, then either $z = u$ or $z = a_i$. In both cases $z \leq xv$ implies $xv = x$, i.e., $z \leq a_i$. Therefore, $z = a_i$ and $z \leq xy$. The same argument holds if $uy = u$.

Now let $uy \neq u$. Then $y \epsilon C_i$, $j \not\rightarrow i$, and $uy = a_j$. Thus $z \leq a_j$. It follows that $z = a_j$, $xy = a_j$, and $z = xy$.

Thus, if $|A_i| \neq 1$ for some $i$ which is not the largest element of $I$ or such that $C_i \neq A_i$, then $\leq$ cannot be extended to a linear fundamental order. Hence conditions of Theorem 3 are necessary. Sufficiency follows from the fact that every semigroup satisfying conditions of Theorem 3 satisfies the above-mentioned condition 3).

The structure of semigroups described by Theorem 3 is the following: let $(I, \to)$ be a linearly irreflexively ordered set, and let $(B_i)_{i \epsilon I}$ be a family of zero semigroups, $a_i$ being the zero of $B_i$. Let $B = \bigcup (B_i)_{i \epsilon I}$. Define the following operation in $B$: if $x \epsilon B_i, y \epsilon B_j$, then $xy = x$ in the case $i \not\rightarrow j$ and $xy = a_i$, otherwise. Let $L$ be a left zero semigroup ($L$ may be empty). Define $S = L \cup B$ and define the binary operation in $S$ as follows: the operation acts in $L$ and in $B$ as the operations of semigroups $L$ and $B$, if $x \epsilon L$ and $y \epsilon B$, then $xy = y^2, yx = y$. The class of just defined semigroups coincides with the class of semigroups on which every fundamental order may be extended to a linear fundamental order.

To prove Theorem 4, suppose $(S, \subseteq)$ is a linearly strictly fundamentally ordered semigroup. Since every strict fundamental order is a fundamental order, $(S, \subseteq)$ has the structure described in Construction II. In the proof of Theorems 1 and 2 we have seen that $xy \leq x$ for all $x \epsilon S$. If $S^*$ is the dual semigroup relative to $S$ (i.e., $xy = z$ in $S^*$ if and only if $yx = z$
in \( S \), then also \((S^*, \leq)\) is a linearly strictly fundamentally ordered semigroup. Hence \( xy \leq y \) for all \( x, y \in S \). If \( x \) and \( y \) are idempotents, then also \( xy \) is. We obtain \( xy = (xy)^2 \leq yx \). In the same way \( yx \leq xy \). Thus \( xy = yx \) and the idempotents of \( S \) commute. It follows that \(|A_i| = 1\) for all \( i \in I \).

Suppose \( x \leq y = y^2 \). Then \( x \leq x \cdot 1, x \leq y \cdot 1, x \leq y \cdot y \). By the steadiness of \( \leq \), \( x \leq xy \). In the same way, \( x \leq yx \). Therefore \( x \leq x \cdot y, x \leq y \cdot y, x \leq y \cdot x \) and, by the steadiness of \( \leq \), \( x \leq xx = x^2 \). Thus \( x = x^2 \).

If \( x \in C_i, y \in A_j \), and \( i \not\rightarrow j \), then \( x \leq y = y^2 \). Thus \( x^2 = x \in A_i \), whence \( C_i = \{a_i\} \).

It follows that \( S \) has the structure described in Theorem 4. Evidently, every semigroup with such a structure is linearly strictly fundamentally ordered.

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