On a system of difference inequalities of parabolic type

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We deal here with a certain system of second order difference inequalities that can be used (cf. [1]) to estimate the convergence of a difference scheme for a system of parabolic partial differential equations of the form

\[ \frac{\partial u_i}{\partial t} = f_i(t, x_1, \ldots, x_n, u_1, \ldots, u_m, \frac{\partial u_i}{\partial x_1}, \ldots, \frac{\partial u_i}{\partial x_n}, \frac{\partial^2 u_i}{\partial x_1^2}, \ldots, \frac{\partial^2 u_i}{\partial x_n^2}), \quad i = 1, \ldots, m. \]

This scheme was investigated by Kowalski in [2]. An analogous theorem for the hyperbolic case was given by Piś in [3].

**Notation.** We shall consider the nodal points \( x^M \) of \( R^{n+1}, 0 \leq M \leq P \), where \( M = (m_0, m_1, \ldots, m_n), \ P = (p_0, p_1, \ldots, p_n) \) are given systems of integers \( m_i, p_i, \ i = 0, 1, \ldots, n, \) and \( 0 \leq M \leq P \) denotes \( 0 \leq m_i \leq p_i \) \( (i = 0, 1, \ldots, n) \). \( x^M = (x_{0}^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}) \), \( x_{0}^{m_0} = m_0 k, x_i^{m_i} = m_i h, i = 1, 2, \ldots, n, \) where \( k = r/N \) and \( k = r/N_1 \) are positive numbers. To each nodal point \( x^M \) there correspond sequences of \( m \) real numbers \( u_i^M \) and \( v_i^M \), \( +M = (m_0+1, m_1, \ldots, m_n), \quad +jM = (m_0, \ldots, m_j-1, m_j+1, m_{j+1}, \ldots, m_n), \quad -jM = (m_0, \ldots, m_j-1, m_j-1, m_{j+1}, \ldots, m_n), \ j = 1, \ldots, n. \)

We shall use the forward and symmetric differences

\[ u_i^{M+} = \frac{1}{h} (u_i^{M+} - u_i^{M-}) , \quad u_i^{Mj} = \frac{1}{2h} (u_i^{+jM} - u_i^{-jM}) \]

and the \( n \)-dimensional vectors

\[ u_i^{Mj} = (u_i^{M1}, u_i^{M2}, \ldots, u_i^{Mn}) . \]

We shall also use the second differences

\[ u_i^{Mjj} = \frac{1}{h} (u_i^{+jM} - 2u_i^{M} + u_i^{-jM}) \]

and the \( n \)-dimensional vectors

\[ u_i^{Mjj} = (u_i^{M11}, u_i^{M22}, \ldots, u_i^{Mnn}) . \]
Theorem 1. Suppose that the scalar functions \( f_i(x, u, q, w), i = 1, \ldots, m, \)
\( x = (x_0, x_1, \ldots, x_n), u = (u_1, \ldots, u_m), q = (q_1, \ldots, q_n), w = (w_1, \ldots, w_n) \) are 
of the class \( C^1 \) in the set \( D = [0, \tau]^{n+1} \times \mathbb{R}^{2m+n} \) and satisfy the conditions

\[
0 \leq \frac{\partial f_i}{\partial u_k} \leq L \quad (i = 1, \ldots, m, k = 1, \ldots, m; i \neq k),
\]

\[
\left| \frac{\partial f_i}{\partial q_j} \right| \leq \Gamma \quad (i = 1, \ldots, m, j = 1, \ldots, n),
\]

\[
0 < g \leq \frac{\partial f_i}{\partial w_j} \leq G \quad (i = 1, \ldots, m, j = 1, \ldots, n).
\]

The numbers \( h \) and \( k \) (mesh sizes) are chosen so as to satisfy

\[
\frac{g}{h} - \frac{\Gamma}{2} > 0,
\]

\[
1 + k \frac{\partial f_i}{\partial u_i} - \frac{2k}{h} \sum_{j=1}^{n} \frac{\partial f_i}{\partial w_j} > 0 \quad (i = 1, \ldots, m).
\]

Let \( u_i^M \) and \( v_i^M, 0 \leq M \leq P \), satisfy the difference inequalities

\[
u_i^M \ll f_i(x^M, u^M, u_i^M, u_i^{MI}), \quad v_i^M \gg f_i(x^M, v^M, v_i^M, v_i^{MI}),
\]

\[
i = 1, \ldots, m
\]

for \( 0 \leq m_0 \leq p_0 - 1, 1 \leq m_i \leq p_i \) \((i = 1, \ldots, n)\), and the inequalities

\[
u_i^M \leq v_i^M \quad \text{for } m_0 = 0, \quad \text{or } m_1 = 0, \ldots, \text{ or } m_n = 0,
\]

\[
\text{or } m_i = N, \ldots, \text{ or } m_n = N.
\]

Then the inequalities

\[
u_i^M \leq v_i^M \quad \text{for } 0 \leq M \leq P
\]

hold true.

Proof. We shall prove our theorem by induction. Write \( r_i^M = u_i^M - v_i^M \). It is sufficient to prove that

\[
r_i^M \leq 0 \quad \text{for } 0 \leq M \leq P, \ i = 1, \ldots, m.
\]

Inequalities (9) are satisfied for \( m_0 = 0 \) in virtue of assumption (7). Suppose it is satisfied for \( m_0 = j \). Let

\[
\max_{m_0 = j+1} \max_{0 \leq M \leq P} r_i^M = r_i^{M(j)}
\]
and $B(i)$ be such a multi-index that $+B(i) = A(i)$ (cf. Fig. 1). The equality

$$r^{A(i)}_i = r^{B(i)}_i + kr^{B(i)}_i$$

follows from the definition of $u^{M(i)}_i$ and $v^{M(i)}_i$.

![Fig. 1](image)

From (6), by the mean value theorem, it follows that

$$r^{B(i)}_i = u^{B(i)}_i - v^{B(i)}_i$$

$$\leq f_i(x^{B(i)}, u^{B(i)}, u^{B(i)I}, u^{B(i)II}) - f_i(x^{B(i)}, v^{B(i)}, v^{B(i)I}, v^{B(i)II})$$

$$= \sum_{i=1}^{m} \frac{\partial f_i}{\partial u_i} r_i^{B(i)} + \frac{1}{2h} \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_i} (r_i^{+jB(i)} - r_i^{-jB(i)}) +$$

$$+ \frac{1}{h^2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial w_i} (r_i^{+jB(i)} - 2r_i^{B(i)} + r_i^{-jB(i)}) .$$

Now we return to (10). Using (11), after a suitable regrouping of terms, we can write

$$r^{A(i)}_i \leq r^{B(i)}_i \left(1 + k \frac{\partial f_i}{\partial u_i} - \frac{2k}{h^2} \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_i} \right) +$$

$$+ \frac{k}{h} \sum_{j=1}^{n} \left( \frac{1}{2} \frac{\partial f_i}{\partial q_i} + \frac{1}{h} \frac{\partial f_i}{\partial w_i} \right) r_i^{+jB(i)} +$$

$$+ \frac{k}{h} \sum_{j=1}^{n} \left( \frac{1}{h} \frac{\partial f_i}{\partial w_i} - \frac{1}{2} \frac{\partial f_i}{\partial q_i} \right) r_i^{-jB(i)} + k \sum_{i=1}^{m} \frac{\partial f_i}{\partial u_i} r_i^{B(i)} .$$
We shall prove that the right-hand side of this inequality is non-positive. In fact, we have
\[ r^M_i \leq 0 \]
for \( M = +B(i), \) \( M = +jB(i), \) \( M = -jB(i) \) (\( i = 1, \ldots, m, \) \( j = 1, \ldots, n \)) because of the induction assumption. We have also:
\[
\frac{1}{h} \frac{\partial f_i}{\partial w_j} + \frac{1}{2} \frac{\partial f_i}{\partial q_j} \geq g \frac{\Gamma}{h} \frac{2}{2} \geq 0, \quad \frac{1}{h} \frac{\partial f_i}{\partial w_j} - \frac{1}{2} \frac{\partial f_i}{\partial q_j} \geq g \frac{\Gamma}{h} \frac{2}{2} \geq 0,
\]
because of (2), (3) and (4). The first term of right-hand side of inequality (12) is non-positive in virtue of (5). Hence (10) hold true for \( m_0 = j+1, \) \( 0 \leq M \leq P. \) This completes the proof.

**Remark.** For \( i = 1, \) \( u \in \mathbb{R}^1, \) and Assumption (1) becomes useless.

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**References**


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